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Stable Decompositions of Coalition Formation Games

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Stable decompositions of coalition formation games*

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Abstract

It is known that a coalition formation game may not have a stable coalition structure. In this study we propose a new solution concept for these games, which we call "stable decomposition", and show that each game has at least one. This solution consists of a collection of coalitions organized in sets that "protect" each other in a stable way. When sets of this collection are singletons, the stable decomposition can be identified with a stable coalition structure. As an application, we study convergence to stability in coalition formation games.

JEL classification: C71, C78.
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1 Introduction

The literature on matching has recently emerged as one of the most successful, widely applied, and policy-relevant branches of economic theory: Understanding and designing mechanisms for school choice or kidney exchange have been significantly enhanced by the insights provided by a wide variety of matching models (see Roth, 2018,

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and references therein). From a theoretical perspective, all these problems can be formalized as coalition formation games in which each agent has preferences over the set of coalitions in which he/she may participate. Depending on the structure of the coalitions that the agents are allowed to form, these problems include hedonic games, one-sided problems such as the roommate problem, and two-sided problems running from the classical marriage problem to many-to-one matching problems with peer effects and externalities.

One of the main goals of analyzing coalition formation games is to predict which coalitions will form and the most widely studied solution to date is that of stability. A coalition structure (i.e. a set of coalitions that partition the set of agents) is stable if it "protects" its coalitions in the following way: Whenever an agent has an incentive to deviate to a different coalition outside the structure, another agent has incentive not to allow that deviation, and thus prevents its formation. However, coalition formation games, in general, may have no stable coalition structure. The question then is whether a more general solution concept that retains the appeal of stability can be provided for situations in which there are no stable coalitions.

This paper tackles that question and provides a positive answer by introducing a novel solution concept for the entire class of coalition formation games, called *stable decomposition*. Our solution is based on the idea of protection, but applied in a more general framework.[1] That is to say, instead of having a partition of the set of agents that protects their coalitions, as in a stable coalition structure, we study how a collection formed by special sets of coalitions protects its members in a way that resembles stability. One of the most important properties of our solution concept is that each coalition formation game has at least one stable decomposition, independently of whether a stable coalition structure exists or not.

The special sets in a stable decomposition, called "parties", are of three different types: (i) A set consisting of singletons; (ii) sets consisting of a unique non-single coalition; and (iii) sets consisting of several non-single coalitions, called "ring components". Ring components exhibit a robust cyclical behavior with respect to the (unanimous) preference of the agents by which one coalition is preferred to another if the first is preferred to the second by each agent at the intersection of the two coalitions.

The way in which a stable decomposition protects its parties depends on the party at hand. When the party is a ring component or a set consisting of a non-single coalition, protection of the party by the stable decomposition implies that if some agents of the party want to form a coalition that is not in the party, then another party of the decomposition prevents such a coalition from being formed. When the party is the set

[1] The concept of stable decomposition is inspired by the notion of "stable partition" in roommate problems due to Tan (1991).

consisting of singletons, protection by the stable decomposition means that no party of the decomposition prevents the formation of non-single coalitions consisting of agents in the set of singletons.

We find that a stable decomposition with no ring components can be identified with a stable coalition structure. Also, any stable coalition structure can be interpreted as a special type of stable decomposition, so our new solution concept identifies stable coalitions structures when they exist.

An important feature of our solution concept is its close link to the known solution concept of the "absorbing set".[2] To formalize this solution and describe its relation to the stable decomposition solution, we first need to determine a dynamic process by means of a domination relation between coalition structures. In this paper, we adopt the standard (myopic) dynamic process which starts from a non-stable coalition structure and forms a new one containing a coalition of better-off agents, in which agents which are abandoned are single and all other coalitions remain unchanged. Given this dynamic process, an absorbing set is a minimal collection of coalition structures which, once entered throughout this dynamic process, is never left. A known result in the literature is that every coalition formation game has at least one absorbing set (see Shenoy, 1979). In particular, a stable coalition structure is not dominated by any other coalition structure, so it coincides with a trivial absorbing set (i.e. a singleton). By contrast, a non-trivial absorbing set is formed by several coalition structures that exhibit a cyclical behavior.

The main result of the paper presents a one to one correspondence between stable decompositions and absorbing sets: Each stable decomposition generates an absorbing set and, conversely, each absorbing set generates a stable decomposition (Theorem 1). As a result, each coalition formation game has a stable decomposition (Corollary 1).

The cyclical behavior of coalitions on agents' preferences has been studied through the concept of a "ring of coalitions". However, the notion of a ring is not sufficiently robust for two reasons: (i) A ring with disjoint coalitions does not always generate such behavior; and (ii) rings can overlap. To incorporate such subtleties, in our definition of stable decomposition we introduce the notion of a ring component that can be seen as the union of overlapping rings that have a joined cyclical behavior.

An auxiliary result in our paper, which is of interest in it self, is the relation between the cyclical behavior of coalitions (rings) and the cyclical behavior of coalition structures, which we call "cycles". We prove that there is a ring of coalitions if and only if there is a cycle of coalition structures (Theorem 2). We further present an algorithm that constructs a ring of coalitions, given a cycle of coalition structures.

[2]As far as we know, Schwartz (1970) was the first to introduce this notion for collective decision making problems.

In spite of presenting a one-to-one correspondence between the stable decompositions and the absorbing sets of the game (Corollary 2), we claim that ours is a more compelling solution than that of an absorbing set. A stable decomposition is a simpler object because (unlike absorbing sets) its definition depends only on agents' preferences and not on the coalition structures, and thus does not require a dynamic process between coalition structures to be established. Furthermore, the notion of stable decomposition has more explanatory power than that of an absorbing set: It identifies the sets of non-single coalitions, ring components, responsible for the cyclical behavior between coalition structures.

Finally, we present some applications of stable decompositions in different models. In roommate problems, the notion of stable partition which is useful for identifying the existence of stable matchings is not sufficient to induce an absorbing set. Inarra et al. (2013) identify the stable partitions that do this job, and call them maximal stable partitions. We show that our proposed of solution coincides with a maximal stable partition (Proposition 2). In marriage problems, we show that no stable decomposition has a ring component, providing an alternative proof that the set of stable matchings is non-empty (Proposition 3). One significant application that we discuss is the problem of convergence to stability. From a market design point of view, this formalizes particular dynamics of coalition formation that may emerge in the absence of a central planner. In cases where decentralized decision making in itself may not suffice to bring about a stable outcome, a centralized coordinating process must be imposed to that end. Decentralized processes can be formalized through the aforementioned dynamic process among coalition structures. A coalition formation game exhibits convergence to stability if, starting from any coalition structure, the dynamic process always leads towards a stable coalition structure. We find that a stable coalition formation game exhibits convergence to stability if and only if non of its stable decompositions has a ring component (Proposition 4). This means that marriage and stable roommate problems exhibit convergence to stability (Corollaries 3 and 4).

Related literature

Our paper is related to several papers in cooperative and matching theory literature. The concept of an absorbing set has appeared in earlier publications in different contexts and under different names. As stated above, Schwartz (1970) was the first to introduce this notion. Shenoy (1979, 1980) proposed it under the name of "elementary dynamic solution" for n-person cooperative games. Jackson and Watts (2002) use the name "closed cycles" and Olaizola and Valenciano (2014) that of absorbing sets, and apply this solution to the network problem.[3] Absorbing sets are analyzed by Inarra

[3]The union of absorbing sets gives the "admissible set" (Kalai and Schmeidler, 1977), a solution defined for abstract systems and applied to various bargaining situations. Recently, Demuynck et al.

et al. (2013) for the roommate problem, showing that a roommate problem has either trivial absorbing sets (stable matchings) or non-trivial absorbing sets. However, this division does not applied for most coalition formation games, where trivial and non-trivial absorbing sets frequently coexist.

It is clear that the definition of an absorbing set depends on what dynamic process is chosen. For instance, following Knuth (1976) for the marriage problem, Tamura (1993) considers problems with equal numbers of men and women, all of them mutually acceptable, in which all agents are always matched. Unlike the standard blocking dynamics, this dynamic process assumes that when a couple satisfies a blocking pair the abandoned partners also match to each other. Knuth (1976) poses the question of whether there is convergence to stability in this model and Tamura (1993) gives a counter-example in which some matchings cannot converge to any stable matching. The example shows the coexistence of five absorbing sets of cardinality one and one of cardinality sixteen.

Numerous papers have studied whether there are decentralized matching markets that converge to stability.[4] Roth and Vande Vate (1990) introduce a process for studying convergence to stability for the marriage problem and Chung (2000) generalizes that process for the roommate problem with weak preferences. Later, Klaus and Klijn (2005) extend it for many-to-one matching with couples and Kojima and Ünver (2008) for many-to-many matching problems. Eriksson and Häggström (2008) show that a stable matching can be attained by means of a decentralized market, even in cases of incomplete information. Following a different approach, Diamantoudi et al. (2004) analyze convergence to stability in the stable roommate problem with strict preferences.

The rest of the paper is organized as follows. Section 2 presents the preliminaries, the dynamics of the domination relation and the notion of absorbing set. Section 3 introduces the notion of the ring component and sets out the definition of stable decomposition. This notion enables us to establish our characterization result of the solution in terms of absorbing sets. We also analyze the relation between rings of coalitions and cycles of coalition structures. Section 4 contains applications of our results to the marriage and roommate problems and to convergence to stability. Some concluding remarks are presented in Section 5. Finally, Appendix A contains some lemmata and proofs omitted in the main text.

(2019) define the "myopic stable set" in a very general class of social environments and study its relation to other solution concepts.

[4]There is also an unpublished manuscript by Pápai (2003) that addresses this problem.

2 Preliminaries

In this section we present the notation, the domination relation and the absorbing sets induced by it. Let $N = \{1, \ldots, n\}$ be a finite set of *agents*. A non-empty subset C of N is called a *coalition*. Each agent $i \in N$ has a strict, transitive *preference relation* over the set of coalitions to which he/she belongs, denoted by $\succ_i$. Given coalitions C and C', when agent $i \in C \cap C'$ prefers coalition C to C' we write $C \succ_i C'$. We say that C *is (unanimously) preferred to* C', and write $C \succ C'$, if $C \succ_i C'$ for each $i \in C' \cap C$.

A preference profile $\succ_N = (\succ_i)_{i \in N}$ defines a *coalition formation game* which is denoted by $(N, \succ_N)$. Let $\mathcal{S} = \{\{i\} : i \in N\}$ be the set of singletons and $\mathcal{K} = \{C \in 2^N \setminus \mathcal{S} : C \succ_i \{i\} \text{ for each } i \in N\}$ be the set of *permissible* coalitions of game $(N, \succ_N)$.[5] Let Π denote the set of partitions of N formed by permissible coalitions or singletons, which we call *coalition structures.* A generic element of Π is denoted by π. For each $\pi \in \Pi$, $\pi(i)$ denotes the coalition in π that contains agent i.

Let $\mathcal{B} \subseteq \mathcal{K} \cup \mathcal{S}$ be a collection of coalitions. Let $|\boldsymbol{\mathcal{B}}|_{\mathcal{K}}$ denote the number of non-single coalitions included in $\mathcal{B}$, i.e., $|\mathcal{B}|_{\mathcal{K}} = |\{C : C \in \mathcal{B} \cap \mathcal{K}\}|$. A set $\mathcal{M}(\mathcal{B}) \subseteq \mathcal{B} \cap \mathcal{K}$ is *maximal for* $\mathcal{B}$:

(i) If $C, C' \in \mathcal{M}(\mathcal{B})$ implies $C \cap C' = \emptyset$.

(ii) If there is $C \in (\mathcal{B} \cap \mathcal{K}) \setminus \mathcal{M}(\mathcal{B})$, then there is $C' \in \mathcal{M}(\mathcal{B})$ such that $C \cap C' \neq \emptyset$.

Notice that, for any coalition structure $\pi \in \Pi$, there is a unique maximal set $\mathcal{M}(\pi)$ consisting of all its non-single coalitions.[6]

Given a collection of non-single coalitions $\mathcal{B} \subset \mathcal{K}$ and a coalition $C \in \mathcal{K} \setminus \mathcal{B}$, we say that C *breaks* $\mathcal{B}$ if there is a maximal set $\mathcal{M}(\mathcal{B})$ such that:

(i) there is a coalition $C' \in \mathcal{M}(\mathcal{B})$ such that $C \cap C' \neq \emptyset$, and

(ii) $C \succ C''$ for each $C'' \in \mathcal{M}(\mathcal{B})$ with $C'' \cap C \neq \emptyset$.

Furthermore, given coalition structure π and a coalition $C \in \mathcal{K} \setminus \pi$, we say that C *blocks* π if $C \succ \pi(i)$ for each $i \in C$. This is equivalent to saying that either (i) C breaks the collection of non-single coalitions of π; or (ii) agents in C are singles in π, i.e. $\pi(i) = \{i\}$ for each $i \in C$. [7]

The main solution concept for a coalition formation game is that of stability, namely a coalition structure that is immune to deviation of coalitions. In such games, a coalition structure $\pi \in \Pi$ is *stable* if the existence of $C \in \mathcal{K}$ and $i \in C$ such that $C \succ_i \pi(i)$

[5] Throughout the paper 2^N denotes the collection of non-empty subsets of N.

[6] Here we only consider coalition structures different from that which fulfills $\pi(i) = \{i\}$ for each $i \in N$.

[7] From now on, when C breaks a collection of non-single coalitions of π, we simply say that C *breaks* π.

implies the existence of $j \in C$ such that $\pi(j) \succ_j C$. Hereafter, a stable coalition structure is denoted by π^N.

2.1 The domination dynamics

As mentioned above, a stable coalition structure is immune to any coalitional deviation. But if a coalition structure is not stable then its blocking by a coalition does not specify its transformation into a new coalition structure. However, once a coalition structure has been blocked there is no single way to define how the new coalition structure emerges. If one or more agents leave a coalition, what happens to the remaining agents? Do they become singletons or do they remain together? Hart and Kurz (1983) argue that if a coalition is understood as an agreement of all its members and then some agents leave, the agreement breaks down and the remaining agents become singletons. In our analysis this interpretation fits well, because our modeling only considers permissible coalitions and singletons, and the coalition of abandoned agents might not be permissible once a new coalition is formed.

Definition 1 *Let $(N, \succ_N)$ be a coalition formation game. The **domination relation** $\gg$ over Π is defined as follows: $\pi' \gg \pi$ if and only if there is $C \in \mathcal{K}$ such that*

(i) $C \in \pi'$ and $C \succ \pi(i)$ for each $i \in C$,

(ii) for each $C' \in \pi$ such that $C' \cap C \neq \emptyset$, $\pi'(j) = \{j\}$ for each $j \in C' \setminus C$,

(iii) for each $C' \in \pi$ such that $C' \cap C = \emptyset$, $C' \in \pi'$.

*To stress the role of coalition C, π' is said to dominate π via C, and $\boldsymbol{\pi' \gg \pi}$ **via** $\boldsymbol{C}$ is written.*

Condition (i) says that each agent i of coalition C improves in π' with respect to his/her position in π. Condition (ii) says that coalitions from which one or more agents depart break into singletons in π'. Condition (iii) says that the coalitions that do not suffer any departure in π, remain unchanged in π'. Notice that the domination relation $\gg$ implies that agents behave myopically, in the sense that they take the decision about blocking a coalition structure by considering just the resulting coalition, i.e. they are unable to foresee their positions even one step ahead.

Remark 1 *The domination relation $\gg$ is irreflexive, antisymmetric and not necessarily transitive.*

Given $\gg$, let $\gg^T$ be the *transitive closure* of $\gg$. That is, $\pi' \gg^T \pi$ if and only if there is a finite sequence of coalition structures $\pi = \pi^0, \pi^1, \ldots, \pi^J = \pi'$ such that, for all $j \in \{1, ..., J\}$, $\pi^j \gg \pi^{j-1}$.

2.2 Absorbing sets

An absorbing set is a minimal set of coalition structures that, once entered through the domination relation, is never left. Formally,

Definition 2 *Let* $(N, \succ_N)$ *be a coalition formation game. A non-empty set of coalition structures* $\mathcal{A} \subseteq \Pi$ *is an* ***absorbing set*** *whenever for each* $\pi \in \mathcal{A}$ *and each* $\pi' \in \Pi \setminus \{\pi\}$,

$$\pi' \gg^T \pi \text{ if and only if } \pi' \in \mathcal{A}.$$

If $|\mathcal{A}| \geq 3$, $\mathcal{A}$ *is said to be a* ***non-trivial absorbing set****. Otherwise, the absorbing set is* ***trivial****.*

Notice that coalition structures in $\mathcal{A}$ are symmetrically connected by the relation $\gg^T$, and that no coalition structure in $\mathcal{A}$ is dominated by a coalition structure not in that set. Next, we introduce a remark containing five facts about absorbing sets.

Remark 2 *Facts on absorbing sets.*

(i) Each coalition formation game has an absorbing set.

(ii) An absorbing set $\mathcal{A}$ *contains no stable coalition structure if and only if* $|\mathcal{A}| \geq 3$.

(iii) π^N *is a stable coalition structure if and only if* $\{\pi^N\}$ *is an absorbing set.*

(iv) For each non-stable coalition structure $\pi \in \Pi$, *there are an absorbing set* $\mathcal{A}$ *and a coalition structure* $\pi' \in \mathcal{A}$ *such that* $\pi' \gg^T \pi$.

(v) For each absorbing set $\mathcal{A}$, *either* $|\mathcal{A}| = 1$ *or* $|\mathcal{A}| \geq 3$.

Remark 2 (i) follows from Definition 2 and the finiteness of Π. (ii) is implied by the antisymmetry of $\gg$. Remark 2 (iii) recalls that each stable coalition structure is in itself an absorbing set. Remark 2 (iv) says that from any non-stable coalition structure there is a finite sequence of such structures that reaches a coalition structure of an absorbing set (this property is called outer stability in Kalai and Schmeidler (1977)). Remark 2 (v) is straightforwardly implied by (i) and (ii).

3 Stable decomposition

In this section we present a new solution concept for the entire class of coalition formation games. We find that our proposed solution always exists and can be characterized in terms of absorbing sets. Furthermore, when the game has stable coalition structures, they can be identified with solutions of our type. First, we present the key ingredient for defining such a solution, which generalizes the well-known concept of a ring of coalitions.

3.1 Ring components

A ring of coalitions is an ordered set of non-single coalitions that behaves cyclically, i.e., for each pair of consecutive coalitions of the ordered set the successor coalition is preferred to its predecessor. Formally,

Definition 3 *An ordered set of non-single coalitions* $(R_1, \ldots, R_J) \subseteq \mathcal{K}$*, with* $J \geq 3$*, is a* ***ring*** *if* $R_{j+1} \succ R_j$ *for* $j = 1, \ldots, J$ *subscript modulo* J*.*

For the sake of convenience, we sometimes identify a ring with the non-ordered set of its coalitions, $\mathcal{R} = \{R_1, \ldots, R_J\}$, and refer to coalitions in $\mathcal{R}$ as *ring coalitions*. Notice that the definition of a ring requires that all agents at the intersection of two consecutive ring coalitions should better off.[8]

A ring of coalitions is not a robust enough notion because in a coalition formation game: (i) there may be multiple rings and some of them may overlap; and (ii) some rings may have a maximal set of coalitions that cannot be broken by any other ring coalition. To address these issues, we present the notion of a "ring component".

Definition 4 *A* ***ring component*** $\mathcal{RC}$ *is a subset of* $\mathcal{K}$ *with* $|\mathcal{RC}|_{\mathcal{K}} \geq 3$ *satisfying:*

(i) $R \succ^T R'$ *for each pair* $R, R' \in \mathcal{RC}$ *with* $R \neq R'$*,*[9]

(ii) for each maximal set $\mathcal{M}(\mathcal{RC})$ *there is* $R \in \mathcal{RC} \setminus \mathcal{M}(\mathcal{RC})$ *such that* R *breaks* $\mathcal{M}(\mathcal{RC})$*.*

Thus, a ring component is a collection of rings such that: (i) each coalition in the collection is transitively preferred to any other coalition in the collection; and (ii) each maximal set of coalitions of the ring component is broken by a coalition of the collection.

Example 1 *Consider the game given by the following table:*

1	**2**	**3**	**4**	**5**	**6**	**7**
12	23	34	467	15	67	467
123	123	123	45	45	467	67
15	12	23	34	5	6	7
1	2	3	4			

[8] There are several ways of defining rings of coalitions. Pycia (2012) and Inal (2015) define cyclicity among coalitions by requiring that only one agent at the intersection of two consecutive coalitions strictly prefer the first of them to the second. In both these definitions, unlike ours, other members of two consecutive coalitions can oppose the transition from one coalition to the next.

[9] Here $\succ^T$ denotes the transitive closure of relation $\succ$.

In this game there are two rings:$\{15, 12, 23, 34, 45\}$ *and* $\{15, 123, 34, 45\}$*. Ring* $\{15, 13, 12, 23, 34, 45\}$ *meets both conditions of Definition 4 and is thus a ring component. However ring* $\{15, 123, 34, 45\}$ *does not meet condition (ii) of Definition 4 because, for instance, the maximal set* $\{123, 45\}$ *cannot be broken. Moreover, their union satisfies condition (i) but not condition (ii) because, again,* $\{123, 45\}$ *cannot be broken.* ♢

Two types of ring components can be distinguished according to how their maximal sets behave. We say that a ring component $\mathcal{RC}$ is *simple* if for each maximal set $\mathcal{M}(\mathcal{RC})$ and each coalition $R \in \mathcal{RC}$ such that R breaks $\mathcal{M}(\mathcal{RC})$ there is only one $R' \in \mathcal{M}(\mathcal{RC})$ such that $R \cap R' \neq \emptyset$. Otherwise, $\mathcal{RC}$ is *not simple*. The following two examples illustrate the two types of ring components.

Example 1 (Continued) *In Example 1, the unique ring component is* $\mathcal{RC} = \{12, 23, 34, 45, 15\}$ *and its maximal sets are* $\{12, 34\}, \{12, 45\}, \{23, 45\}, \{23, 15\}$*, and* $\{34, 15\}$*. It is easy to see that each coalition in* $\mathcal{RC}$ *that breaks a maximal set has a non-empty intersection with only one coalition of the maximal set, so* $\mathcal{RC}$ *is simple.* ♢

Example 2 *Consider the game given by the following table:*

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
12	23	356	145	356	678	78	678
145	12	23	46	145	46	678	78
1	2	3	4	5	356	7	8
					6		

The unique ring component is $\mathcal{RC} = \{145, 12, 23, 356, 46\}$ *and its maximal sets are* $\{145, 23\}$*,* $\{12, 356\}, \{12, 46\}$*, and* $\{23, 46\}$*. Notice that the maximal set* $\{145, 23\}$ *is broken by coalition* 356 *that has non-empty intersection with coalitions* 145 *and* 23*. Then,* $\mathcal{RC}$ *is not simple.* ♢

3.2 A new general solution concept

Inspired by the idea of the "stable partition" introduced by Tan (1991) for roommate problems, we next present our solution concept for general coalition formation games, which we call "stable decomposition". This concept applies the idea of "protection" not to coalitions but to special sets of coalitions that we call "parties". A stable decomposition, then, is a collection of parties that involves every agent of the game and that protects its parties. This means that if an external coalition breaks a party, then another party of the decomposition "prevents the formation" of such an external coalition. Below we give precise definitions of all the notions involved in our solution concept.

Formally, a set of coalitions $\mathcal{B} \subset \mathcal{K} \cup \mathcal{S}$ is a *party* if one of the following conditions holds:

(i) $\mathcal{B} \subset \mathcal{S}$,

(ii) $\mathcal{B} \subset \mathcal{K}$ and $|\mathcal{B}|_{\mathcal{K}} = 1$,

(iii) $\mathcal{B}$ is a ring component.

Given a ring component $\mathcal{RC}$, its *compact collection* $\mathcal{C}(\mathcal{RC})$ is defined as follows. If $\mathcal{RC}$ is simple, then $\mathcal{C}(\mathcal{RC})$ is equal to the collection of maximal sets of $\mathcal{RC}$. If $\mathcal{RC}$ is not simple, then $\mathcal{C}(\mathcal{RC}) = \{\{R\} : R \in \mathcal{RC}\}$.

Now, given a party $\mathcal{B} \subset \mathcal{K}$ and a coalition $C \in \mathcal{K} \setminus \mathcal{B}$ such that $N(\mathcal{B}) \cap C \neq \emptyset$, we say that $\mathcal{B}$ *impedes coalition* C *to be formed* if:[10]

(i) $\mathcal{B} = \{C'\}$ and there is an agent $i \in C' \cap C$ such that $C' \succ_i C$, or

(ii) $\mathcal{B}$ is a ring component and for each $\mathcal{E} \in \mathcal{C}(\mathcal{B})$ there are a coalition $C' \in \mathcal{E}$ with $C' \cap C \neq \emptyset$ and an agent $i \in C' \cap C$ such that $C' \succ_i C$.

Condition (i) states that when a party is formed by only one coalition, there is an agent that prefers to stay in the coalition of the party rather than move to the external coalition. Condition (ii) says that when the party is a ring component, for each set of the compact collection of the ring component there is an agent that belongs to a coalition in that set that prefers to stay in that coalition rather than move to the external coalition. Notice that when a ring component is simple the maximal sets of the ring component are the objects that prevent the formation of external coalitions, but if the ring component is not simple the individual coalitions of the ring component are the objects that prevent the formation of external coalitions.

Given a party $\mathcal{B}$, denote by $N(\mathcal{B})$ the set of agents that belong to (at least) one coalition in $\mathcal{B}$, that is, $N(\mathcal{B}) \equiv \bigcup_{C \in \mathcal{B}} C$. A collection of parties $\{\mathcal{B}_1, \ldots, \mathcal{B}_L\}$ *partitions* N if $\{N(\mathcal{B}_1), \ldots, N(\mathcal{B}_L)\}$ forms a partition of N. Furthermore, given a party $\mathcal{B}$ and a collection of parties $\mathscr{D}$ that partitions N, $\mathcal{B}$ *is said to be protected by* $\mathscr{D}$ if for each coalition C that breaks $\mathcal{B}$ there is $\mathcal{B}' \in \mathscr{D}$ such that $\mathcal{B}'$ prevents C from being formed.[11] Our solution concept for coalition formation games can now be defined.

Definition 5 *A* ***stable decomposition*** *is a collection of parties* $\mathscr{D}$ *that partitions* N *and satisfies the following:*

(i) Each $\mathcal{B} \in \mathscr{D}$ *such that* $\mathcal{B} \subset \mathcal{K}$ *is protected by* $\mathscr{D}$.

(ii) There is at most one $\mathcal{B}^\star \in \mathscr{D}$ *such that* $\mathcal{B}^\star \subset \mathcal{S}$. *Moreover, for each party* $\mathcal{B}' \subset \mathcal{K}$ *with* $N(\mathcal{B}') \subseteq N(\mathcal{B}^\star)$, $\mathcal{B}'$ *is not protected by* $\mathscr{D}$.

[10]From now on, if there is no confusions we sometimes write $\mathcal{B}$ with $|\mathcal{B}|_{\mathcal{K}} > 1$ instead of $\mathcal{RC}$ to denote a ring component.

[11]Notice that party $\mathcal{B}$ need not be included in $\mathscr{D}$.

Condition (i) is the stability condition within parties in $\mathscr{D}$. Condition (ii) does not allow the formation of parties formed solely by the agents in party $\mathcal{B}^\star$. The reason for this exclusion is that those parties cannot be protected by $\mathscr{D}$.

Remark 3 *When $\mathscr{D}$ is such that each $\mathcal{B} \in \mathscr{D}$ satisfies $|\mathcal{B}|_{\mathcal{K}} \leq 1$, a stable decomposition can be identified with a stable coalition structure. Moreover, if $\pi^N = \{C_1, \ldots, C_L\}$ is a stable coalition structure then $\mathscr{D}$ is set as follows. Assume w.l.o.g. that the first k coalitions of π^N with $k \leq L$ satisfy the requirement that $C_k \in \mathcal{K}$ and the remaining coalitions belong to $\mathcal{S}$. Thus,*

$$\mathscr{D} = \{\{C_1\}, \ldots, \{C_k\}, \{C_{k+1}, \ldots, C_L\}\}$$

is a stable decomposition.

The following definition singles out a special type of coalition structure induced by a stable decomposition.

Definition 6 *Let $\mathscr{D}$ be a stable decomposition. A $\mathscr{D}$ – **coalition structure** is a coalition structure $\pi_{\mathscr{D}}$ such that:*

(i) for each $\mathcal{B} \in \mathscr{D}$ with $|\mathcal{B}|_{\mathcal{K}} \leq 1$, $\pi_{\mathscr{D}} \cap \mathcal{B} = \mathcal{B}$,

(ii) for each $\mathcal{B} \in \mathscr{D}$ with $|\mathcal{B}|_{\mathcal{K}} \geq 3$, $\pi_{\mathscr{D}} \cap \mathcal{B} = \mathcal{E}$ for some $\mathcal{E}$ in $\mathcal{C}(\mathcal{B})$. Furthermore, $\pi_{\mathscr{D}}(i) = \{i\}$ for each $i \in N(\mathcal{B}) \setminus N(\mathcal{E})$.

Condition (i) says that each party of $\mathscr{D}$ that is not a ring component must be included in the $\mathscr{D}$ – coalition structure. Condition (ii) says that for each party that is a ring component, the $\mathscr{D}$ – coalition structure includes a set of its compact collection and the rest of the agents of the ring component are single in this coalition structure.

Given a stable decomposition $\mathscr{D}$ and a $\mathscr{D}$– coalition structure $\pi_{\mathscr{D}}$, the *set generated by* $\pi_{\mathscr{D}}$, denoted by $\mathcal{A}_{\mathscr{D}}$, is the set formed by $\pi_{\mathscr{D}}$ together with all the coalition structures that transitively dominate it. Formally,

$$\mathcal{A}_{\mathscr{D}} \equiv \{\pi_{\mathscr{D}}\} \cup \{\pi \in \Pi : \pi \gg^T \pi_{\mathscr{D}}\}.$$

The following result states that each set generated by a $\mathscr{D}$– coalition structure is actually an absorbing set.

Proposition 1 *If $\mathscr{D}$ is a stable decomposition, then $\mathcal{A}_{\mathscr{D}}$ is an absorbing set. Furthermore, if there is a ring component in $\mathscr{D}$, $\mathcal{A}_{\mathscr{D}}$ is a non-trivial absorbing set.*

Proof. See proof in Appendix A. □

The absorbing set $\mathcal{A}_{\mathscr{D}}$ depends only on the stable decomposition $\mathscr{D}$, and not on the specific $\mathscr{D}-$ coalition structure selected to construct it (see Lemmata 4 and 5 in Appendix A).

We are now in a position to present the main result of the paper, which is that absorbing sets and sets generated by $\mathscr{D}-$ coalition structures are equivalent.

Theorem 1 *$\mathcal{A}$ is an absorbing set if and only if $\mathcal{A} = \mathcal{A}_{\mathscr{D}}$ for a stable decomposition $\mathscr{D}$.*

To prove Theorem 1, the relation between cyclical behavior in agents' preferences and cyclical behavior of coalition structures must first be understood. That relation is studied in the next subsection. The proof of Theorem 1 is thus relegated to Appendix A.

Each coalition formation game has an absorbing set and, by Theorem 1, each absorbing set is the set generated by a $\mathscr{D}-$ coalition structure for a stable decomposition $\mathscr{D}$, so the following corollaries hold:

Corollary 1 *Each coalition formation game has a stable decomposition.*

Corollary 2 *For each coalition formation game there is a bijection between absorbing sets and stable decompositions.*

These corollaries, together with Remark 3, mean that our solution concept is always non-empty and that it generalizes the concept of the stable coalition structure. The following examples illustrate these results. Notice that when a stable decomposition includes a ring component $\mathcal{RC}$ that is simple each coalition structure of the absorbing set generated includes a maximal set of $\mathcal{RC}$. By contrast, when a stable decomposition includes a ring component $\mathcal{RC}$ that is not simple there are coalition structures in the absorbing set generated that contain only one coalition of the ring component (and the remaining agents involved in the ring component are single in such coalition structures).

Example 1 (Continued) *In Example 1 there are three stable decompositions: $\mathscr{D} = \{\{123\}, \{45\}, \{67\}\}$, $\mathscr{D}' = \{\{15\}, \{23\}, \{467\}\}$, and $\mathscr{D}'' = \{\{12, 23, 34, 45, 15\}, \{67\}\}$. Stable decompositions $\mathscr{D}$ and $\mathscr{D}'$ induce the stable coalition structures $\{123, 45, 67\}$ and $\{15, 23, 467\}$, respectively. Stable decomposition $\mathscr{D}''$ includes as a party the unique ring component of the game, $\mathcal{RC} = \{12, 23, 34, 45, 15\}$. Notice that although coalition 467 breaks party $\mathcal{RC}$ in $\mathscr{D}''$, party $\{67\}$ in $\mathscr{D}''$ prevents 467 from being formed. Furthermore, the unique non-trivial absorbing set is $\mathcal{A} = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5\}$ where, $\pi_1 = \{12, 34, 5, 67\}$, $\pi_2 = \{12, 3, 45, 67\}$, $\pi_3 = \{1, 23, 45, 67\}$, $\pi_4 = \{15, 23, 4, 67\}$ and $\pi_5 = \{15, 2, 34, 67\}$. Notice that, since $\mathcal{RC}$ is simple, each coalition structure of $\mathcal{A}$ includes a maximal set.* $\Diamond$

Example 2 (Continued) *In Example 2 there are two stable decompositions:* $\mathscr{D} = \{\{145\}, \{23\}, \{678\}\}$*, and* $\mathscr{D}' = \{\{145, 12, 23, 356, 46\}, \{78\}\}$*. Stable decomposition* $\mathscr{D}$ *induces the stable coalition structure* $\{145, 23, 678\}$ *and stable decomposition* $\mathscr{D}'$ *includes as a party the unique ring component of the game,* $\mathcal{RC} = \{145, 12, 23, 356, 46\}$*. Notice that although coalition* 678 *breaks party* $\mathcal{RC}$ *in* $\mathscr{D}'$*, party* $\{78\}$ *in* $\mathscr{D}'$ *prevents* 678 *from being formed. Furthermore,* $\{145, 23, 678\}$ *is the unique stable coalition structure, and the non-trivial absorbing set is formed by coalition structures* $\{12, 3, 46, 5, 78\}$*,* $\{1, 23, 46, 5, 78\}$*,* $\{145, 23, 6, 78\}$*,* $\{1, 2, 356, 4, 78\}$*,* $\{12, 356, 4, 78\}$*,* $\{1, 2, 3, 46, 5, 78\}$*,* $\{145, 2, 3, 6, 78\}$*,* $\{12, 3, 4, 5, 6, 78\}$*, and* $\{1, 23, 4, 5, 6, 78\}$*. Notice that since* $\mathcal{RC}$ *is not simple, there are coalition structures in the absorbing set that only contain one coalition of the ring component. For instance, coalition structure* $\{1, 2, 356, 4, 78\}$ *only contains coalition* 356*.* ♢

Example 3 *Consider the game given by the following table:*

1	**2**	**3**	**4**	**5**	**6**
12	23	34	45	56	46
13	12	13	46	45	56
1	2	23	34	5	6
		3	4		

This game has no stable coalition structure and two ring components: $\{12, 23, 13\}$ *and* $\{45, 46, 56\}$*. Consider the collection of ring components* $\{\{12, 23, 13\}, \{45, 46, 56\}\}$*. Ring component* $\{12, 23, 13\}$ *is not protected by the collection, since coalition* 34 *breaks* $\{12, 23, 13\}$ *and ring component* $\{45, 46, 56\}$ *does not prevent* 34 *from being formed. Therefore, this collection is not a stable decomposition. However, the collection* $\mathscr{D} = \{\{1, 2, 3\}, \{45, 46, 56\}\}$ *is the unique stable decomposition of the game because: (i) each non-single coalition formed by agents of the set of singles* $\{1, 2, 3\}$*, namely* 12, 23*, and* 13*, is not protected by* $\mathscr{D}$*; and (ii) no coalition breaks ring component* $\{45, 46, 56\}$*, so it is protected by* $\mathscr{D}$*. This stable decomposition induces the following* $\mathscr{D}$*– coalition structures:* $\{1, 2, 3, 45, 6\}$*,* $\{1, 2, 3, 46, 5\}$ *and* $\{1, 2, 3, 56, 4\}$*, which generate the unique absorbing set of this game* $\mathcal{A} = \{\pi_1, \pi_2, \ldots, \pi_{14}\}$ *where:*

$\pi_1 = \{1, 2, 3, 45, 6\}$ $\pi_2 = \{1, 2, 3, 4, 56\}$ $\pi_3 = \{1, 2, 3, 5, 46\}$ $\pi_4 = \{12, 3, 5, 46\}$
$\pi_5 = \{12, 3, 45, 6\}$ $\pi_6 = \{12, 3, 4, 56\}$ $\pi_7 = \{1, 23, 5, 46\}$ $\pi_8 = \{1, 23, 45, 6\}$
$\pi_9 = \{1, 23, 4, 56\}$ $\pi_{10} = \{2, 13, 4, 56\}$ $\pi_{11} = \{2, 13, 5, 46\}$ $\pi_{12} = \{2, 13, 45, 6\}$
$\pi_{13} = \{1, 2, 34, 56\}$ $\pi_{14} = \{12, 34, 56\}$.

Notice that in each coalition structure of $\mathcal{A}$ *there is a coalition of the ring component that belongs to* $\mathscr{D}$*.* ♢

3.3 Relation between rings and cycles

In this subsection we analyze how cyclical behavior that may arise in agents' preferences induces cyclical behavior of coalition structures. A cycle of coalitions structures is an ordered set of coalition structures that presents cyclical behavior. That is, for each pair of consecutive coalition structures of the ordered set, the successor coalition structure dominates its predecessor. Formally,

Definition 7 *An ordered set of coalition structures $(\pi_1, \ldots, \pi_J) \subset \Pi$, with $J \geq 3$, is a **cycle** if $\pi_{j+1} \gg \pi_j$ for $j = 1, \ldots, J$ subscript modulo J.*

Next, we present an algorithm that constructs a ring of coalitions from a cycle of coalition structures. Let $\mathscr{C} = (\pi_1, \ldots, \pi_J)$ be a cycle of coalition structures, let C_j denote the coalition that is formed in π_j, i.e., $\pi_j \gg \pi_{j-1}$ via C_j, and consider the ordered set $\mathcal{C} = (C_1, \ldots, C_J)$. To construct a ring, proceed as follows:

Algorithm:

Step 1 Set $\overline{R}_1$ as any coalition in $\mathcal{C}$.

Step t Set

$\overline{R}_t \equiv \min_{r\geq 1}\{C_{j+r}$ such that $C_j = \overline{R}_{t-1}$ and $C_j \cap C_{j+r} \neq \emptyset$ with $j + r \bmod J\}$.

`IF` $\overline{R}_t = \overline{R}_s$ for $s < t$,

`THEN` set $(\overline{R}_{s+1}, \ldots, \overline{R}_t)$, and `STOP`.

`ELSE` continue to Step $t + 1$.

Notice that in each step of the algorithm a different coalition of $\mathcal{C}$ is selected except in the last step, where one of the previously selected coalitions is singled out. Therefore, the algorithm stops in at most $J + 1$ steps (recall that $J = |\mathcal{C}|$).

The following lemma shows that the ordered set $(\overline{R}_{s+1}, \ldots, \overline{R}_t)$, where s is identified in the above algorithm, is actually a ring. To simplify notation, we rename the elements of the ordered set and write $(R_1, \ldots, R_\ell) = (\overline{R}_{s+1}, \ldots, \overline{R}_t)$.

Lemma 1 *Let $\mathscr{C}$ be a cycle of coalition structures. Then, cycle $\mathscr{C}$ induces a ring.*

Proof. Let $\mathscr{C}$ be a cycle of coalition structures. Applying the above algorithm results in the ordered set $(R_1, \ldots, R_\ell)$. We claim that the ordered set $(R_1, \ldots, R_\ell)$ thus constructed is a ring, i.e. for each R_{j+1} and R_j in the ordered set, $R_{j+1} \succ R_j$ and $\ell \geq 3$. Take any coalition R_j. Coalition R_{j+1} (modulo ℓ) is the closest coalition that has a nonempty intersection with R_j (following the modular order of the coalition structures in cycle $\mathscr{C}$), so all the coalition structures between the one in which R_j breaks and the one in which R_{j+1} breaks contain coalition R_j. Let π and π' be the two consecutive coalition structures in $\mathscr{C}$ such that $\pi' \gg \pi$ via R_{j+1}. R_{j+1} is the breaking coalition, so R_{j+1} belongs to π'. Furthermore, since R_j belongs to π and $R_{j+1} \cap R_j \neq \emptyset$, by Definition 1 $R_{j+1} \succ R_j$. Furthermore, $\ell \geq 3$. This holds for the following two facts: (i) there are at least two coalitions in the ordered set, because all the coalitions that break in a cycle are also broken; (ii) if there are only two coalitions, say R_1 and R_2, then there is an agent $i \in R_1 \cap R_2$ such that $R_1 \succ_i R_2 \succ_i R_1$, which by transitivity implies $R_1 \succ_i R_1$, which is a contradiction. □

The following theorem establishes the relationship between a ring of coalitions in preferences and a cycle of coalition structures of the game.

Theorem 2 *A coalition formation game has a ring of coalitions if and only if it has a cycle of coalition structures.*

Proof. ($\Longleftarrow$) This is proven by Lemma 1.
($\Longrightarrow$) Let $(R_1, \ldots, R_J)$ be a ring in coalition formation game $(N, \succ_N)$. This ring induces a cycle of coalition structures $\mathscr{C} = (\pi_1, \ldots, \pi_J)$ where π_j is defined as follows:

$$\pi_j(i) = \begin{cases} R_j & \text{for } i \in R_j \\ \{i\} & \text{otherwise.} \end{cases}$$

Note that π_j is obtained from π_{j-1} by forming coalition R_j for each $j = 1, \ldots, J$. □
Next, we illustrate the above result with an example.

Example 1 (Continued) *In Example 1 there are two rings:*$\{15, 12, 23, 34, 45\}$ *and* $\{15, 123, 34, 45\}$. *The collection* $\mathscr{C} = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$ *where,* $\pi_1 = \{12, 34, 5, 67\}$, $\pi_2 = \{12, 3, 45, 67\}$, $\pi_3 = \{1, 23, 45, 67\}$, $\pi_4 = \{15, 23, 4, 67\}$ *and* $\pi_5 = \{15, 2, 34, 67\}$ *is a cycle of coalition structures. Starting from* π_1, *the set of blocking coalitions between coalition structures is* $\mathcal{C} = (45, 23, 15, 34, 12)$. *Assume that Step 1 of the previous algorithm selects coalition* 45. *The following steps select coalitions* 15, 12, 23 *and* 34, *respectively. The algorithm ends when coalition* 45 *is reached again and ring* $(45, 15, 12, 23, 34)$ *is obtained.* ◊

4 Some applications

In Subsection 4.1 we analyze the special characteristics of a stable decomposition in roommate and marriage problems. In Subsection 4.2 we discuss convergence to stability.

4.1 Stable decompositions in the roommate and marriage problems

In the roommate problem, introduced by Gale and Shapley (1962), each agent has preferences over all coalitions of cardinality two to which he/she belongs. As is known, a roommate problem may not admit stable matchings. Tan (1991) proves that a roommate problem has no stable matchings if and only if there is a stable partition with an odd ring.[12]

Consider the relation between stable decompositions and stable partitions. Parties in Tan (1991) are similar to ours: A set of singletons, sets of one non-single coalitions, and rings. However, the notion of protection in Tan (1991) is weaker than ours. In our terminology, a collection of parties $\mathscr{P}$ is a stable partition if, whenever coalition C breaks party $\mathcal{B} \in \mathscr{P}$ there is party $\mathcal{B}' \in \mathscr{P}$ such that $C \setminus N(\mathcal{B}) \subset N(\mathcal{B}')$ and $R \succ C$ for each $R \in \mathcal{B}'$ with $R \cap C \neq \emptyset$. This can be illustrated with the following example:

Example 4 (Example 2 in Inarra et al., 2013) *Consider the game given by this table:*

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	***a***
12	23	13	47	58	69	57	68	49	a
13	12	23	48	59	67	67	48	59	
14	24	34	49	57	68	17	58	69	
15	25	35	45	45	46	47	78	79	
16	26	36	46	56	6	79	89	89	
17	27	37	14	5		78	8	9	
18	28	38	24			7			
19	29	39	34						
1	2	3	4						

This example has three stable partitions: $\{\{12,23,13\},\{48\},\{59\},\{67\},\{a\}\}$, $\{\{12,23,13\},\{49\},\{57\},\{68\},\{a\}\}$, *and* $\{\{12,23,13\},\{47\},\{58\},\{69\},\{a\}\}$. *The first two are stable decompositions but the third is not. This is because party* $\{47\}$ *is not protected: Coalition* 17 *breaks party* $\{47\}$ *and there is no party that prevents the formation of coalition* 17. *In fact, when agent* 1 *is abandoned by agent* 2 *then* 17 *can be formed. By contrast,* $\{\{12,23,13\},\{47\},\{58\},\{69\},\{a\}\}$ *is a stable partition because no coalition breaks parties* $\{12,23,13\}$ *and* $\{a\}$, *and it can be seen that the protection criterion of Tan is met for the rest of the parties. An analysis of all coalitions that break party* $\{47\}$ *reveals the following: coalition* 17 *breaks party* $\{47\}$ *but party* $\{12,23,13\}$ *satisfies the requirement that* $12 \succ 13 \succ 17$. *Coalition* 67 *breaks party* $\{47\}$ *but party* $\{69\}$ *satisfies the requirement that* $69 \succ 67$. *Lastly, coalition* 57 *breaks party* $\{47\}$ *but party* $\{58\}$ *satisfies the requirement that* $58 \succ 57$. *The*

[12]In this subsection, as usual in roommate and marriage problems, we talk about "matchings" rather than coalition structures.

analysis for parties $\{58\}$ *and* $\{69\}$ *is similar and we omit it. Thus, for the roommate problem, the notion of stable partition is weaker than the notion of stable decomposition.* $\Diamond$

Next, following Inarra et al. (2013), we use the term *maximal stable partition* to refer to those stable partitions with the maximal set of satisfied agents, i.e. agents with no incentive to change partners. The following result can then be established:

Proposition 2 *For any roommate problem there is a bijection between maximal stable partitions and stable decompositions.*

Proof. Theorem 1 in Inarra et al. (2013) proves that there is a bijection between maximal stable partitions and absorbing sets. Our Corollary 2 states that there is a bijection between absorbing sets and stable decompositions. Therefore, the result follows straightforwardly. $\square$

In the marriage problem, agents can be divided into two types: Men and women. An agent of one type can only be matched to an agent of the other, or can remain single. Thus, the marriage problem is a special case of the roommate problem with specific restrictions on preferences. Gale and Shapley (1962) show that stable matching may not exist in the roommate problem, but always exist in the marriage problem. Chung (2000) raises the question of why the marriage problem always admits stable matchings while the roommate, a generalization of the marriage problem, may not. The results of Tan (1991) and Chung (2000), who analyze the roommate problem for weak preferences show that a marriage problem, unlike the roommate problem, has no odd rings, which explains why the marriage problem always has stable matchings. In our terms the specific form of a stable decomposition can be shown in the marriage problem.

Proposition 3 *For any marriage problem no stable decomposition has a ring component.*

Proof. Consider a marriage problem. By Proposition 2, there is a bijection between stable decompositions and (maximal) stable partitions of the problem. By Propositions 3.1 and 3.2 in Tan (1991), there is a bijection between (maximal) stable partitions and stable matchings of the problem. Also, by Remark 3, there is a bijection between stable matchings and stable decompositions with no ring component. This implies that no stable decomposition has a ring component. $\square$

The following example illustrates the result.

Example 5 *Consider the game given by this table:*

m_1	m_2	m_3	w_1	w_2	w_3
m_1w_1	m_2w_1	m_3w_3	m_3w_1	m_2w_2	m_1w_3
m_1w_3	m_2w_2	m_3w_2	m_1w_1	m_3w_2	m_3w_3
		m_3w_1	m_2w_1		

This example has two even rings: $(m_1w_1, m_3w_1, m_3w_3, m_1w_3)$ and $(m_3w_1, m_3w_2, m_2w_2, m_2w_1)$. The union of these two rings satisfies Condition (i) but not Condition (ii) of Definition 4. To see this, take the three maximal sets that can be formed with this collection: $\{m_1w_3, m_3w_1, m_2w_2\}$, $\{m_1w_3, m_2w_1, m_3w_2\}$ and $\{m_1w_1, m_2w_2, m_3w_3\}$. It is easy to verify that no coalition of these rings breaks any of these maximal sets. Hence, this collection of coalitions is not a ring component. ♢

4.2 Convergence to stability

We say that a coalition formation game $(N, \succ_N)$ exhibits *convergence to stability* if for each non stable coalition structure $\pi \in \Pi$ there is a stable coalition structure $\pi^N \in \Pi$ such that $\pi^N \gg^T \pi$.

As claimed in the Introduction, the stable decomposition solution provides a tool for analyzing convergence to stability. If no stable decomposition of a coalition formation game has a ring component, then the game exhibits convergence to stability. However, a coalition formation game in which there are stable decompositions both with and without ring components does not exhibit convergence to stability.

Proposition 4 *A stable coalition formation game exhibits convergence to stability if and only if no stable decomposition has a ring component.*

Proof. Let $(N, \succ_N)$ be a stable coalition formation game.
$(\Longrightarrow)$ Assume that $(N, \succ_N)$ has a stable decomposition with a ring component. By Proposition 1, the stable decomposition induces a non-trivial absorbing set $\mathcal{A}$. Let π^N be a stable coalition structure, so $\pi^N \in \Pi \setminus \mathcal{A}$. Thus, by Definition 2 there is no $\pi \in \mathcal{A}$ such that $\pi^N \gg^T \pi$. Therefore, $(N, \succ_N)$ does not exhibit convergence to stability.
$(\Longleftarrow)$ Assume that $(N, \succ_N)$ has a stable decomposition with no ring component. By Remark 3 this means that each stable decomposition can be identified with a stable coalition structure, so the game has only trivial absorbing sets. By Remark 2 (iii) and (iv), for each non stable coalition structure $\pi \in \Pi$ there is a stable coalition structure $\pi^N \in \Pi$ such that $\pi^N \gg^T \pi$. □

Roth and Vande Vate (1990) prove that the marriage problem exhibits convergence to stability. Here, using our results, we give an alternative argument as to how this follows. Note that by Proposition 3, no stable decomposition of the marriage problem has a ring component. By Proposition 4, the following result emerges straightforwardly.

Corollary 3 *The marriage problem exhibits convergence to stability.*

Diamantoudi et al. (2004) prove that when the roommate problem is stable (i.e. has a stable matching) it exhibits convergence to stability. We argue that following Proposition 2 in Inarra et al. (2013), the only absorbing sets of a stable roommate problem are stable matchings. Thus, by Remark 2 (iv), each non-stable matching is transitively dominated by a stable one. Therefore, we can state the following.

Corollary 4 *If a roommate problem is stable, it exhibits convergence to stability.*

5 Concluding remarks

To conclude, we first emphasize the results obtained and then propose some further research.

We introduce a new solution concept called stable decomposition, for the entire class of coalition formation games. As said, the set of stable decompositions of a game is always non-empty and encompasses stable coalition structures when they exist. When a stable decomposition is not related to a stable coalition structure, it incorporates a new ingredient –the ring component– which is the source of the cyclical behavior of the coalition structures. Our solution is characterized in terms of absorbing sets, i.e. there is a bijection between absorbing sets and stable decompositions. However, although a stable decomposition conveys the same information regarding the cyclical behavior of some coalition structures as an absorbing set, it is a much simpler object that can be derived exclusively from the preferences of the agents. As applications, we restate some important results about stability on roommate and marriage problems and analyze convergence to stability.

Our approach opens up a number of interesting research directions, including the following: The paper relies on a dynamic process among coalition structures which is consistent with the standard blocking definition in that all members of the blocking coalition become strictly better off, and assumes that abandoned agents remain single in the newly formed coalition structure. However, another possibility is for the abandoned agents to get together as in, for instance, Tamura (1993). How our solution concept adapts to this new dynamics is an open question. Furthermore, Pycia (2012) and Gallo and Inarra (2018), in different contexts, study what sharing rules induce stable coalition formation games, i.e., they assume that coalitions produce an output to be divided among their members according to a pre-specified sharing rule. In such environments, the sharing rule naturally induces a game in which each agent ranks the coalitions to which he/she belongs according to the payoffs that he/she could get. Here, the question to be answered is what rules can generate coalition formation games that exhibit convergence to stability.

A Appendix

In this section we provide proofs to some of the results from Section 3. The following lemma states that given a ring component, a maximal set of the ring component and a set of its compact collection, a coalition structure that contains the maximal set, is transitively dominated by a coalition structure that contains the set of the compact collection.

Lemma 2 *Let $\mathcal{B}$ be a ring component and $\mathcal{C}(\mathcal{B})$ its compact collection. Consider $\mathcal{M}(\mathcal{B})$ and $\mathcal{E} \in \mathcal{C}(\mathcal{B})$ with $\mathcal{E} \subseteq \mathcal{M}(\mathcal{B})$. If $\pi \in \Pi$ is such that $\mathcal{M}(\mathcal{B}) \subset \pi$ and $\pi^\star \in \Pi$ is such that*

$$\pi^\star(i) = \begin{cases} \{i\} & \text{for } i \in N(\mathcal{B}) \setminus N(\mathcal{E}) \\ \pi(i) & \text{for } i \in N \setminus (N(\mathcal{B}) \setminus N(\mathcal{E})). \end{cases}$$

Then, $\pi^\star \gg^T \pi$.

Proof. Note that if $\mathcal{E} = \mathcal{M}(\mathcal{B})$, then $\pi^\star = \pi$ and the proof is complete. Assume that $\mathcal{E} = \{R\} \subset \mathcal{M}(\mathcal{B})$. By the definitions of ring component and compact collection, it is possible to construct a sequence of ring coalitions $R_0, \ldots, R_m$ and coalition structures $\pi_0, \ldots, \pi_m$ with $\pi_0 = \pi$ fulfilling the following conditions for each $\ell = 1, \ldots, m-1$:

(i) $\pi_\ell \gg \pi_{\ell-1}$ via R_ℓ.

(ii) $|\pi_m \cap \mathcal{B}|_{\mathcal{K}} = |\mathcal{E}|_{\mathcal{K}}$.

If $\pi_m \cap \mathcal{B} = \mathcal{E}' \neq \mathcal{E}$ for some $\mathcal{E}' \in \mathcal{C}(\mathcal{B})$, by the definitions of ring component and compact collection, there is $\pi^\star$ with $\pi^\star \cap \mathcal{B} = \mathcal{E}$ such that $\pi^\star \gg^T \pi_m$. Therefore, $\pi^\star \gg^T \pi$. □

The following lemmata deal with the transitive dominance by the $\mathscr{D}$ – coalition structures derived from a stable decomposition $\mathscr{D}$: Lemma 3 says that a $\mathscr{D}$ – coalition structure transitively dominates any coalition structure that contains its non-single coalitions. Lemma 4 says that any two $\mathscr{D}$ – coalition structures transitively dominate each other. Lemma 5 says that if a coalition structure transitively dominates a $\mathscr{D}$ – coalition structure then the converse also follows.

Lemma 3 *Let $\mathscr{D}$ be a stable decomposition and let $\pi_{\mathscr{D}}$ be a $\mathscr{D}$ – coalition structure. If π is a coalition structure such that $\mathcal{M}(\pi_{\mathscr{D}}) \subseteq \mathcal{M}(\pi)$, then $\pi = \pi_{\mathscr{D}}$ or $\pi_{\mathscr{D}} \gg^T \pi$.*

Proof. Let $\mathscr{D}$ be a stable decomposition, let $\pi_{\mathscr{D}}$ be a $\mathscr{D}$ – coalition structure, and let π be a coalition structure such that $\mathcal{M}(\pi_{\mathscr{D}}) \subseteq \mathcal{M}(\pi)$. By Lemma 2, there is $\pi^\star$ with $\pi^\star \gg^T \pi$ such that there is $\mathcal{E}_t \in \mathcal{C}(\mathcal{B}_t)$ with $\pi \cap \mathcal{B}_t = \mathcal{E}_t$ for each ring component $\mathcal{B}_t \in \mathscr{D}$.

If $\mathscr{D}$ has no party of single agents or $\pi^\star = \pi_{\mathscr{D}}$ there is nothing to prove, so assume that $\mathscr{D}$ has a party of single agents $\widetilde{\mathcal{B}}$ and $\pi^\star \neq \pi_{\mathscr{D}}$.

Claim: there is a coalition structure $\widetilde{\pi}$ with $\mathcal{M}(\pi_{\mathscr{D}}) \subseteq \mathcal{M}(\widetilde{\pi})$ such that $\widetilde{\pi} \gg^T \pi^\star$ and $|\widetilde{\pi}|_{\mathcal{K}} < |\pi^\star|_{\mathcal{K}}$.

To prove the Claim, notice that in $\pi^\star$ there is a non-single coalition with a non-empty intersection with agents in $N(\widetilde{\mathcal{B}})$, say C_0. First, assume that $C_0 \subseteq N(\widetilde{\mathcal{B}})$. Thus, C_0 is a party in itself or is part of a ring component and Definition 5 (ii) implies the existence of a coalition C_1 that breaks the party to which C_0 belongs and no other party prevents C_1 from being formed. If there is no party $\mathcal{B}_t \in \mathscr{D}$ such that $\mathcal{B}_t \subset \mathcal{K}$ and $C_1 \cap N(\mathcal{B}_t) \neq \emptyset$, then C_1 is a party in itself or belongs to a ring component formed by agents in $N(\widetilde{\mathcal{B}})$. In either case, the definition of stable decomposition implies the existence of a coalition C_2 that breaks the party to which C_1 belongs and no other party prevents C_2 from being formed. If there is no party $\mathcal{B}_t \in \mathscr{D}$ such that $\mathcal{B}_t \subset \mathcal{K}$ and $C_2 \cap N(\mathcal{B}_t) \neq \emptyset$, repeat the previous argument. Continuing this reasoning, it is possible to construct a sequence of coalitions $C_0, C_1, \ldots, C_m$ and a sequence of coalition structures $\pi_0, \pi_1, \ldots, \pi_m$ with $\pi_0 = \pi^\star$ fulfilling the following conditions for each $\ell = 1, \ldots, m-1$:

(i) $C_\ell \succ C_{\ell-1}$,

(ii) $C_{\ell-1} \subseteq N(\widetilde{\mathcal{B}})$,

(iii) $\pi_\ell \gg \pi_{\ell-1}$ via C_ℓ, and

(iv) there is a party $\mathcal{B}_t \in \mathscr{D}$ such that $\mathcal{B}_t \subset \mathcal{K}$ and $C_m \cap N(\mathcal{B}_t) \neq \emptyset$.

Notice that, by conditions (i) and (ii) above, $C_m \cap N(\widetilde{\mathcal{B}}) \neq \emptyset$. Also, the existence of party $\mathcal{B}_t$ in (iv) is ensured by the finiteness of the number of coalitions of the game and by the definition of stable decomposition. Since $\mathcal{B}_t \in \mathscr{D}$ is a ring component[13] and does not prevent C_m from being formed, there is a compact collection $\mathcal{C}(\mathcal{B}_t)$ and $\mathcal{E} \in \mathcal{C}(\mathcal{B}_t)$ such that $C_m \cap N(\mathcal{E}) = \emptyset$. By the definition of ring component, there is a coalition structure π' such that $\pi' \gg^T \pi_{m-1}$ with $\pi'(i) = \pi_{m-1}(i)$ for each $i \in N \setminus N(\mathcal{B}_t)$, and either $\pi'(i) = \{i\}$ or $\pi'(i) \subseteq \mathcal{E}$ for each $i \in N(\mathcal{B}_t)$. Thus, there is π_m

[13]Notice that $\mathcal{B}_t$ is a ring component. Otherwise, since $\mathcal{B}_t$ does not prevent C_m from being formed, C_m would dominate $\mathcal{B}_t$. This contradicts the definition of stable decomposition.

such that $\pi_m \gg \pi'$ via C_m. Since $\mathcal{B}_t$ is a ring component, $C_m \cap N(\mathcal{B}_t) \neq \emptyset$, and $\mathcal{B}_t$ does not prevent C_m from being formed, there is $\mathcal{E}' \in \mathcal{C}(\mathcal{B}_t)$, a coalition $R_0 \in \mathcal{E}'$ such that R_0 breaks $\mathcal{E}$ and $R_0 \succ C_m$, and a coalition structure π'' such that $\pi'' \gg \pi_m$ via R_0.[14] Since $\mathcal{E}'$ may not be included in $\pi^\star$, and $\mathcal{B}_t$ is a ring component, there are a sequence of coalitions $R_1, \ldots, R_s$ in $\mathcal{B}_t$, and a sequence of coalition structures $\widetilde{\pi}_0, \widetilde{\pi}_1, \ldots, \widetilde{\pi}_s$ with $\widetilde{\pi}_0 = \pi''$ fulfilling the following conditions for each $\ell = 1, \ldots, s$:

(i) $R_\ell \succ R_{\ell-1}$,

(ii) $\widetilde{\pi}_\ell \gg \widetilde{\pi}_{\ell-1}$ via R_ℓ, and

(iii) $\widetilde{\pi}_s(i) = \pi^\star(i)$ for each $i \in N(\mathcal{B}_t)$.

Let $\widetilde{\pi} \equiv \widetilde{\pi}_s$. Notice that, since $C_m \in \pi_m$, $R_0 \succ C_m$ and R_0 breaks the maximal set included in π_m, it follows that $|\pi_m|_{\mathcal{K}} > |\widetilde{\pi}|_{\mathcal{K}}$. Also, by construction of the sequence $\pi_0, \ldots, \pi_m$, $|\pi^\star|_{\mathcal{K}} \geq |\pi_m|_{\mathcal{K}}$. Therefore, $|\pi^\star|_{\mathcal{K}} > |\widetilde{\pi}|_{\mathcal{K}}$. Moreover, by condition (iii) of the definition of the sequence $\widetilde{\pi}_0, \widetilde{\pi}_1, \ldots, \widetilde{\pi}_s$, we have that $\mathcal{M}(\pi_{\mathscr{D}}) \subseteq \mathcal{M}(\widetilde{\pi})$. Thus, $\widetilde{\pi}$ fulfills the conditions of the Claim. Second, assume that $C_0 \cap \left(N \setminus N(\widetilde{\mathcal{B}})\right) \neq \emptyset$. Thus, C_0 can be considered as the coalition C_m of the previous case, and the proof follows similarly. This completes the proof of the Claim.

To conclude the proof of Lemma 3, first notice that by the Claim a coalition structure $\widetilde{\pi}$ is obtained such that $\widetilde{\pi} \gg^T \pi^\star$, $\mathcal{M}(\pi_{\mathscr{D}}) \subseteq \mathcal{M}(\widetilde{\pi})$ and $|\widetilde{\pi}|_{\mathcal{K}} < |\pi^\star|_{\mathcal{K}}$. If $\widetilde{\pi} = \pi_{\mathscr{D}}$, then the proof is completed. Otherwise, there is a non-single coalition $C \in \widetilde{\pi}$ such that $C \subseteq N(\widetilde{\mathcal{B}})$. Applying the Claim to coalition structure $\widetilde{\pi}$ gives a new coalition structure $\widehat{\pi}$ such that $\widehat{\pi} \gg^T \widetilde{\pi}$, $\mathcal{M}(\pi_{\mathscr{D}}) \subseteq \mathcal{M}(\widehat{\pi})$ and $|\widehat{\pi}|_{\mathcal{K}} < |\widetilde{\pi}|_{\mathcal{K}} < |\pi^\star|_{\mathcal{K}}$. If $\widehat{\pi} = \pi_{\mathscr{D}}$, then the proof is completed. Otherwise, continue applying the Claim until coalition structure $\pi_{\mathscr{D}}$ is obtained. □

Lemma 4 *Let $\mathscr{D}$ be a stable decomposition and let $\pi_{\mathscr{D}}$ and $\pi'_{\mathscr{D}}$ be two $\mathscr{D}$– coalition structures. Then $\pi_{\mathscr{D}} \gg^T \pi'_{\mathscr{D}}$.*

Proof. Let $\mathscr{D}$ be a stable decomposition and let $\pi_{\mathscr{D}}$ and $\pi'_{\mathscr{D}}$ be two $\mathscr{D}$– coalition structures. By the definition of ring component, there is a sequence of coalitions $R_1, \ldots, R_J$ and coalition structures $\pi_0, \ldots, \pi_J$ such that:

(i) $\pi_0 = \pi'_{\mathscr{D}}$ and $\pi_J = \pi_{\mathscr{D}}$,

(ii) $R_j \in \mathcal{B}$ for some $\mathcal{B} \in \mathscr{D}$ and $R_j \cap C \neq \emptyset$ for some $C \in \mathcal{K} \cap \pi_{j-1}$ for $j = 1, \ldots, J$,

(iii) $\pi_j \gg \pi_{j-1}$ via R_j for each $j = 1, \ldots, J$.

[14] If $\mathcal{E}$ is the unique set such that $C_m \cap N(\mathcal{E}) = \emptyset$ the existence of R_0 such that $R_0 \succ C_m$ is guaranteed. If it is not unique, one can be selected such that $\pi'' \gg \pi_m$ via R_0.

Therefore, $\pi_{\mathscr{D}} \gg^T \pi'_{\mathscr{D}}$. $\square$

Lemma 5 *Let $\mathscr{D}$ be a stable decomposition, let $\pi_{\mathscr{D}}$ be a $\mathscr{D}$– coalition structure and let $\pi \in \Pi$. If $\pi \gg^T \pi_{\mathscr{D}}$, then $\pi_{\mathscr{D}} \gg^T \pi$.*

Proof. Let $\mathscr{D}$ be a stable decomposition, let $\pi_{\mathscr{D}}$ be a $\mathscr{D}$– coalition structure and let $\pi \in \Pi$ such that $\pi \gg^T \pi_{\mathscr{D}}$. By the definition of stable decomposition, there is a $\mathscr{D}$– coalition structure $\pi'_{\mathscr{D}}$ such that $\mathcal{M}(\pi'_{\mathscr{D}}) \subseteq \mathcal{M}(\pi)$. Thus, by Lemma 3, $\pi'_{\mathscr{D}} = \pi$ or $\pi'_{\mathscr{D}} \gg^T \pi$. Then, by Lemma 4, $\pi_{\mathscr{D}} \gg^T \pi'_{\mathscr{D}}$ and, therefore, $\pi_{\mathscr{D}} \gg^T \pi$. $\square$

Proof of Proposition 1. Let $\mathscr{D} = \{\mathcal{B}_1, \ldots, \mathcal{B}_L\}$ be a stable decomposition. First, assume that either $\mathcal{B}_\ell = \{C_\ell\}$ and $C_\ell \in \mathcal{K}$, or $\mathcal{B}_\ell \subseteq \mathcal{S}$ for each $\ell = 1, \ldots, L$. There is thus a unique $\mathscr{D}$– coalition structure $\pi_{\mathscr{D}}$. Then, $\pi_{\mathscr{D}}$ is a stable coalition structure and, by Remark 2 (ii), $\mathcal{A}_{\mathscr{D}} = \{\pi_{\mathscr{D}}\}$ is an absorbing set. Second, assume there is $\mathcal{B} \in \mathscr{D}$ such that $|\mathcal{B}| \geq 3$. Thus, $|\mathcal{A}_{\mathscr{D}}| \geq 3$. Let $\pi \in \mathcal{A}_{\mathscr{D}}$ and consider $\pi' \in \Pi \setminus \{\pi\}$ such that $\pi' \gg^T \pi$. As $\pi \in \mathcal{A}_{\mathscr{D}}$, it follows that $\pi \gg^T \pi_{\mathscr{D}}$. Therefore, by transitivity of $\gg^T$, $\pi' \gg^T \pi_{\mathscr{D}}$ and $\pi' \in \mathcal{A}_{\mathscr{D}}$. Next, let π and π' be different elements of $\mathcal{A}_{\mathscr{D}}$. By definition of $\mathcal{A}_{\mathscr{D}}$, it follows that $\pi \gg^T \pi_{\mathscr{D}}$ and $\pi' \gg^T \pi_{\mathscr{D}}$. By Lemma 5, it follows that $\pi_{\mathscr{D}} \gg^T \pi$. Thus, $\pi' \gg^T \pi_{\mathscr{D}}$ together with $\pi_{\mathscr{D}} \gg^T \pi$ and the transitivity of $\gg^T$ imply $\pi' \gg^T \pi$. This proves that $\mathcal{A}_{\mathscr{D}}$ satisfies the conditions of Definition 2. Therefore, $\mathcal{A}_{\mathscr{D}}$ is a non-trivial absorbing set. $\square$

The following lemma is an important tool for relating absorbing sets and the ring components of a game.

Lemma 6 *A non-trivial absorbing set induces a collection of ring components.*

Proof. Let $\mathcal{A}$ be a non-trivial absorbing set. Notice that, given any two different coalition structures in $\mathcal{A}$, by Definition 2 there is a cycle of coalition structures in $\mathcal{A}$ that includes those structures. $\mathcal{A}$ can therefore be seen as the union of all such cycles. Thus, for each cycle of coalition structures in $\mathcal{A}$, the algorithm developed in this Section 3.3 constructs a ring. By merging overlapping rings, all ring components in $\mathcal{A}$ are constructed. $\square$

Proof of Theorem 1. ($\Longleftarrow$) It follows from Proposition 1.
($\Longrightarrow$) Let $\mathcal{A}$ be an absorbing set. If $|\mathcal{A}| = 1$, the unique element of $\mathcal{A}$ is a stable coalition structure π^N. Then, by Remark 3, a stable decomposition can be induced. If $|\mathcal{A}| > 1$, the stable decomposition $\mathscr{D}$ is constructed as follows. First, by Lemma 6, each of the ring components involved in the absorbing set $\mathcal{A}$ can be identified. Let $\mathscr{R}$ be the collection of all such ring components. Next, consider the subcollection

$$\mathscr{R}^\star = \{\mathcal{RC} \in \mathscr{R} : \text{ for each } \pi \in \mathcal{A} \text{ there is } R \in \mathcal{RC} \text{ such that } R \in \pi\}.$$

and put each ring component of $\mathscr{R}^\star$ as a party of $\mathscr{D}$. Second, let $\mathcal{F} = \{C \in \mathcal{K} : C \in \pi \text{ for each } \pi \in \mathcal{A}\}$ and for each $C \in \mathcal{F}$ put $\{C\}$ as a party of $\mathscr{D}$. Finally, to complete the decomposition, put all the agents not involved in the parties defined above into a party of singletons. Now, it remains to show that $\mathscr{D}$ is a stable decomposition. First, take any $\mathcal{B} \in \mathscr{D}$ such that $\mathcal{B} \subset \mathcal{K}$. The goal is to confirm whether $\mathcal{B}$ is protected by $\mathscr{D}$. Assume otherwise. Thus, there is $C \in \mathcal{K}$ that breaks $\mathcal{B}$ and no other party $\mathcal{B}' \in \mathscr{D}$ prevents C from being formed. This implies that there are $\pi, \pi' \in \mathcal{A}$ and $R \in \mathcal{B}$ such that $\pi' \gg \pi$ via C, $R \in \pi$ and $C \succ R$. By the definition of absorbing set it also follows that $\pi \gg^T \pi'$. Then, there is a cycle $\mathcal{C}$ of coalition structures that contains π and π'. By Theorem 2, cycle $\mathcal{C}$ induces a ring of coalitions that contains both coalitions R and C. Therefore, C belongs to ring component $\mathcal{B}$, which is absurd since C breaks $\mathcal{B}$. Hence, $\mathcal{B}$ is protected by $\mathscr{D}$ and Condition (i) in Definition 5 holds. To see that Condition (ii) in Definition 5 holds, let $\mathcal{B} \in \mathscr{D}$ be such that $\mathcal{B} \subseteq \mathcal{S}$ and let $\widetilde{\mathcal{B}}$ be a party with $N(\widetilde{\mathcal{B}}) \subseteq N(\mathcal{B})$. Assume that Condition (ii) is not true. There are two cases to consider:

1. **$\widetilde{\mathcal{B}}$ is undominated**. Thus, by construction of $\mathscr{D}$, $\widetilde{\mathcal{B}}$ belongs either to $\mathscr{R}^\star$ or $\mathcal{F}$, contradicting the requirement that $\mathcal{B} \subseteq \mathcal{S}$.

2. **There is a coalition C that breaks $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}$ is protected by $\mathscr{D}$.** Let $\pi_{\mathscr{D}}$ be a $\mathscr{D}$–coalition structure. Thus $\pi_{\mathscr{D}}(i) = \{i\}$ for each $i \in N(\widetilde{\mathcal{B}})$ (since $\widetilde{\mathcal{B}} \subseteq \mathcal{S}$). Let $R \in \widetilde{\mathcal{B}}$ such that $\pi \gg \pi_{\mathscr{D}}$ via R. Since $\widetilde{\mathcal{B}}$ is protected by $\mathscr{D}$, $\widetilde{\mathcal{B}}$ belongs either to $\mathscr{R}^\star$ or $\mathcal{F}$, contradicting the requirement that $\mathcal{B} \subseteq \mathcal{S}$.

Therefore, $\mathscr{D}$ is a stable decomposition. □

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