RedNIE | Red Nacional de Investigadores en Economía

Regret-Free Truth-Telling Voting Rules

R. Pablo Arribillaga (Universidad Nacional de San Luis/CONICET)

Agustín G. Bonifacio (Universidad Nacional de San Luis/CONICET)

Marcelo A. Fernandez (Johns Hopkins University)

DOCUMENTO DE TRABAJO N° 166

Agosto de 2022

Citar como:
Arribillaga, R. Pablo, Agustín G. Bonifacio y Marcelo A. Fernandez (2022). Regret-Free Truth-Telling Voting Rules. *Documento de trabajo RedNIE N°166.*

Regret-free truth-telling voting rules*

R. Pablo Arribillaga† Agustín G. Bonifacio†
Marcelo A. Fernandez‡

July 7, 2022

Abstract

We study the implications of regret-free truth-telling for voting rules. Regret-freeness, a weakening of strategy-proofness, provides incentives to report preferences truthfully if agents want to avoid regret. We first show that for tops-only rules regret-freeness is equivalent to strategy-proofness. Then, we focus on three families of (non-tops-only) voting methods: maxmin, scoring, and Condorcet consistent ones. We show positive and negative results for both neutral and anonymous versions of maxmin and scoring rules. We also show that Condorcet consistent rules that satisfy a mild monotonicity requirement are not regret-free, and neither are successive elimination rules. Furthermore, we provide full characterizations for the case of three alternatives and two agents.

JEL classification: D71.

Keywords: Strategy-proofness, Regret-freeness, Voting Rules, Social Choice.

1 Introduction

In the context of a standard voting problem in which several agents must jointly choose one among several alternatives, we study procedures to associate an outcome to each

*Thanks to be added. We acknowledge financial support from UNSL through grants 032016, 030120, and 030320, from Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) through grant PIP 112-200801-00655, and from Agencia Nacional de Promoción Científica y Tecnológica through grant PICT 2017-2355.

†Instituto de Matemática Aplicada San Luis, Universidad Nacional de San Luis and CONICET, San Luis, Argentina, and RedNIE. Emails: `rarribi@unsl.edu.ar` and `abonifacio@unsl.edu.ar`

‡Department of Economics, Johns Hopkins University, Baltimore, USA. Email: `fernandez@jhu.edu`

possible profile of agents' preferences, known as voting rules. In order to control the strategic behavior of the agents, the property of strategy-proofness has played a vital role. A voting rule is strategy-proof if it is always in the best interest of the agents to reveal their true preferences. Unfortunately, the celebrated Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) states that, with more than two alternatives and universal domain of preferences, there is no efficient, strategy-proof, and non-dictatorial voting rule. A large part of the literature has focused on finding suitable domain restrictions in order to avoid this impossibility (see, for example, Moulin, 1980).

Although we recognize that strategy-proofness is a relevant property, it presents some drawbacks that the literature has highlighted: in some sense, the property assumes that an agent that manipulates has complete knowledge about other agents' preferences. Recent literature, in different contexts, has presented several incentive compatibility concepts in which an agent that manipulates does so having only some partial information about other agents' preferences. This lack of information makes this new concepts weaker than strategy-proofness, so positive results could be expected. We study the implications of the notion of regret-free truth-telling for voting rules defined on the universal domain of (strict) preferences. Regret-free truth-telling is introduced by Fernandez (2020) in the context of two-sided matching models. Here, an agent sees the precise outcome of the rule and with that information and his own preference infers which were the possible preferences profiles of the other agents. An agent suffers regret if he takes an action and ex-post he finds it to be dominated. Therefore, agents may optimally choose not to manipulate a rule if they wish to avoid regret.

In this paper, we first analyze the relation between regret-freeness and strategy-proofness for tops-only rules (i.e., rules that are sensible only to the top alternatives of the preference profiles) and show that under this informational simplicity requirement both properties are equivalent. This equivalence implies that: (i) for problems with only two alternatives, extended majority voting rules are the only regret-free rules; and (ii) with more than two alternatives and universal domain of preferences, there is no tops-only, efficient, and non-dictatorial voting rule that satisfies regret-freeness. Therefore, when there are more than two alternatives, we must examine rules that are not tops-only. We review some of the best known families of non-tops-only rules studied in the literature and determine whether they satisfy the regret-free truth-telling requirement or not. Three of the most important families of voting methods are the maxmin methods, that select those alternatives that "make the least happy agents as happy as possible";[1] the scoring methods, that assign points to each alternative according to the rank it has in agents' preferences and selects one of the alternatives with highest score; and the Condorcet consistent methods, that select a Condorcet winner whenever one exists.

The scoring and maxmin methods, in general, could choose more than one alternative (they are correspondences). To resolve this multiplicity, we will consider two classical tie-breakings. One is defined by picking the preference of a fixed agent. The other is defined by a fixed order of the alternatives. In this way, we get a neutral version and an anonymous version of each of this two methods, respectively. We perform a thorough account of all these rules and determine in which situations (if any) regret-free truth-telling obtains.

Consider a problem with n agents and m alternatives. First, we show that all neutral maxmin rules are regret-free and that anonymous maxmin rules are regret-free if and only if $n \geq m-1$ or n divides $m-1$. We also obtain general positive results for the negative plurality rule, a special scoring rule in which all the rank positions get one point except the last one that gets zero. The results are similar to those of maxmin rules: we prove that all neutral negative plurality rules satisfy the property and that an anonymous negative plurality rule is regret-free if and only if $n \geq m-1$.

For more general scoring rules, the results depend heavily on $k^\star$, the highest position where the score is not maximal.[2] The case $k^\star = 1$ corresponds to negative plurality, already discussed. When $k^\star = m-1$, no scoring rule (neither anonymous nor neutral) is regret-free. This case includes both anonymous and neutral versions of Borda, plurality, and Dowdall rules; and all efficient and anonymous scoring rules. When $1 < k^\star < m-1$, we consider two cases. First, the case $k^\star n < m$, which encompasses the class of scoring rules where, in any preference profile, there is always an alternative that gets maximal score. Here, anonymous scoring rules are regret-free if and only if $k^\star n = m-1$, whereas all neutral scoring rules satisfy the property. Second, when $k^\star n \geq m$, we get a particular result when $s_{k^\star-1} = s_{k^\star}$, which states that no rule is regret-free. As a consequence of the results for these two cases it follows that, for $k < m-1$, anonymous k-approval rules are regret-free if and only if $(m-k)n = m-1$ and that neutral k-approval rules are regret free if and only if $(m-k)n < m$.

Negative results apply to Condorcet consistent rules. This class of rules is very large because they are only required to choose a precise alternative when a Condorcet

[1]This methods are presented and defended, among others, by John Rawls in its classic Theory of Justice (Rawls, 1971).

[2]This is, $k^\star$ is such that $s_1 \leq s_2 \leq s_{k^\star} < s_{k^\star+1} = \ldots = s_m$.

winner exists. In our paper, an additional mild condition of monotonicity, compatible with both neutrality and anonymity, is imposed on rules. This condition says that if an alternative is below the outcome of the rule for an agent and he changes his preferences modifying only the ordering of alternatives above the outcome, then such alternative continues not to be chosen. Under this monotonicity, our result states that no Condorcet consistent rule is regret-free. In particular, we get that the six famous Condorcet consistent rules associated with the names of Simpson, Copeland, Young, Dodgson, Fishburn and Black (in both anonymous and neutral versions) are monotone, and therefore not regret-free.[3] We also show that a family of non-monotone Condorcet consistent rules, the successive elimination rules, do not satisfy regret-freeness either.

Finally, for the case with two agents and three alternatives, we present two characterization results. The first one says that a rule is regret-free, efficient, and anonymous if and only if it is either successive elimination or an anonymous maxmin rule in which the tie-breaking device is an antisymmetric and complete (not necessarily transitive) binary relation. The second one says that a rule is regret-free and neutral if and only if it is a dictatorship or a N-maxmin rule.

Weak versions of strategy-proofness are introduced in Reijngoud and Endriss (2012) and Endriss et al. (2016). They present the concept of information function in order to vary the amount of information an agent has about the preferences of the rest.[4] For different voting rules, they study when an agent has an incentive to manipulate subject to the restricted information available. Recently, Gori (2021) studies in detail a special case of information function, where the information about the preferences of the other individuals in the society is limited to the knowledge, for every pair of alternatives, of the number of people preferring the first alternative to the second one. This notion is called WMG-strategy-proofness by Gori (2021). In that paper, there are a positive result showing a class of Pareto optimal, WMG-strategy-proof and non-dictatorial voting functions; and a negative result proving that, when at least three alternatives are considered, no Pareto optimal and anonymous voting function is WMG-strategy-proof.[5] Another related work is Osborne and Rubinstein (2003), where they assume that the partial information about the other agents' behavior that an agent has is acquired from

[3]For the anonymous Simpson and Copeland rules, these results have been previously obtained by Endriss et al. (2016).

[4]A particular information function analyzed in Reijngoud and Endriss (2012) and Endriss et al. (2016), called the *winner information function*, leads to a property equivalent to regret-freeness. In what follows, we will be more specific in this respect.

[5]As we will prove, there is also some conflict between anonymity and regret-freeness. However, in some cases, we can identify rules that satisfy both properties.

a small random sample of the population. The real distribution of votes in the population is assumed to be the same as the distribution of votes in his sample, and on this basis the agent decides how to vote strategically. A complementary approach in the same direction is undertaken by Troyan and Morrill (2020). They introduce the concept of non-obvious manipulability, which is another weakening of strategy-proofness, and apply it to several market design environments to determine whether known mechanisms are non-obvious manipulable or not. It is assumed that an agent knows the possible outcomes of the mechanism conditional on his own declaration of preferences. Troyan and Morrill (2020) define a deviation to be an obvious manipulation if either the best possible outcome under the deviation is strictly better than the best possible outcome under truth-telling, or the worst possible outcome under the deviation is strictly better than the worst possible outcome under truth-telling. A mechanism that does not allow any obvious manipulation is called non-obvious manipulable. In the context of voting, the notion of obvious manipulability has been studied recently by Aziz and Lam (2021). They present a general sufficient condition for not obvious manipulability and also show that Condorcet consistent and some strict scoring rules are not obviously manipulable. Furthermore, for the class of k-approval voting rules, they give necessary and sufficient conditions for obvious manipulability.

The rest of the paper is organized as follows. In Section 2, we introduce the model and the property of regret-freeness. We show the equivalence of regret-freeness and strategy-proofness for tops-only rules in Section 3, where we also characterize extended majority voting rules as the only regret-free rules when there are only two alternatives to choose from. In Section 4, we study some positive and negative results for maxmin rules and scoring rules. In Section 5, we present negative results for Condorcet consistent rules. The special case with two agents and three alternatives is analyzed in Section 6, where two characterizations are presented. Some final remarks are gathered in Section 7.

2 Preliminaries

2.1 Model

A set of *agents* $N = \{1, \ldots, n\}$, with $n \geq 2$, has to choose an alternative from a finite and given set X, with $|X| = m$. Each agent $i \in N$ has a strict *preference* P_i (a linear order) over X. We denote by $t(P_i)$ the best alternative according to P_i, to which we will refer to as the *top* of P_i. We denote by R_i the weak preference over X associated to P_i; i.e., for all $x, y \in X$, xR_iy if and only if either $x = y$ or xP_iy. Let $\mathcal{P}$ be the set of all strict preferences over X. A *(preference) profile* is a n-tuple $P = (P_1, \ldots, P_n) \in \mathcal{P}^n$, an ordered list of n preferences, one for each agent. Given a profile P and an agent i, P_{-i} denotes the subprofile obtained by deleting P_i from P. For each $P_i \in \mathcal{P}$, denote by $t_k(P_i)$ to the alternative in the k-th position from bottom to top. Many times we write $t(P_i)$ instead of $t_m(P_i)$. We often also write P_i as an ordered list

$$P_i : t_m(P_i), t_{m-1}(P_i), \ldots, t_1(P_i).$$

A *(voting) rule* on $\mathcal{P}^n$ is a function $f : \mathcal{P}^n \to X$ that selects for each preference profile $P \in \mathcal{P}^n$ an alternative $f(P) \in X$. Next, we define several classical properties that a rule may satisfy and that we will use in the sequel. The rule $f : \mathcal{P}^n \to X$ is *strategy-proof* if agents can never induce a strictly preferred alternative by misrepresenting their preferences; namely, for each $P \in \mathcal{P}^n$, each $i \in N$ and each $P'_i \in \mathcal{P}$,

$$f(P) R_i f(P'_i, P_{-i}).$$

The rule $f : \mathcal{P}^n \to X$ is *efficient* if, for each $P \in \mathcal{P}^n$, there is no $y \in X$ such that $yP_if(P)$ for each $i \in N$. This requirement imposes the classical Pareto optimality criterion on the alternatives selected by the rule.

The rule $f : \mathcal{P}^n \to X$ is *tops-only* if $P, P' \in \mathcal{P}^n$ such that $t(P_i) = t(P'_i)$ for each $i \in N$ imply $f(P) = f(P')$. Tops-onlyness constitutes a basic simplicity requirement.

The rule $f : \mathcal{P}^n \to X$ is *dictatorial* if there exists $i \in N$ such that for each $P \in \mathcal{P}^n$, $f(P) = t(P_i)$. In a dictatorial rule, in each profile of preferences, the same agent selects his most preferred outcome.

The rule $f : \mathcal{P}^n \to X$ is *unanimous* if $t(P_i) = x$ for each $i \in N$ imply $f(P) = x$. Unanimity is a natural and weak form of efficiency: if all agents consider an alternative as being the most-preferred one, the rule should select it.

Anonymity requires that the rule treats all agents equally because the social outcome is selected without paying attention to the identities of the agents. Formally, the rule $f : \mathcal{P}^n \to X$ is *anonymous* if for each $P \in \mathcal{P}^n$ and each bijection $\pi : N \to N$, $f(P) = f(P^\pi)$ where for each $i \in N$, $P_i^\pi = P_{\pi(i)}$.

Finally, the rule $f : \mathcal{P}^n \to X$ is *neutral* if for each $P \in \mathcal{P}^n$ and each bijection $\pi : X \to X$, $\pi(f(P)) = f(\pi P)$ where $\pi P_i : \pi(t(P_i)), \pi(t_{m-1}(P_i)), ..., \pi(t_1(P_i))$.

In general, the axioms of anonymity and neutrality are incompatible. A classical way to give a partial solution to such incompatibility is to consider rules defined in two stages as follows:[6]

[6]"In practice, we will be happy with (voting) correspondences that respect the three principles (ef-

(i) First, consider a *voting correspondence* $\mathcal{Y} : \mathcal{P} \to 2^X \setminus \{\emptyset\}$ that for each preference profile $P \in \mathcal{P}^n$ chooses a (non-empty) subset $\mathcal{Y}(P) \subseteq X$, and assume that $\mathcal{Y}$ satisfies both anonymity and neutrality.[7]

(ii) Second, given $P \in \mathcal{P}^n$, consider a strict order on X and choose the first element according to that order in $\mathcal{Y}(P)$. There are two classical selections of such an order, one to preserve anonymity and the other to preserve neutrality:

(a) The strict order $\succ$ is independent of P and is part of the rule's definition. In this case anonymity is preserved and the rule is defined by[8]

$$f(P) = \max_{\succ} \mathcal{Y}(P).$$

(b) There exists an agent $i \in N$ such that, for each $P \in \mathcal{P}$, the strict order we consider is P_i. In this case, neutrality is preserved and the rule is defined by

$$f(P) = \max_{P_i} \mathcal{Y}(P).$$

From Section 4 onwards, we will study rules that can be defined by this two-stage procedure. This way to define a rule is flexible enough to encompass many well known and long studied families of rules.

2.2 Regret-freeness

As previously said, a regret-free rule provides incentives to report preferences truthfully if agents want to avoid regret. This means that, for each agent and each possible misrepresentation of preferences the agent could try, there is a scenario where the agent regrets deviating; this is, for those deviations that may be profitable in some situation, there is another preference profile consistent with the observed alternative such that the deviation would yield a detrimental outcome. This ensures that the agent will never suffer regret. Consequently, truth is a regret-free report. Formally,

ficiency, anonymity and neutrality). If a deterministic election is called for, we will use either a non-anonymous tie-breaking rule or a non-neutral one" (see Moulin, 1991, p.234).

[7]The extension of the definitions of anonymity and neutrality to correspondences is immediate. It is clear that the incompatibility problem between anonymity and neutrality disappears if we consider voting correspondences.

[8]Throughout the paper, given a strict order $>$ defined on a set A and a subset $B \subseteq A$, we denote by $\max_{>} B$ to the maximum element of $>$ in B.

Definition 1 *The rule* $f : \mathcal{P}^n \to X$ *is* ***regret-free (truth-telling)*** *whenever for each* $i \in N$, *each* $P \in \mathcal{P}^n$, *and each* $P_i' \in \mathcal{P}$ *such that* $f(P_i', P_{-i}) P_i f(P)$, *there is* $P_{-i}^\star \in \mathcal{P}^{n-1}$ *such that*

$$f(P_i, P_{-i}^\star) = f(P) \text{ and } f(P_i, P_{-i}^\star) P_i f(P_i', P_{-i}^\star).$$

3 Tops-only rules and the case of two alternatives

It is clear that strategy-proofness implies regret-freeness. Our first result states that the converse is also true for tops-only rules.

Proposition 1 *If a rule is regret-free and tops-only, then it is strategy-proof.*

Proof. Let $f : \mathcal{P}^n \to X$ be a regret-free and tops-only rule. Assume f is not strategy-proof. Then, there are $P \in \mathcal{P}^n$, $i \in N$, and $P_i' \in \mathcal{P}$ such that $f(P_i', P_{-i}) P_i f(P)$. Let $\widetilde{P}_i \in \mathcal{P}$ be such that $t(\widetilde{P}_i) = t(P_i)$ and $t_1(\widetilde{P}_i) = f(P)$. By tops-onlyness, $f(\widetilde{P}_i, P_{-i}) = f(P)$. Therefore, since $t_1(\widetilde{P}_i) = f(P)$, it follows that

$$f(P_i', P_{-i}) \widetilde{P}_i f(\widetilde{P}_i, P_{-i}). \tag{1}$$

Let $P_{-i}^\star \in \mathcal{P}^{n-1}$ be such that $f(\widetilde{P}_i, P_{-i}^\star) = f(\widetilde{P}_i, P_{-i})$. Since $t_1(\widetilde{P}_i) = f(P) = f(\widetilde{P}_i, P_{-i})$, we have

$$f(P_i', P_{-i}^\star) \widetilde{R}_i f(\widetilde{P}_i, P_{-i}^\star). \tag{2}$$

By (1) and (2), f is not regret-free. □

Consider the simplest social choice problem where $X = \{x, y\}$. In this problem, there is a complete characterization of the class of regret-free rules. In order to obtain it, we first need to define the family of extended majority voting rules on $\{x, y\}$.[9] Fix $w \in \{x, y\}$ and let 2^N denote the family of all subsets of N, referred to as *coalitions*. A family $\mathcal{C}_w \subseteq 2^N$ of coalitions is a *committee for* w if it satisfies the following monotonicity property: $S \in \mathcal{C}_w$ and $S \subsetneq T$ imply $T \in \mathcal{C}_w$. The elements in $\mathcal{C}_w$ are called *winning coalitions (for* w*)*.

Definition 2 *A rule* $f : \mathcal{P}^n \to \{x, y\}$ *is an* ***extended majority voting*** *rule if there is a committee* $\mathcal{C}_x$ *for* x *with the property that, for each* $P \in \mathcal{P}^n$,

$$f(P) = x \text{ if and only if } \{i \in N \mid t(P_i) = x\} \in \mathcal{C}_x.$$

[9] These rules are equivalent to the ones presented in Moulin (1980), where fixed ballots are used to describe them instead of committees.

The following corollary provides the characterization result.

Corollary 1 *Assume $m = 2$. Then,*

(i) A rule is regret-free if and only if it is strategy-proof,

(ii) A rule is regret-free if and only it is an extended majority voting rule.

Proof. (i) If f is strategy-proof it is clear that f is regret-free. If f is regret-free, since when $m = 2$ every rule is tops-only, f is strategy-proof by Proposition 1.
(ii) It follows from (i) and Moulin (1980). □

When there are more than two alternatives, tops-onlyness and efficiency lead to an impossibility result.

Corollary 2 *Assume $m > 2$. A rule is regret-free, efficient, and tops-only if and only if it is a dictatorship.*

Proof. It follows from Proposition 1 and Gibbard-Satterthwaite's Theorem. □

In the rest of the paper we assume that there are three or more alternatives ($m > 2$). Since a general characterization result with more than two alternatives seems hard to obtain, we study rules that fulfill regret-freeness in three well known classes of non-tops-only rules: the maxmin rules, the scoring rules and the Condorcet-consistent rules. We believe that the class of all regret-free rules is quite large, and leave the full characterization of this class as an open problem for future research.

4 Maxmin rules and scoring rules: positive and negative results

The first rules we study consist of the maxmin methods, that select those alternatives that "make the least happy agents as happy as possible". This methods are presented and defended, among others, by John Rawls in its classic Theory of Justice (Rawls, 1971). Given $P \in \mathcal{P}^n$ and $x \in X$, the *minimal position* of x according to P is defined by

$$mp(x, P) = \min\{k : \text{there exists } i \in N \text{ such that } x = t_k(P_i)\}.$$

An alternative is a *maxmin winner* if there is no other alternative with higher minimal position. We denote the set of *maxmin winners* according to P as $\mathcal{M}(P)$. Namely,

$$\mathcal{M}(P) = \{x : mp(x, P) \geq mp(y, P) \text{ for each } y \in X\}.$$

The idea of making the least happy agents as happy as possible is captured by rules that pick, for each preference profile, a maxmin winner for that profile. We study both anonymous and neutral versions of these rules. Formally,

Definition 3 *A rule $f : \mathcal{P}^n \to X$ is*

(i) ***A-maxmin*** *if there is a strict order $\succ$ on X such that, for each $P \in \mathcal{P}^n$,*

$$f(P) = \max_{\succ} \mathcal{M}(P).$$

(ii) ***N-maxmin*** *if there is an agent $i \in N$ such that, for each $P \in \mathcal{P}^n$,*

$$f(P) = \max_{P_i} \mathcal{M}(P).$$

The following theorem summarizes the positive results concerning regret-freeness for these rules:[10]

Theorem 1 *(i) An A-maxmin rule is regret-free if and only if $n \geq m-1$ or n divides $m-1$.*

(ii) Any N-maxmin rule is regret-free.

Proof. See Appendix A.1. □

Next, we present the family of scoring rules. Given $P \in \mathcal{P}^n$ and $x \in X$, let $N(P,k,x) = \{i \in N : t_k(P_i) = x\}$ be the set of agents that have x in the k-th position (from bottom to top) in their preferences, and let $n(P,k,x) = |N(P,k,x)|$. Let s_k be the score associated to the k-th position (from bottom to top) with $s_1 \leq s_2 \leq \ldots \leq s_m$ and $s_1 < s_m$. The *score of $x \in X$ according to P* is defined by

$$s(P,x) = \sum_{k=1}^{m} [s_k \cdot n(P,k,x)].$$

The set of *scoring winners according to P* is

$$\mathcal{S}(P) = \{x \in X : s(P,x) \geq s(P,y) \text{ for all } y \in X\}.$$

For future reference, given scores $s_1 \leq s_2 \leq \ldots \leq s_m$, let denote by $k^\star$ the highest position where the score is not maximal, i.e., $k^\star$ is such that $s_1 \leq s_2 \leq s_{k^\star} < s_{k^\star+1} = \ldots = s_m$.

[10] All the proofs of this section are relegated to Appendix A.

Definition 4 *A rule $f : \mathcal{P}^n \to X$ is*

(i) ***A-scoring*** *associated to $s_1 \leq s_2 \leq \ldots \leq s_m$ if there is an order $\succ$ on X such that, for each $P \in \mathcal{P}^n$,*

$$f(P) = \max_{\succ} \mathcal{S}(P).$$

(ii) ***N-scoring*** *associated to $s_1 \leq s_2 \leq \ldots \leq s_m$ if there is an agent $i \in N$ such that, for each $P \in \mathcal{P}^n$,*

$$f(P) = \max_{P_i} \mathcal{S}(P).$$

Remark 1 *Some of the most well known scoring rules are:*

*(i) the **Borda** rule, in which $s_k = k$ for $k = 1, \ldots, m$;*

*(ii) the **Dowdall** rule (see Reilly, 2002), in which $s_k = \frac{1}{m-k+1}$ for $k = 1, \ldots, m$;*

*(iii) the **k-approval** rules, in which $0 = s_1 = s_2 = \ldots = s_{m-k}, s_{m-k+1} = \ldots = s_{m-1} = s_m = 1$ for some k such that $m - 1 \geq k \geq 1$, i.e., the top k scores are 1 and the rest are 0. In these rules, agents are asked to name their k best alternatives, and the alternative with most votes wins. Note that in this rule $k^\star = m - k$.*

Within this rules, two subclasses stand out:

*(iii.a) the **plurality** rule, where $k = 1$ and therefore $s_1 = s_2 = \ldots = s_{m-1} = 0$ and $s_m = 1$ (note that $k^\star = m - 1$);*

*(iii.b) the **negative plurality** rule, where $k = m - 1$ and therefore $s_1 = 0$ and $s_2 = \ldots = s_{m-1} = s_m = 1$ (note that $k^\star = 1$).*

Remark 2 *If an A-scoring rule is efficient, then $s_{m-1} < s_m$ (i.e., $k^\star = m - 1$).*

Observe that, by definition, $k^\star \in \{1, 2, \ldots, m-1\}$. The next theorems consider the extreme cases in which $k^\star = 1$ and $k^\star = m - 1$, and allows us to present conclusive results about efficient A-scoring rules, the Borda rule, the Dowdall rule, and plurality and negative plurality rules.

Theorem 2 *(i) An A-scoring rule with $k^\star = 1$ (i.e., an A-negative plurality rule) is regret-free if and only if $n \geq m - 1$.*[11]

[11] For A-negative plurality rules, Theorem 6 in Reijngoud and Endriss (2012) presents only a sufficient condition ($n + 2 \geq 2m$) guaranteeing regret-freeness. We present a general and independent proof that encompasses their result as well.

(ii) Any N-scoring rule with $k^\star = 1$ (i.e., any N-negative plurality rule) is regret-free.

Proof. See Appendix A.2. □

Theorem 3 *Assume $n > 2$. Then, no (anonymous or neutral) scoring rule with $k^\star = m - 1$ is regret-free.*

Proof. See Appendix A.3. □

Corollary 3 *Assume $n > 2$. Then, both anonymous and neutral versions of Borda, plurality, and Dowdall rules are not regret-free. Moreover, no efficient A-scoring rule is regret-free.*

From now on, we assume that $k^\star$ is such that $1 < k^\star < m - 1$. Next, we present some results for scoring rules by means of two complementary theorems, one of which can be considered as positive and the other one as negative. These theorems allow us to present conclusive results about approval rules and scoring rules in which $s_{k^\star-1} = s_{k^\star}$. Theorem 4 below focuses on the case $k^\star n < m$, which encompasses the class of scoring rules where, in any preference profile, there is always an alternative that gets maximal score. This positive result gives a necessary and sufficient condition for an A-scoring rule to be regret-free and also states that any N-scoring rule is regret-free.

Theorem 4 *Assume that $n > 2$ and $k^\star n < m$. Then,*

(i) An A-scoring rule is regret-free if and only if $k^\star n = m - 1$.

(ii) Any N-scoring rule is regret-free.

Proof. See Appendix A.4. □

On the other hand, Theorem 5 below gives a negative result for the case $k^\star n \geq m$ when $s_{k^\star-1} = s_{k^\star}$. When $s_{k^\star-1} \neq s_{k^\star}$, we believe that the existence of regret-free rules depends sensibly on the specific scores defining each rule.

Theorem 5 *Assume that $n > 2$ and $k^\star n \geq m$. Then, there is no regret-free scoring rule (neither anonymous nor neutral) with $s_{k^\star-1} = s_{k^\star}$.*

Proof. See Appendix A.5. □

The previous theorem extends the result of Theorem 3 in Reijngoud and Endriss (2012) to the case $n = 3$ and also to the neutral scoring rules (their result only applies when $n > 3$ in the anonymous case). Furthermore, our proof is general and independent of theirs.

Corollary 4 *Assume $n > 2$ and $k^\star > 1$. Then,*

(i) An A-scoring rule with $s_{k^\star-1} = s_{k^\star}$ is regret-free if and only if $k^\star n = m - 1$. In particular, an anonymous $(m - k^\star)$-approval rule is regret-free if and only if $k^\star n = m - 1$.

(ii) An N-scoring rule with $s_{k^\star-1} = s_{k^\star}$ is regret-free if and only if $k^\star n < m$. In particular, a neutral $(m - k^\star)$-approval rule is regret-free if and only if $k^\star n < m$.

Corollary 4 (i) contradicts Theorem 2 of Endriss et al. (2016) which states that there is no anonymous approval rule satisfying regret-freeness. We think that their proof is incorrect because they assume that there is a preference profile in which the outcome is below the position $k^\star + 1$ for some agent. However, this assumption cannot be met in the case where $k^\star n < m$.

5 Condorcet consistent rules: negative results

Let $P \in \mathcal{P}^n$ and consider two alternatives $a, b \in X$. Denote by $C_P(a, b)$ the number of agents that prefer a to b according to P, i.e., $C_P(a, b) = \{i \in N : aP_ib\}$. An alternative $a \in X$ is a *Condorcet winner* according to P if for each alternative $b \in X \setminus \{a\}$,

$$C_P(a, b) > C_P(b, a). \tag{3}$$

Notice that a Condorcet winner may not always exist but when it does, it is unique. If (3) holds with weak inequality for each alternative $b \in X \setminus \{a\}$, then a is called a *weak Condorcet winner.*

Definition 5 *A rule $f : \mathcal{P}^n \to X$ is* ***Condorcet consistent*** *if it chooses the Condorcet winner whenever it exists.*

Next, we define a minimal monotonicity condition to control the rule whenever a Condorcet winner does not exist. This condition says that if an alternative is below the outcome for an agent and he changes his preferences modifying only the ordering of alternatives above the outcome, then such alternative continues not to be chosen. Formally,

Definition 6 *Let $P_i, P'_i \in \mathcal{P}$ and let $a \in X$ be such that $a = t_k(P_i)$. We say that P'_i is a* ***monotonic transformation of P_i with respect to a*** *if $t_{k'}(P_i) = t_{k'}(P'_i)$ for each $k' \leq k$. A rule $f : \mathcal{P}^n \to X$ is* ***monotone*** *if, for each $P \in \mathcal{P}^n$, each $i \in N$, and each $b \in X$ such that $f(P)P_ib$,*

$$f(P'_i, P_{-i}) \neq b$$

for each P'_i that is a monotonic transformation of P_i with respect to $f(P)$.

Notice that $C_P(x,b) = C_{(P'_i,P_{-i})}(x,b)$ for each $x \in X \setminus \{b\}$. Thus, our monotonicity condition is fully compatible with Condorcet consistency.

Furthermore, our notion of monotonicity is weaker than the well-known Maskin monotonicity. Remember that $P'_i \in \mathcal{P}$ is a *Maskin monotonic transformation* of $P_i \in \mathcal{P}$ with respect to $a \in X$ if xP'_ia implies xP_ia. Then, $f : \mathcal{P}^n \to X$ is *Maskin monotonic* if, for each $P \in \mathcal{P}^n$, $f(P'_i, P_{-i}) = f(P)$ for each $P'_i \in \mathcal{P}$ that is a Maskin monotonic transformation of P_i with respect to $f(P)$. It is clear that a monotonic transformation of P_i (according to our definition) is a Maskin monotonic transformation of P_i.

Besides the intrinsic appeal of our monotonicity condition, this weakening of Maskin's property is necessary since Maskin's monotonicity is incompatible with Condorcet consistency. To see this, let $X = \{a, b, c\}$ and consider a Condorcet consistent $f : \mathcal{P}^3 \to X$ and a profile $P \in \mathcal{P}^3$ given by the following table:

P_1	P_2	P_3
a	b	c
b	c	a
c	a	b

Since there is no Condorcet winner, and without loss of generality, assume $f(P) = a$. Now, let $P'_1 \in \mathcal{P}$ be such that $cP'_1bP'_1a$. It follows, by Condorcet consistency, that $f(P'_1, P_{-1}) = c$. If f is also Maskin monotonic, then $f(P'_1, P_{-1}) = c$ implies $f(P) = c$, a contradiction.

With our mild requirement of monotonicity, we obtain the following negative result concerning regret-freeness.

Theorem 6 *Assume $n \neq 4, 2$ or $n = 4$ and $m > 3$. Then, there is no Condorcet consistent, monotone and regret-free rule.*

Proof. See Appendix A.6. □

Remark 3 *When $n = 4$ and $m = 3$, the previous impossibility result does not apply. Let $X = \{a, b, c\}$ and consider a rule $f : \mathcal{P}^4 \to X$ that selects the Condorcet winner when it exists and, otherwise, chooses an alternative within the set of alternatives that appear less times in the bottom of the preference profile. If there are two alternatives in this set, the rule selects the one that wins at least two times to the other and, in case of a tie, the tie-breaking is given by $a \succ b \succ c$. This rule is monotone since, given $P \in \mathcal{P}^4$ and $i \in N$, when there are three alternatives a monotonic transformation of P_i with respect to $f(P)$ is different from P_i only when $t_1(P_i) = f(P)$, and in this case there is no $x \in X$ such that $f(P)P_ix$. Then, monotonicity* *is trivially satisfied. To see that this rule is also regret-free, consider profile* $P \in \mathcal{P}^4$ *such that, w.l.o.g.,* $P_1 : x, y, z$. *If* $f(P) = x$, *agent* 1 *does not manipulate* f. *If* $f(P) = z$, *then* $f(P_1', P_{-1}) = t_1(P_1)$ *for each* $P_1' \in \mathcal{P}$ *by definition of the rule, so agent* 1 *cannot manipulate either. If* $f(P) = y$ *and agent* 1 *manipulates* f *via* P_1', *then* $f(P_1', P_{-1}) = x$ *and, by definition of the rule,* $t_1(P_1') = y$. *Consider* $P_2^\star : y, z, x$, $P_3^\star = P_2^\star$, *and* $P_4^\star : z, y, x$. *Then,* $f(P_1, P_{-1}^\star) = y$ *and* $f(P_1', P_{-1}^\star) = z$. *Therefore, agent 1 regrets manipulating* f *via* P_1'.

Six of the most important Condorcet consistent rules are Simpson, Copeland, Young, Dodgson, Fishburn and Black rules (see Fishburn, 1977). Each one of these rules uses pairwise comparison of alternatives in a specific way in order to get a *winner* alternative for each profile of preferences. Their definitions are as follows. Given $P \in \mathcal{P}^n$,

(i) the *Simpson score* of alternative $a \in X$ is the minimum number $C_P(a, b)$ for $b \neq a$,

$$Simpson(P, a) = \min_{b \neq a} C_P(a, b)$$

and a *Simpson winner* is an alternative with highest such score.[12]

(ii) the *Copeland score* of alternative $a \in X$ is the number of pairwise victories minus the number of pairwise defeats against all other alternatives

$$Copeland(P, a) = |\{b : C_P(a, b) > C_P(b, a)\}| - |\{b : C_P(b, a) > C_P(a, b)\}|$$

and a *Copeland winner* is an alternative with highest such score.

(iii) the *Young score* of alternative $a \in X$ is the largest cardinality of a subset of voters for which alternative a is a weak Condorcet winner

$$Young(P, a) = \max_{N' \subseteq N} \left\{ |N'| : \{|i \in N' : aP_i b\}| \geq \frac{|N'|}{2} \text{ for all } b \in X \setminus \{a\} \right\}$$

and a *Young winner* is an alternative with highest such score.

(iv) the *Dodgson score* of alternative $a \in X$, $Dodgson(P, a)$, is the fewest inversions[13] in the preferences in P that will make a tie or beat every other alternative in X on the basis of simple majority, and a *Dodgson winner* is an alternative with lowest such score.

[12] Simpson rule is also known as *Simpson-Kramer* rule.

[13] Let $P_i, P_i' \in \mathcal{P}$ and let $x, y \in X$. P_i' is an *inversion of* P_i *with respect* x *and* y if $xP_i y$ implies $yP_i' x$.

(v) the *Fishburn partial order* on X, F_P, is defined as follows: aF_Pb if and only if for each $x \in X$, $C_P(x,a) > C_P(a,x)$ implies $C_P(x,b) > C_P(b,x)$ and there is $w \in X$ such that $C_P(w,b) > C_P(b,w)$ and $C_P(a,w) \geq C_P(w,a)$. A *Fishburn winner* is a maximal alternative for F_P.

(vi) a *Black winner* is a Condorcet winner whenever it exists and, otherwise, a *Borda winner*.[14]

An *anonymous (neutral) Simpson, (Copeland, Young, Dodgson, Fishburn, Black) rule* always chooses a Simpson, (Copeland, Young, Dodgson, Fishburn, Black) winner and uses a fixed order (agent) as tie-breaker when there are more than one. The following result shows that the six rules are monotonic.

Corollary 5 *Assume $n > 2$. Then Simpson, Copeland, Young, Dodgson, Fishburn and Black rules (in both their anonymous and neutral versions) are not regret-free.*

Proof. See Appendix A.7. □

Another interesting class of Condorcet consistent rules which are widely used in practice, for instance, by the United States Congress to vote upon a motion and its proposed amendments, is the class of succesive elimination rules (see Chapter 9 of Moulin, 1991, for more detail). These rules, which consider an order among alternatives an consist of sequential majority comparisons, are defined as follows.

Definition 7 *A rule $f : \mathcal{P}^n \to X$ is a **successive elimination** rule with respect to an order $\succ$ such that $a_1 \succ a_2 \succ \ldots \succ a_m$ if it operates in the following way. First, a majority vote decides to eliminate a_1 or a_2, then a majority vote decides to eliminate the survivor from the first round or a_3, and so on. The same order $\succ$ is used as tie-breaker in each pairwise comparison, if necessary.*

It is clear that a successive elimination rule is Condorcet consistent but it may be not monotone, as the next example shows.

Example 1 *(The successive elimination rule is not monotone). Let $P \in \mathcal{P}^5$ be given by the following table:*

P_1	P_2	P_3	P_4	P_5
a	a	c	c	d
b	c	d	b	b
d	d	a	d	a
c	b	b	a	c

[14] A *Borda winner* is an alternative with highest Borda score.

Then, $f(P) = d$. Now, let $P_1' \in \mathcal{P}$ be such that $P_1' : b, a, d, c$. Then P_1' is a monotonic transformation of P_1 with respect to d but $f(P_1', P_{-1}) = c$, so f is not monotone.

Unfortunately, we also obtain a negative result concerning regret-freeness for this class of rules.

Theorem 7 *Assume $n > 2$. Then, no successive elimination rule is regret-free.*

Proof. See Appendix A.8. □

6 The case with two agents and three alternatives

In what follows, we focus in the case where we have only two agents, $N = \{1, 2\}$, and three alternatives, $X = \{a, b, c\}$. In this case we can obtain characterizations of the classes of all: (i) regret-free and neutral, and (ii) regret-free, efficient, and anonymous rules. Notice that for the first characterization efficiency is not needed since it is implies by regret-freeness and neutrality, as we prove next in Theorem 8.

First, observe that with two agents and three alternatives a N-maxmin rule (associated to agent i) coincides with:

(i) the N- negative plurality rule (associated to agent i), and

(ii) the N-scoring rule (associated to agent i) corresponding to $\bar{s} = (\bar{s}_1, \bar{s}_2, \bar{s}_3) = (1, 3, 4)$.

The following theorem shows that N-maxmin and dictatorships are the only regret-free and neutral rules.

Theorem 8 *Assume $n = 2$ and $m = 3$. Then, a rule is regret-free and neutral if and only if it is a N-maxmin rule or a dictatorship.*

Proof. See Appendix A.9. □

A similar result to the previous theorem can be obtained changing neutrality for anonymity. As efficiency is not a consequence of regret-freeness and anonymity we require it in the next theorem.[15]

In this case, we need to enlarge the class of A-maxmin rules by dropping the requirement of transitivity for the tie-breaking associated to the rules and to add the successive elimination rules into the picture, as we did with dictatorial rules in Theorem 8.

[15] For example, a constant rule is regret-free and anonymous but not efficient.

Definition 8 *A rule $f : \mathcal{P}^2 \to X$ is an **A-maxmin*** rule if there is an antisymmetric and complete (not necessarily transitive) binary relation $\succ^\star$ on X such that, for each $P \in \mathcal{P}^2$,*

$$f(P) = \max_{\succ^\star} \mathcal{M}(P).$$

Observe that, since $|N| = 2$, $|\mathcal{M}(P)| \leq 2$ and therefore $\max_{\succ^\star} \mathcal{M}(P)$ is well defined. In a similar way to Definition 8 we can define the A-scoring* rule associated to $\succ^\star$. Notice that the A-maxmin* rule associated to $\succ^\star$ coincides with the A-scoring* rule with $\bar{s} = (\bar{s}_1, \bar{s}_2, \bar{s}_3) = (1, 3, 4)$ associated to $\succ^\star$.

Theorem 9 *Assume $n = 2$ and $m = 3$. Then, a rule is regret-free, efficient, and anonymous if and only if it is a successive elimination rule or an A-maxmin* rule.*

Proof. See Appendix A.10. □

Concerning the independence of axioms in the characterizations, it is clear that regret-freeness and neutrality in Theorem 8 are independent. Successive elimination rules are regret-free and not neutral, and the rule that always chooses the bottom of agent 1 is neutral and not regret-free. On the other hand, in Theorem 9, a constant rule is regret-free, anonymous, and not efficient and a dictatorship is regret-free, efficient, and not anonymous. Now, given order $a \succ b \succ c$, consider the rule $f(P) = \max_{\succ}\{t(P_1), t(P_2)\}$. This rule is anonymous, efficient, and not regret-free.

$m = 2$	regret-free $\iff$ ext. majority voting	Cor. 1
$n = 2, m = 3$	regret-free + neutral $\iff$ N-maxmin or dictatorship	Th. 8
$n = 2, m = 3$	regret-free + eff. + anon. $\iff$ A-maxmin* or succ. elim.	Th. 9

Table 1: *Characterization results with $m = 2$ or $n = 2$ and $m = 3$.*

7 Concluding remarks

Table 1 summarizes the characterization results when there are only two alternatives, or two agents and three alternatives. Table 2 summarizes our main findings about tops-only, maxmin, scoring, and Condorcet consistent rules.

We finish noticing that when there are two agents and more than three alternatives: (i) anonymous maxmin rules are regret-free if and only if the number of alternatives is odd (Theorem 1), (ii) no anonymous negative plurality rules are regret-free (Theorem

<table>
<tr><td>Tops-only</td><td colspan="3">strategy-proof $\iff$ regret-free</td><td>Pr. 1</td></tr>
<tr><td>A-maxmin</td><td colspan="3">$n \geq m-1$ or n divides $m-1$ $\iff$ regret-free</td><td>Th. 1</td></tr>
<tr><td>N-maxmin</td><td colspan="3">all regret-free</td><td>Th. 1</td></tr>
<tr><td rowspan="4">A-scoring[†]
$(n > 2)$</td><td>$k^\star = 1$</td><td colspan="2">$n \geq m-1$ $\iff$ regret-free</td><td>Th. 2</td></tr>
<tr><td rowspan="2">$1 < k^\star < m-1$</td><td>$k^\star n < m$</td><td>$k^\star n = m-1$ $\iff$ regret-free</td><td>Th. 4</td></tr>
<tr><td>$k^\star n \geq m$</td><td>$s_{k^\star-1} = s_{k^\star}$ $\implies$ none regret-free</td><td>Th. 5</td></tr>
<tr><td>$k^\star = m-1$</td><td colspan="2">none regret-free</td><td>Th. 3</td></tr>
<tr><td rowspan="4">N-scoring[†]
$(n > 2)$</td><td>$k^\star = 1$</td><td colspan="2">all regret-free</td><td>Th. 2</td></tr>
<tr><td rowspan="2">$1 < k^\star < m-1$</td><td>$k^\star n < m$</td><td>all regret-free</td><td>Th. 4</td></tr>
<tr><td>$k^\star n \geq m$</td><td>$s_{k^\star-1} = s_{k^\star}$ $\implies$ none regret-free</td><td>Th. 5</td></tr>
<tr><td>$k^\star = m-1$</td><td colspan="2">none regret-free</td><td>Th. 3</td></tr>
<tr><td rowspan="2">Condorcet consistent
$(n > 2)$</td><td>Monotone</td><td colspan="2">$n \neq 4$ or $m > 3$ $\implies$ none regret-free</td><td>Th. 6</td></tr>
<tr><td>Succesive elimination</td><td colspan="2">none regret-free</td><td>Th. 7</td></tr>
</table>

[†] Remember that $k^\star$ is such that $s_1 \leq s_2 \leq s_{k^\star} < s_{k^\star+1} = \ldots = s_m$. The results for $k^\star = 1$ also apply when $n = 2$.

Table 2: *Summary of results for tops-only, maxmin, scoring, and Condorcet consistent rules.*

2), and (iii) neutral maxmin rules and neutral negative plurality rules are always regret-free (Theorems 1 and 2). However, it is an open question which other scoring rules (if any) are regret-free.

References

AZIZ, H. AND A. LAM (2021): "Obvious Manipulability of Voting Rules," in *International Conference on Algorithmic Decision Theory*, Springer, 179–193.

ENDRISS, U., S. OBRAZTSOVA, M. POLUKAROV, AND J. S. ROSENSCHEIN (2016): "Strategic voting with incomplete information," in *Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence*, 236–242.

FERNANDEZ, M. A. (2020): "Deferred acceptance and regret-free truth-telling," Working Paper.

FISHBURN, P. C. (1977): "Condorcet social choice functions," *SIAM Journal on Applied Mathematics*, 33, 469–489.

GIBBARD, A. (1973): "Manipulation of voting schemes: a general result," *Econometrica*, 587–601.

GORI, M. (2021): "Manipulation of social choice functions under incomplete information," *Games and Economic Behavior*, 129, 350–369.

MOULIN, H. (1980): "On strategy-proofness and single peakedness," *Public Choice*, 35, 437–455.

——— (1991): *Axioms of Cooperative Decision Making*, Cambridge University Press.

OSBORNE, M. J. AND A. RUBINSTEIN (2003): "Sampling equilibrium, with an application to strategic voting," *Games and Economic Behavior*, 45, 434–441.

RAWLS, J. (1971): *A Theory of Justice*, Harvard University Press.

REIJNGOUD, A. AND U. ENDRISS (2012): "Voter response to iterated poll information," in *Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems-Volume 2*, 635–644.

REILLY, B. (2002): "Social choice in the south seas: Electoral innovation and the borda count in the pacific island countries," *International Political Science Review*, 23, 355–372.

SATTERTHWAITE, M. A. (1975): "Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions," *Journal of Economic Theory*, 10, 187–217.

TROYAN, P. AND T. MORRILL (2020): "Obvious manipulations," *Journal of Economic Theory*, 185, 104970.

A Appendix

A.1 Proof of Theorem 1

We first show the equivalence in part (i). Let $f : \mathcal{P}^n \to X$ be a A-maxmin rule.
($\Longrightarrow$) Assume that $n < m - 1$ and that n does not divide $m - 1$. Then, there exists $h \geq 1$ and $1 \leq s < n$ such that $nh + s = m - 1$. Then, $nh + r = m$ with $h \geq 1$ and $2 \leq r \leq n$.

Let $X = \{x_1, \ldots, x_m\}$ and assume that f has associated order $x_m \succ x_{m-1} \succ \ldots \succ x_1$. Let $P \in \mathcal{P}^n$ be given by the following table:

	P_1	P_2	P_3	$\dots$	P_r	P_{r+1}	$\dots$	P_n
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
last	x_m	x_{m-1}	x_{m-2}	$\dots$	$x_{m-(r-1)}$	$\dots$	$\dots$	$\dots$
$(h+1)$	x_1	x_{h+1}	x_{2h+1}	$\dots$	$x_{(r-1)h+1}$	x_{rh+1}	$\dots$	$x_{(n-1)h+1}$
positions	x_2	x_{h+2}	x_{2h+2}	$\dots$	$x_{(r-1)h+2}$	x_{rh+2}	$\dots$	$x_{(n-1)h+2}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	x_h	x_{2h}	x_{3h}	$\dots$	x_{rh}	$x_{(r+1)h}$	$\dots$	x_{nh}

Note that $nh = m - r$, so $x_{m-(r-1)} = x_{nh+1}$. Then, $\mathcal{M}(P) = \{x_m, x_{m-1}, \dots, x_{m-(r-1)}\}$ and $f(P) = x_m$. Now, consider agent 1 and preference $P_1' \in \mathcal{P}$ that differs from P_1 only in that the positions of x_m and x_1 are interchanged. We have that $\mathcal{M}(P_1', P_{-1}) = \{x_1, x_{m-1,...,} x_{m-(r-1)}\}$, $f(P_1', P_{-1}) = x_{m-1}$ and, therefore,

$$f(P_1', P_{-1}) P_1 f(P_1, P_{-1}). \tag{4}$$

Let $P_{-1}^{\star} \in \mathcal{P}^{n-1}$ be such that $f(P_1, P_{-1}^{\star}) = f(P) = x_m$. Since $hn + r = m$, it follows that $mp((P_1', P_{-1}^{\star}), f(P_1', P_{-1}^{\star}) \geq m - (h+1)$. There are two cases to consider:

1. $\boldsymbol{mp((P_1', P_{-1}^{\star}), f(P_1', P_{-1}^{\star}) > m - (h+1)}$. By the definition of P_1', $f(P_1', P_{-1}^{\star}) \notin \{x_1, ..., x_h\}$ and, therefore,

$$f(P_1', P_{-1}^{\star}) R_1 f(P_1, P_{-1}^{\star}). \tag{5}$$

   By (4) and (5), f is not regret-free.

2. $\boldsymbol{mp((P_1', P_{-1}^{\star}), f(P_1', P_{-1}^{\star})) = m - (h+1)}$. As $nh + r = m$, $|\mathcal{M}(P_1', P_{-1}^{\star})| = r \geq 2$ and $f(P_1', P_{-1}^{\star}) \neq x_1$ (because x_1 is the last one in order $\succ$). Now, by the definition of P_1', $\mathcal{M}(P_1', P_{-1}^{\star}) \cap \{x_2, ..., x_h\} = \varnothing$. Then, $f(P_1', P_{-1}^{\star}) \in X \setminus \{x_1, x_2, ..., x_h\}$. Thus, both (4) and (5) hold in this case as well, and f is not regret-free.

($\Longleftarrow$) Assume that there exist $i \in N$, $(P_i, P_{-i}) \in \mathcal{P}^n$ and $P_i' \in \mathcal{P}$ such that

$$f(P_i', P_{-i}) P_i f(P_i, P_{-i}). \tag{6}$$

We will prove that there is $P_{-i}^{\star} \in \mathcal{P}^{n-1}$ such that $f(P) = f(P_i, P_{-i}^{\star})$ and $f(P_i, P_{-i}^{\star}) P_i f(P_i', P_{-i}^{\star})$. Let $\widehat{P} = (P_i', P_{-i})$. As f is an A-maxmin rule,

$$mp(P, f(P)) \geq mp(P, f(\widehat{P})) \tag{7}$$

and

$$mp(\widehat{P}, f(\widehat{P})) \geq mp(\widehat{P}, f(P)). \tag{8}$$

Let $\bar{k}$ be such that $t_{\bar{k}}(P_i) = f(P)$. By (7) and since $f(\widehat{P})P_i f(P)$,

$$mp(P, f(\widehat{P})) = k^{\star} \leq \bar{k} \leq m-1, \tag{9}$$

where $t_{k^{\star}}(\widehat{P}_j) = f(\widehat{P})$ for some $j \in N \setminus \{i\}$. Then, as $\widehat{P}_j = P_j$ and $k^{\star} \leq \bar{k}$,

$$mp(P, f(\widehat{P})) \geq mp(\widehat{P}, f(\widehat{P})). \tag{10}$$

If $mp(P, f(P)) = mp(P, f(\widehat{P}))$ and $mp(\widehat{P}, f(\widehat{P})) = mp(\widehat{P}, f(P))$, then

$$f(P), f(\widehat{P}) \in \mathcal{M}(\widehat{P}) \cap \mathcal{M}(P), \tag{11}$$

contradicting that $f(P) \neq f(\widehat{P})$. Therefore, by (7) and (8), $mp(\widehat{P}, f(\widehat{P})) > mp(\widehat{P}, f(P))$ or $mp(P, f(P)) > mp(P, f(\widehat{P}))$. By (10),

$$mp(P, f(P)) > mp(\widehat{P}, f(P)). \tag{12}$$

Let $\widehat{k}$ be such that $mp(\widehat{P}, f(P)) = \widehat{k}$. Then, by (12), $t_{\widehat{k}}(\widehat{P}_i) = f(P)$ and $f(P)\widehat{P}_j t_{\widehat{k}}(\widehat{P}_j)$ for all $j \in N \setminus \{i\}$. If $\bar{k} \leq \widehat{k}$, then

$$mp(\widehat{P}, f(P)) = \widehat{k} \geq \bar{k} \geq mp(P, f(P)),$$

which contradicts (12). Therefore,

$$\bar{k} > \widehat{k}. \tag{13}$$

This implies that there exists an alternative $c \in X$ such that

$$f(P) = t_{\bar{k}}(P_i) P_i c \text{ and } c R_i' t_{\bar{k}}(P_i') P_i' f(P). \tag{14}$$

There are two cases to consider:

1. **$n \geq m-1$.** Let $P_{-i}^{\star} \in \mathcal{P}^{n-1}$ be such that $t(P_j^{\star}) = f(P)$, $t_{m-1}(P_j^{\star}) = c$ for each $j \in N \setminus \{i\}$, and for each $x \in X \setminus \{f(P), c\}$ choose an agent j^x such that $t_1(P_{j^x}^{\star}) = x$ (this is feasible because $n-1 \geq m-2$). Now, let $P^{\star} = (P_i, P_{-i}^{\star})$. Then, $mp((P_i, P_{-i}^{\star}), x) = 1$ for all $x \in X \setminus \{f(P), c\}$ and from definition of $P_{-i}^{\star}$, (14) and the fact that $\bar{k} \leq m-1$, we have $mp((P_i, P_{-i}^{\star}), f(P)) = \bar{k} > mp((P_i, P_{-i}^{\star}), c)$. Therefore, $f(P_i, P_{-i}^{\star}) = f(P)$. Furthermore, $mp((P_i', P_{-i}^{\star}), x) = 1$ for each $x \in X \setminus \{f(P), c\}$ and from definition of $P_{-i}^{\star}$, (14) and the fact that $\bar{k} \leq m-1$, we have $mp((P_i', P_{-i}^{\star}), c) > mp((P_i', P_{-i}^{\star}), f(P))$. Therefore, $f(P_i', P_{-i}^{\star}) = c$.

   We conclude that $f(P) = f(P_i, P_{-i}^{\star})$ and, by (14) and the fact that $f(P_i', P_{-i}^{\star}) = c$, $f(P_i, P_{-i}^{\star}) P_i f(P_i', P_{-i}^{\star})$. Hence, f is regret-free.

2. **n divides $m-1$**. Thus, $m-1 = hn$ with $h \geq 1$. Therefore,

$$mp(P, f(P)) \geq h+1. \tag{15}$$

Let $Y = \{x \in X : xP_i f(P)\}$. Then,

$$|Y| < m - mp(P, f(P)) \leq m - (h+1) = hn + 1 - h - 1 = h(n-1). \tag{16}$$

Let $P^\star_{-i} \in \mathcal{P}^{n-1}$ be such that $t(P^\star_j) = f(P)$, $t_{m-1}(P^\star_j) = c$ for each $j \in N \setminus \{i\}$, and for each $x \in Y$ choose an agent j and a position $u \leq h$ such that $t_u(P^\star_j) = x$ (the construction of $P^\star_{-i}$ is feasible by (16) and the fact that $m-2 = hn-1 \geq h(n-1)$). Now, let $P^\star = (P_i, P^\star_{-i})$. Then, $mp((P_i, P^\star_{-i}), x) \leq h$ for each $x \in Y$, $mp((P_i, P^\star_{-i}), f(P)) \geq mp(P, f(P)) \geq h+1$ (this holds by (15) and the definition of $P^\star_{-i}$), and $mp((P_i, P^\star_{-i}), f(P)) = \bar{k} > mp((P_i, P^\star_{-i}), z)$ for each $z \in X \setminus Y$ (this follows from the definitions of $P^\star_{-i}$ and Y). Hence, $f(P_i, P^\star_{-i}) = f(P)$. Furthermore, $mp((P'_i, P^\star_{-i}), x) \leq h$ for each $x \in Y$ and $mp((P'_i, P^\star_{-i}), c) > mp((P'_i, P^\star_{-i}), f(P))$ (this follows from (9), (14), and the definition of $P^\star_{-i}$). Therefore, $f(P'_i, P^\star_{-i}) \in X \setminus Y$ and $f(P'_i, P^\star_{-i}) \neq f(P)$.

We conclude that $f(P) = f(P_i, P^\star_{-i})$ and $f(P_i, P^\star_{-i}) P_i f(P'_i, P^\star_{-i})$. Hence, f is regret-free.

Next, we show part (ii). Assume that $f : \mathcal{P}^n \to X$ is a N-maxmin rule. Then, there exists $\bar{j} \in N$ such that

$$f(\widetilde{P}) = \max_{\widetilde{P}_{\bar{j}}} \mathcal{M}(\widetilde{P}) \text{ for each } \widetilde{P} \in \mathcal{P}^n. \tag{17}$$

Let $P, P'_i, \widehat{P}, \bar{k}$, and $\widehat{k}$ be as in ($\Longleftarrow$) of part (i). It is easy to see that equations (6), (7), (8), (9) and (10) also hold here.

If $mp(P, f(P)) = mp(P, f(\widehat{P}))$ and $mp(\widehat{P}, f(\widehat{P})) = mp(\widehat{P}, f(P))$, then (11) holds as in the proof of part (i). As $f(P'_i, P_{-i}) P_i f(P_i, P_{-i})$, we have $\bar{j} \neq i$. But then (11) contradicts $f(P) \neq f(\widehat{P})$ since $P_{\bar{j}} = \widehat{P}_{\bar{j}}$. Therefore, by (7) and (8), $mp(\widehat{P}, f(\widehat{P})) > mp(\widehat{P}, f(P))$ or $mp(P, f(P)) > mp(P, f(\widehat{P}))$. Now, it is easy to see that equations (12), (13) and (14) hold in this proof as well, so there exists $c \in X$ such that $f(P) = t_{\bar{k}}(P_i) P_i c$ and $c R'_i t_{\bar{k}}(P'_i) P'_i f(P)$.

Now, we define profile $P^\star_{-i} \in \mathcal{P}^{n-1}$ where, for each $j \in N \setminus \{i\}$, $P^\star_j$ is differs from P_j in that $f(P)$ is now in the top of $P^\star_j$ and c is in the second place, while all the other alternatives keep their relative ranking. Formally, let $P^\star_{-i} \in \mathcal{P}^{n-1}$ be such that, for each

$j \in N \setminus \{i\}$, $t(P^\star_j) = f(P)$, $t_{m-1}(P^\star_j) = c$, and if k' and k'' are such that $t_{k'}(P_j) = f(P)$ and $t_{k''}(P_j) = c$, if we let $k_1 = \max\{k', k''\}$ and $k_2 = \min\{k', k''\}$, define

$$t_k(P^\star_j) = \begin{cases} t_{k+2}(P_j) & \text{if } m-2 \geq k \geq k_1 - 1, \\ \\ t_{k+1}(P_j) & \text{if } \bar{k} - 1 > k \geq k_2. \end{cases}$$

Next, we present two claims.

Claim 1: $f(P_i, P^\star_{-i}) = f(P)$. Let $P^\star = (P_i, P^\star_{-i})$. Since $f(P)P_ic$ and by definition of $P^\star_{-i}$,

$$mp(P^\star, f(P)) > mp(P^\star, c) \tag{18}$$

Then, $f(P^\star) \neq c$. As $f(P'_i, P_{-i})P_if(P) = t_{\bar{k}}(P_i)$,

$$mp(P^\star, f(P)) = \bar{k}. \tag{19}$$

Now, let $b \in X \setminus \{f(P), c\}$. By definition of $P^\star$ and the fact that f is a N-maxmin rule,

$$mp(P^\star, b) \leq mp(P, b) \leq mp(P, f(P)) \leq \bar{k}. \tag{20}$$

Therefore, $f(P) \in \mathcal{M}(P^\star)$ and $mp(P^\star, f(P)) = \bar{k}$. On the one hand, if $\bar{j} \neq i$, then $t(P^\star_{\bar{j}}) = f(P)$ and, by definition of f, $f(P) = f(P^\star)$. On the other hand, if $\bar{j} = i$ and there is $b \in \mathcal{M}(P^\star) \setminus \{f(P)\}$, then by (40) and (20), $mp(P, b) = mp(P, f(P)) = \bar{k}$. Thus, by (17) and the fact that $\bar{j} = i$, $f(P)P_ib$. Therefore, as $P^\star_i = P_i$, $f(P) = f(P^\star)$. This proves the Claim.

Claim 2: $f(P_i, P^\star_{-i})P_if(P'_i, P^\star_{-i})$. If $f(P'_i, P^\star_{-i}) = c$, then by Claim 1 and (14) the proof is trivial. Now assume $f(P'_i, P^\star_{-i}) \neq c$. First, we will prove that $f(P'_i, P^\star_{-i}) \neq f(P)$. As $f(P) = t_{\widehat{k}}(P'_i)$,

$$\widehat{k} = mp((P'_i, P^\star_{-i}), f(P)).$$

Furthermore, as $cR'_it_{\bar{k}}(P'_i)$ and $\bar{k} \leq m-1$, by definition of $P^\star_{-i}$,

$$mp((P'_i, P^\star_{-i}), c) \geq \bar{k}. \tag{21}$$

Then, by (13),

$$mp((P'_i, P^\star_{-i}), c) > \widehat{k} = mp((P'_i, P^\star_{-i}), f(P)),$$

implying that $f(P'_i, P^\star_{-i}) \neq f(P)$.

Now, let $b \in X \setminus \{f(P), c\}$ be such that $bP_if(P_i, P^\star_{-i})$. Since $f(P_i, P^\star_{-i}) = f(P) = t_{\bar{k}}(P_i)$, by definition of f there exists $j \in N \setminus \{i\}$ such that $t_{\bar{k}}(P_j)R_jb$. By definition of $P^\star_{-i}$, $t_{\bar{k}}(P^\star_j)R^\star_jb$. Therefore,

$$mp((P'_i, P^\star_{-i}), b) \leq \bar{k}. \tag{22}$$

On the one hand, if $\bar{j} \neq i$, since $t_{m-1}(P^\star_{\bar{j}}) = c$ the definition of f, (21), and (22) imply that $f(P'_i, P^\star_{-i}) \neq b$. On the other hand, if $\bar{j} = i$, since $bP_i f(P_i, P^\star_{-i})$ the definition of f implies $mp((P_i, P^\star_{-i}), b) < mp((P_i, P^\star_{-i}), f((P_i, P^\star_{-i})))$. Then,

$$mp((P_i, P^\star_{-i}), b) < \bar{k}.$$

Therefore, as $bP_i f(P_i, P^\star_{-i}) = t_{\bar{k}}(P_i)$,

$$mp((P'_i, P^\star_{-i}), b) < \bar{k}$$

Then, by the definition of f and (21), $f(P'_i, P^\star_{-i}) \neq b$ in this case as well. Therefore, we conclude that

$$f(P_i, P^\star_{-i}) P_i f(P'_i, P^\star_{-i}),$$

proving the Claim.

By Claims 1 and 2 we conclude that f is regret-free. □

A.2 Proof of Theorem 2

We first show the equivalence in part (i). Let $f : \mathcal{P}^n \to X$ be an A-scoring rule with $k^\star = 1$.

($\Longrightarrow$) Suppose that $n < m - 1$ (this implies $m > 3$). Assume that a, b are the first two alternatives in the tie-breaking with $a \succ b$ and let z the last alternative in the tie-breaking. Let $P \in \mathcal{P}^n$ be such that $t_3(P_i) = b$, $t_2(P_i) = a$, $t_1(P_i) = z$, and $t_m(P_j) = b$, $t_{m-1}(P_j) = a$, and $t_{m-2}(P_j) = z$ for each $j \in N \setminus \{i\}$. Then, $f(P) = a$. Now, let $P'_i \in \mathcal{P}$ be such that $t_1(P'_i) = a$. Then, $f(P'_i, P_{-i}) = b$ and, therefore,

$$f(P'_i, P_{-i}) P_i f(P). \tag{23}$$

Now, let $P^\star_{-i} \in \mathcal{P}^{n-1}$ be such that $f(P) = f(P_i, P^\star_{-i})$. As $n + 1 < m$, $\left|\mathcal{S}(P'_i, P^\star_{-i})\right| \geq 2$. Therefore, as z is the last alternative in the order $\succ$, $f(P'_i, P^\star_{-i}) \neq z$ and

$$f(P'_i, P^\star_{-i}) R_i f(P_i, P^\star_{-i}). \tag{24}$$

Hence, by (23) and (24), f is not regret-free.

($\Longleftarrow$) Assume $n \geq m - 1$ and there exist $i \in N$, $P \in \mathcal{P}^n$ and $P'_i \in \mathcal{P}$ such that

$$f(P'_i, P_{-i}) P_i f(P). \tag{25}$$

Next, we show there is $P^\star_{-i} \in \mathcal{P}^{n-1}$ such that $f(P) = f(P_i, P^\star_{-i})$ and $f(P_i, P^\star_{-i}) P_i f(P'_i, P^\star_{-i})$. Let $\widehat{P} = (P'_i, P_{-i})$. If $t_1(P_i) = t_1(P'_i)$, then $\mathcal{S}(P) = \mathcal{S}(\widehat{P})$, contradicting the definition of f and the fact that $f(\widehat{P}) \neq f(P_i, P_{-i})$. Therefore,

$$t_1(P_i) \neq t_1(P'_i). \tag{26}$$

If $t_1(P_i) = f(P)$, then as $t_1(P_i) \neq t_1(P'_i)$, $s(\widehat{P}, f(P)) > s(P, f(P))$. By definition of f, $s(P, f(P)) \geq s(P, x)$ for each $x \in X$. Then, $s(\widehat{P}, f(P)) > s(P, x)$ for each $x \in X \setminus \{f(P)\}$. Now, as $\widehat{P} = (P'_i, P_{-i})$, $s(\widehat{P}, f(P)) > s(\widehat{P}, x)$ for each $x \in X \setminus \{t_1(P_i)\}$. Therefore, as $t_1(P_i) = f(P)$, $f(\widehat{P}) = f(P)$ which contradicts (25). Thus,

$$t_1(P_i) \neq f(P). \tag{27}$$

Furthermore,

$$\begin{aligned} s(\widehat{P}, x) &= s(P, x) \text{ for each } x \notin \{t_1(P_i), t_1(P'_i)\}, \\ s(\widehat{P}, t_1(P_i)) &= s(P, t_1(P_i)) + 1, \text{and} \\ s(\widehat{P}, t_1(P'_i)) &= s(P, t_1(P'_i)) - 1. \end{aligned}$$

Then, as $s(P, x) \leq s(P, f(P))$ for each $x \in X$,

$$\mathcal{S}(\widehat{P}) = \{t_1(P_i)\} \text{ or } \mathcal{S}(P) \setminus \{t_1(P'_i)\} \subset \mathcal{S}(\widehat{P}) \subset \mathcal{S}(P) \cup \{t_1(P_i)\}.$$

Thus, by (25),

$$\mathcal{S}(P) \setminus \{t_1(P'_i)\} \subset \mathcal{S}(\widehat{P}) \subset \mathcal{S}(P) \cup \{t_1(P_i)\}. \tag{28}$$

Next, we claim that

$$t_1(P'_i) = f(P) \tag{29}$$

holds. Assume otherwise that $t_1(P'_i) \neq f(P)$. Then, by (28) and the definition of f, $f(\widehat{P}) = f(P)$ or $f(\widehat{P}) = t_1(P_i)$, which contradicts that $f(\widehat{P}) P_i f(P)$. Then, (29) holds.

Now, let $P^\star_{-i}$ be such that $\{t_1(P^\star_j) : j \neq i\} = X \setminus \{f(P), t_1(P_i)\}$ ($P^\star_{-i}$ exists because $n \geq m - 1$). As $t_1(P_i) \neq f(P)$, $\mathcal{S}(P_i, P^\star_{-i}) = \{f(P)\}$ and, therefore,

$$f(P) = f(P_i, P^\star_{-i})$$

By (29), $t_1(P'_i) = f(P)$. Then, $\mathcal{S}(P'_i, P^\star_{-i}) = \{t_1(P_i)\}$ implying $f(P'_i, P^\star_{-i}) = t_1(P_i)$ and, therefore,

$$f(P_i, P^\star_{-i}) P_i f(P'_i, P^\star_{-i}). \tag{30}$$

By (25) and (30), f is regret-free.

In order to see (ii), Assume that f is a N-scoring rule with $k^\star = 1$. Then, there exists $\bar{j}$ such that

$$f(\widetilde{P}) = \max_{\widetilde{P}_{\bar{j}}} \mathcal{S}(\widetilde{P}) \text{ for each } \widetilde{P} \in \mathcal{P}^n. \tag{31}$$

Let $P, P'_i, \widehat{P}$, be as in ($\Longleftarrow$) of part (i). By definition, (25) also holds here.

If $t_1(P_i) = t_1(P_i')$, then $\mathcal{S}(P) = \mathcal{S}(\widehat{P})$. As $f(P_i', P_{-i})P_i f(P_i, P_{-i})$, we have $\bar{j} \in N \setminus \{i\}$. Now, $\mathcal{S}(P) = \mathcal{S}(\widehat{P})$ contradicts $f(P) \neq f(\widehat{P})$ since $P_{\bar{j}} = \widehat{P}_{\bar{j}}$. Therefore, (26) holds here and it follows that both (27) and (28) hold as well. If $\bar{j} = i$, we get a contradiction with $f(\widehat{P})P_i f(P)$ and $\mathcal{S}(\widehat{P}) \subset \mathcal{S}(P) \cup \{t_1(P_i)\}$, so $\bar{j} \neq i$.

Now, let $P^\star_{-i} \in \mathcal{P}^{n-1}$ be such that $t(P^\star_{\bar{j}}) = t_1(P_i)$, $t_{m-1}(P^\star_{\bar{j}}) = f(P)$ and, for each $j \in N \setminus \{i, \bar{j}\}$, $t(P^\star_j) = f(P)$ and $t_2(P^\star_j) = t_1(P_i)$. Therefore, by (27), $f(P) \in \mathcal{S}(P_i, P^\star_{-i})$ and $t_1(P_i) \notin \mathcal{S}(P_i, P^\star_{-i})$. By definition of f and $P^\star_{\bar{j}}$ it follows that

$$f(P) = f(P_i, P^\star_{-i}).$$

Then, by (26), $t_1(P_i) \in \mathcal{S}(P_i', P^\star_{-i})$. By definition of f and $P^\star_{\bar{j}}$ we have

$$f(P_i', P^\star_{-i}) = t_1(P_i)$$

Therefore,

$$f(P_i, P^\star_{-i})P_i f(P_i', P^\star_{-i}). \tag{32}$$

By (25) and (32), f is regret-free. □

A.3 Proof of Theorem 3

Let $f : \mathcal{P}^n \to X$ be a scoring rule with $k^\star = m - 1$ (this implies that $s_{m-1} < s_m$). Let $a, b, c \in X$ and assume w.l.o.g. that if f is an A-scoring then the tie-breaking is given by order $\succ$ with $a \succ b \succ c \succ \ldots$, whereas if f is a N-scoring rule agent 1 break ties. There are two cases to consider:

1. **$n = 2t$ with $t \geq 2$.** Let $P \in \mathcal{P}^n$ be given by the following table:

P_1	P_2	P_3	P_4	$\cdots$	P_{t+1}	P_{t+2}	$\cdots$	P_{2t}
a	c	c	a	$\cdots$	a	b	$\cdots$	b
b	b	a	b	$\cdots$	b	a	$\cdots$	a
c	a	b	c	$\cdots$	c	c	$\cdots$	c
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
			$t-2$ agents			$t-1$ agents		

As $k^\star = m - 1$, $f(P) \in \{a, b, c\}$. Furthermore, as $s(P, a) = s(P, b)$, $a \succ b$ and aP_1b, $f(P) \in \{a, c\}$. There are two subcases to consider:

1.1. $\boldsymbol{f(P) = a}$. Then, $s(P,a) = s(P,b) \geq s(P,c)$. Let $P_2' \in \mathcal{P}$ be such that $P_2' : b, c, a, \ldots$, and let $\widehat{P} = (P_2', P_{-2})$. This implies that

$$s(\widehat{P}, b) > s(P,b) = s(P,a) = s(\widehat{P}, a)$$

and

$$s(\widehat{P}, b) > s(P,b) \geq s(P,c) > s(\widehat{P}, c).$$

Therefore,

$$f(\widehat{P}) = bP_2 a = f(P). \tag{33}$$

Next, consider $P_{-2}^{\star} \in \mathcal{P}^{n-1}$ such that $f(P_2, P_{-2}^{\star}) = a$. Then, $f(P_2', P_{-2}^{\star}) \in \{a, b\}$ because $s((P_2, P_{-2}^{\star}), b) < s((P_2', P_{-2}^{\star}), b)$, $s((P_2, P_{-2}^{\star}), c) > s((P_2', P_{-2}^{\star}), c)$, and $s((P_2, P_{-2}^{\star}), x) = s((P_2', P_{-2}^{\star}), x)$ for each $x \in X \setminus \{b, c\}$. Therefore,

$$f(P_2', P_{-2}^{\star}) R_2 f(P_2, P_{-2}^{\star}) \tag{34}$$

By (33) and (34), f is not regret-free.

1.2. $\boldsymbol{f(P) = c}$. Then, $s(P,c) \geq s(P,a) = s(P,b)$. Consider agent j such that $t + 2 \leq j \leq 2t$ (i.e., $P_j : b, a, c, \ldots$) and let $P_j' \in \mathcal{P}$ be such that $P_j' : a, b, c, \ldots$ and $\widehat{P} = (P_j', P_{-j})$. As $s(\widehat{P}, a) \geq s(\widehat{P}, c)$, $s(\widehat{P}, a) > s(\widehat{P}, b)$, $a \succ c$ and $aP_1 c$,

$$f(\widehat{P}) = aP_j c = f(P). \tag{35}$$

Next, consider $P_{-j}^{\star} \in \mathcal{P}^{n-1}$ such that $f(P_j, P_{-j}^{\star}) = c$. Then, $f(P_j', P_{-j}^{\star}) \in \{c, a\}$, because $s((P_j, P_{-j}^{\star}), a) < s((P_j', P_{-j}^{\star}), a)$, $(s(P_j, P_{-j}^{\star}), b) > s((P_j', P_{-j}^{\star}), b)$, and $s((P_j, P_{-j}^{\star}), x) = s((P_j', P_{-j}^{\star}), x)$ for each $x \in X \setminus \{b, a\}$. Therefore,

$$f(P_j', P_{-j}^{\star}) R_j f(P_j, P_{-j}^{\star}) \tag{36}$$

By (35) and (36), f is not regret-free.

2. $\boldsymbol{n = 2t + 3}$ **with** $\boldsymbol{t \geq 0}$. Let $P \in \mathcal{P}^n$ be given by the following table:

P_1	P_2	P_3	P_4	$\cdots$	P_{t+3}	P_{t+4}	$\cdots$	P_{2t+3}
a	c	b	a	$\cdots$	a	b	$\cdots$	b
c	b	a	b	$\cdots$	b	a	$\cdots$	a
b	a	c	c	$\cdots$	c	c	$\cdots$	c
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
			t agents			t agents		

As $s(P,a) = s(P,b) \geq s(P,c)$, $a \succ b \succ c$ and $aP_1 bP_1 c$, it follows that $f(P) = a$. Now, the proof proceeds similarly to Case 1.1.

□

A.4 Proof of Theorem 4

Assume $n > 2$ and let $f : \mathcal{P}^n \to X$ be a scoring rule such that $k^\star n < m$. We first show the equivalence in part (i). Let further assume that f is an A-scoring rule.

($\Longrightarrow$) Assume that $k^\star n < m-1$, we will prove that f is not regret-free. Assume that a and b are the first two alternatives in the tie-breaking $\succ$ with $a \succ b$ and let z the last alternative in the tie-breaking. Let $P \in \mathcal{P}^n$ be such that $t_{k^\star+2}(P_i) = b$, $t_{k^\star+1}(P_i) = a$, $t_{k^\star}(P_i) = z$ and, for each $j \in N \setminus \{i\}$, $P_j : b, a, z, \ldots$. Then, $f(P) = a$. Now, let $P_i' \in \mathcal{P}$ be such that $t_{k^\star}(P_i') = a$, $t_{k^\star+1}(P_i') = z$ and $t_k(P_i') = t_k(P_i)$ for each $k \neq k^\star, k^\star+1$. Therefore, $f(P_i', P_{-i}) = b$ and

$$f(P_i', P_{-i}) P_i f(P). \tag{37}$$

Now, let $P^\star_{-i} \in \mathcal{P}^{n-1}$ be such that $f(P_i, P^\star_{-i}) = f(P)$. As $k^\star n < m-1$, $\left|\mathcal{S}(P_i', P^\star_{-i})\right| \geq 2$. Therefore, as z is the last alternative in order $\succ$, $f(P_i', P^\star_{-i}) \neq z$ and

$$f(P_i', P^\star_{-i}) R_i f(P_i, P^\star_{-i}). \tag{38}$$

By (37) and (38), f is not regret-free.

($\Longleftarrow$) Assume that $k^\star n = m-1$. Let $P \in \mathcal{P}^n$. As $m > k^\star n$, there exists $x \in X$ such that $s(P, x) = n \cdot s_m$. Therefore, $f(P) = z$ where $s(P, z) = n \cdot s_m$ and $z P_j t_{k^\star}(P_j)$ for each $j \in N$.

Let $P_i' \in \mathcal{P}$ be such that

$$f(P_i', P_{-i}) P_i f(P), \tag{39}$$

and let $y = f(P_i', P_{-i})$. Then, $y P_i z P_i t_{k^\star}(P_i)$. By definition of $k^\star$,

$$s(P, y) \geq s((P_i', P_{-i}), y) \tag{40}$$

Also,

$$t_{k^\star}(P_i) R_i' z. \tag{41}$$

Otherwise, $z P_i' t_{k^\star}(P_i)$ implies $s(P, z) = s((P_i', P_{-i}), z)$. By (40), $s(P, y) = s(P, z)$ and $s((P_i', P_{-i}), y) = s((P_i', P_{-i}), z)$, contradicting the definition of f since $y \neq z$. So (41) holds.

Therefore, there exists $w \in X$ such that $t_{k^\star}(P_i) R_i w$ and $w P_i' t_{k^\star}(P_i)$. As $k^\star n = m-1$ is equivalent to $(n-1)k^\star = m - k^\star - 1$, we can consider $P^\star_{-i} \in \mathcal{P}^{n-1}$ such that the two following requirements hold: (i) $P^\star_j : w, z, \ldots$ for each $j \in N \setminus \{i\}$, and (ii) for each $x \in X \setminus \{z\}$ such that $x P_i t_{k^\star}(P_i)$ there exist $j \in N \setminus \{i\}$ such that $t_{k^\star}(P^\star_j) P^\star_j x$. Therefore, $\mathcal{S}(P_i, P^\star_{-i}) = \{z\}$ and

$$f(P_i, P^\star_{-i}) = z.$$

As $s((P_i', P_{-i}^{\star}), w) = n \cdot s_m$ and, by (41) and the definition of $P_{-i}^{\star}$, $s((P_i', P_{-i}^{\star}), r) < n \cdot s_m$ for each r such that $rP_i t_{k^{\star}}(P_i)$, we have

$$f(P_i, P_{-i}^{\star}) P_i f(P_i', P_{-i}^{\star}). \tag{42}$$

By (39) and (42), f is regret-free.

To see part (ii), let $f : \mathcal{P}^n \to X$ be a N-scoring rule. Then, there exists $\bar{j} \in N$ such that

$$f(\widetilde{P}) = \max_{\widetilde{P}_{\bar{j}}} \mathcal{S}(\widetilde{P}) \text{ for each } \widetilde{P} \in \mathcal{P}^n. \tag{43}$$

Let $P \in \mathcal{P}^n$. As $m > k^{\star}n$, there exists $x \in X$ such that $s(P, x) = n \cdot s_m$. Therefore, $f(P) = z$ where $s(P, z) = n \cdot s_m$and $zP_j t_{k^{\star}}(P_j)$ for each $j \in N$.

Let $P_i' \in \mathcal{P}$ be such that

$$f(P_i', P_{-i}) P_i f(P). \tag{44}$$

Then $i \in N \setminus \{\bar{j}\}$. Let $y = f(P_i', P_{-i})$. Therefore, $yP_i z P_i t_{k^{\star}}(P_i)$. By definition of $k^{\star}$,

$$s(P, y) \geq s((P_i', P_{-i}), y).$$

Notice that $t_{k^{\star}}(P_i) R_i' z$. Otherwise, $zP_i' t_{k^{\star}}(P_i)$ implies $s(P, z) = s((P_i', P_{-i}), z)$, contradicting $y = f(P_i', P_{-i})$. Therefore, $t_{k^{\star}}(P_i) P_i' z$ and there exists $w \in X$ such that $t_{k^{\star}}(P_i) P_i w$ and $wP_i' t_{k^{\star}}(P_i)$.

Now, let $P_{-i}^{\star} \in \mathcal{P}^{n-1}$ be such that $P_j^{\star} : w, z, \ldots$ for each $j \in N \setminus \{i\}$. Then, $s((P_i, P_{-i}^{\star}), z) = n \cdot s_m$ and $f(P_i, P_{-i}^{\star}) = z$, $s((P_i', P_{-i}^{\star}), w) = n \cdot s_m$ and $f(P_i', P_{-i}^{\star}) = w$. As $zP_i w$,

$$f(P_i, P_{-i}^{\star}) P_i f(P_i', P_{-i}^{\star}). \tag{45}$$

By (44) and (45), f is regret-free. □

A.5 Proof of Theorem 5

Assume $n > 2$ and let $f : \mathcal{P}^n \to X$ be a scoring rule such that $k^{\star}n \geq m$ and $s_{k^{\star}-1} = s_{k^{\star}}$ (this implies $k^{\star} > 1$). If $k^{\star} = m - 1$, the result follows from Theorem 3, so assume that $k^{\star} < m - 1$. If f is and A-scoring rule, assume that a and b are the first two alternatives in the tie-breaking $\succ$ with $a \succ b$ and let z the last alternative in the tie-breaking, whereas if f is a N-scoring rule, let agent 1 be the one who break ties. By the definition of $k^{\star}$, $s_{k^{\star}-1} = s_{k^{\star}} < s_{k^{\star}+1} = s_{m-1} = s_m$. Let $a, b \in X$. As $k^{\star}n \geq m$ and $k^{\star} > 1$, $k^{\star}(n-1) \geq m - k^{\star}$. Then, there exists $P \in \mathcal{P}^n$ such that:

(i) $a = t_{k^{\star}-1}(P_2)$ and $b = t_{k^{\star}}(P_2)$,

(ii) for each $j \in N \setminus \{2\}$, $t(P_j) = a$ and $t_{m-1}(P_j) = b$,

(iii) for each $x \in X$ such that xP_2b, there exist $j \in N \setminus \{2\}$ such that $t_{k^\star}(P_j)R_jx$.

Since $s(P,a) \geq s(P,x)$ for each $x \in X$ such that xP_2b, $a \succ x$ and aP_1b, it follows that $bP_2f(P)$. Let $P_2' \in \mathcal{P}$ be such that $t_{k^\star+1}(P_2') = b = t_{k^\star}(P_2)$, $t_{k^\star}(P_2') = t_{k^\star+1}(P_2)$, and $t_k(P_2') = t_k(P_2)$ for each $k \neq k^\star, k^\star+1$. Let $\widehat{P} = (P_2', P_{-2})$. Then, by the definition of $k^\star$,

$$s(\widehat{P}, b) > s(P,b) = s(P,a) = s(\widehat{P}, a)$$

and $s(\widehat{P}, b) \geq s(\widehat{P}, x)$ if bP_2x. Therefore,

$$f(P_2', P_{-2})P_2f(P). \tag{46}$$

Let $P_{-2}^\star \in \mathcal{P}^{n-1}$ be such that $f(P_2, P_{-2}^\star) = f(P)$. Since $s((P_2, P_{-2}^\star), f(P)) = s((P_2', P_{-2}^\star), f(P))$ and $s((P_2, P_{-2}^\star), x) \geq s((P_2', P_{-2}^\star), x)$ for each $x \in X \setminus \{b\}$, it follows that $f(P_2', P_{-2}^\star) \in \{f(P), b\}$. Therefore,

$$f(P_2', P_{-2}^\star)R_2f(P_2, P_{-2}^\star). \tag{47}$$

By (46) and (47), f is not regret-free. □

A.6 Proof of Theorem 6

Let $f : \mathcal{P}^n \to X$ be a Condorcet consistent and monotone rule. There are two cases to consider:

1. $n \neq 2, 4$. Then, there are $t \geq 1$ and $s \geq 0$ such that $n = 3t + 2s$. Let $P \in \mathcal{P}^n$ be given by the following table:

P_1	$\cdots$	P_t	P_{t+1}	$\cdots$	P_{2t+s}	P_{2t+s+1}	$\cdots$	P_{3t+2s}
a	$\cdots$	a	b	$\cdots$	b	c	$\cdots$	c
b	$\cdots$	b	c	$\cdots$	c	a	$\cdots$	a
c	$\cdots$	c	a	$\cdots$	a	b	$\cdots$	b
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
t agents			$t+s$ agents			$t+s$ agents		

Since $C_P(a,c) = t < \frac{3t+2s}{2}$, $C_P(c,b) = t+s < \frac{3t+2s}{2}$, and $C_P(b,a) = t+s < \frac{3t+2s}{2}$, it follows that there is no Condorcet winner according to P.

Let $x = f(P)$. Then, there exists $i^\star \in N$ such that $x = t_k(P_{i^\star})$ with $k \leq m-2$. Assume first that $i^\star$ is such that $t+1 \leq i^\star \leq 2t+s$. Let $N' = \{j \in N : t+1 \leq$

$j \leq 2t+s\}$ and consider the subprofile $P'_{N'} \in \mathcal{P}^{t+s}$ where, for each $j \in N'$, $P'_j \in \mathcal{P}$ is such that $t(P'_j) = c$, $t_{m-1}(P'_j) = b$, $t_{m-2}(P'_j) = a$, and $t_k(P'_j) = t_k(P_j)$ for each $k \leq m-3$. Then, c is the Condorcet winner in $(P'_{N'}, P_{-N'})$. As $i^\star \in N'$, $x \neq c$. This implies the existence of $S \subset N'$ and $j^\star \in N' \setminus S$ such that

$$f(P'_S, P_{-S}) = x \tag{48}$$

and

$$f(P'_{S\cup\{j^\star\}}, P_{-S\cup\{j^\star\}}) \neq x. \tag{49}$$

Now, by monotonicity and (49), $f(P'_{S\cup\{j^\star\}}, P_{-S\cup\{j^\star\}}) P_{j^\star} x$, implying

$$f(P'_{S\cup\{j^\star\}}, P_{-S\cup\{j^\star\}}) P_{j^\star} f(P'_S, P_{-S}). \tag{50}$$

Now let, $P^\star_{-j^\star} \in \mathcal{P}^{n-1}$ be such that $f(P_{j^\star}, P^\star_{-j^\star}) = f(P'_S, P_{-S})$. By (48), $f(P_{j^\star}, P^\star_{-j^\star}) = x$. Then, by monotonicity, $f(P'_{j^\star}, P^\star_{-j^\star}) R_{j^\star} x$. Hence

$$f(P'_{j^\star}, P^\star_{-j^\star}) R_{j^\star} f(P_{j^\star}, P^\star_{-j^\star}). \tag{51}$$

By (50) and (51), f is not regret-free. The cases where $i^\star$ is such that $1 \leq i^\star \leq t$ or $2t+s+1 \leq i^\star \leq 3t+2s$ are similar and therefore we omit them.

2. **$n = 4$ and $m > 3$**. Let $P \in \mathcal{P}^n$ be given by the following table:

P_1	P_2	P_3	P_4
a	b	c	d
b	c	d	a
c	d	a	b
d	a	b	c
$\vdots$	$\vdots$	$\vdots$	$\vdots$

As $C_P(a,c) = C_P(a,b) = C_P(b,d) = 2$, there is no Condorcet winner according to P. Let $f(P) = x$. Assume that $f(P) \notin \{b,c,d\}$ (the other 3 cases in which $f(P) \notin \{w,u,h\}$ with $\{w,u,h\} \subset \{a,b,c,d\}$ follow a similar argument). Next, let $P'_2 \in \mathcal{P}$ be such that $t(P'_2) = d$, $t_{m-1}(P'_2) = b$, $t_{m-2}(P'_2) = c$, $t_{m-3}(P'_2) = a$, and $t_k(P'_2) = t_k(P_2)$ for each $k \leq m-4$. Similarly, let $P'_3 \in \mathcal{P}$ be such that $t(P'_3) = d$, $t_{m-1}(P'_3) = c$, $t_{m-2}(P'_3) = a$, $t_{m-3}(P'_3) = b$, and $t_k(P'_3) = t_k(P_3)$ for each $k \leq m-4$. Then, d is the Condorcet winner according to $(P'_{\{2,3\}}, P_{-\{2,3\}})$. There are two cases to consider:

2.1. $f(P'_2, P_{-2}) \neq x$. Then, by monotonicity, $f(P'_2, P_{-2})P_2x$. Hence,

$$f(P'_2, P_{-2})P_2f(P). \tag{52}$$

Now, let $P^{\star}_{-2} \in \mathcal{P}^{n-1}$ be such that $f(P_2, P^{\star}_{-2}) = f(P)$. Then, by monotonicity,

$$f(P'_2, P^{\star}_{-2})R_2f(P_2, P^{\star}_{-2}). \tag{53}$$

By (52) and (53), f is not regret-free.

2.2. $f(P'_2, P_{-2}) = x$. Then, $f(P'_{\{2,3\}}, P_{-\{2,3\}}) = dP_3x = f(P'_2, P_{-2})$ and an analogous reasoning to the one presented in Case 2.1 for agent 2, now performed with agent 3, shows that f is not regret-free.

□

A.7 Proof of Corollary 5

We first show that each of the rules is monotone.

Lemma 1 *Simpson, Copeland, Young, Dodgson, Fishburn and Black rules (both anonymous and neutral) satisfy monotonicity.*

Proof. Let $x \in X$, $P \in \mathcal{P}^n$ and $P'_i \in \mathcal{P}$ be such that P'_i is a monotonic transformation of P_i with respect to x. Let $z \in X$ be such that xP_iz and let $y \in \{x, z\}$. Then, $C_P(y, a) = C_{(P'_i, P_{-i})}(y, a)$ for each $a \in X$. Therefore, (both anonymous and neutral) Simpson, Copeland and Fishburn rules are monotonic. To see that Young and Dodgson rules are monotonic, simply note that yP_ia if and only if yP'_ia for each $a \in X \setminus \{y\}$. Finally, to see that Black rule is monotonic, note that (i) y is a Cordorcet winner in P if and only if y is a Condorcet winner in (P'_i, P_{-i}), and (ii) the Borda score for y is the same in profiles P and (P'_i, P_{-i}). □

Proof of Corollary 5. Assume first that $N = \{1, 2, 3, 4\}$ and $X = \{a, b, c\}$. In all of the cases that we consider in what follows, w.l.o.g., we assume that the tie-breaking is given by $a \succ b \succ c$ in the anonymous case, or by agent 1 in the neutral case.

Let $f : \mathcal{P}^4 \to \{a, b, c\}$ be a Simpson (Young, Dodgson, Fishburn) rule. Le $P \in \mathcal{P}^4$ be given by the following table:

P_1	P_2	P_3	P_4
b	c	c	a
a	b	b	c
c	a	a	b

Then, c is the only Simpson (Young, Dodgson, Fishburn) winner at P and $f(P) = c$. Now, consider $P_1' \in \mathcal{P}$ such that $P_1' : a, b, c$. Then, a is a Simpson (Young, Dodgson, Fishburn) winner at (P_1', P_{-1}). Therefore, $f(P_1', P_{-1}) = aP_1c = f(P)$. Let $P_{-1}^{\star} \in \mathcal{P}^{n-1}$ be such that $f(P_1, P_{-1}^{\star}) = f(P)$. Since $f(P) = c = t_1(P_1)$, $f(P_1', P_{-1}^{\star}) R_1 f(P_1, P_{-1}^{\star})$. Hence, f is not regret-free.

Next, let $f : \mathcal{P}^4 \to \{a, b, c\}$ be a Copeland (Black) rule. Let $P \in \mathcal{P}^4$ be given by

P_1	P_2	P_3	P_4
b	c	c	a
a	a	b	c
c	b	a	b

Then, c is the only Copeland (Black) winner at P and $f(P) = c$. Now, consider $P_1' \in \mathcal{P}$ such that $P_1' : a, b, c$. Then, a is a Copeland (Black) winner at (P_1', P_{-1}) and a similar reasoning to the one presented for Simpson' rule shows that f is not regret-free.

Finally, assume $n \neq 4, 2$ or $n = 4$ and $m > 3$. By Lemma 1, Simpson, Copeland, Young, Dodgson, Fishburn and Black rules are monotonic. Since all of them are also Condorcet consistent, the result follows from Theorem 6. □

A.8 Proof of Theorem 7

Let $f : \mathcal{P}^n \to X$ be a succesive elimination rule with associated order $a \succ b \succ c \succ \ldots$ and let $t \geq 1$ and $1 \geq s \geq 0$ be such that $n = 2t + s$. Next, let $P \in \mathcal{P}^n$ be given by the following table:[16]

P_1	P_2	P_3	$\cdots$	P_{t+2}	P_{t+3}	$\cdots$	P_{2t+s}
a	c	b	$\cdots$	b	a	$\cdots$	a
b	a	c	$\cdots$	c	b	$\cdots$	b
$\vdots$	b	a	$\cdots$	a	c	$\cdots$	c
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
c	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
		t agents			$t+s-2$ agents		

Since $C_P(a, b) = t + s \geq t = C_P(b, a)$, $C_P(a, c) = t + s - 1 < t + 1 = C_P(c, a)$, and $C_P(c, x) = n - 1 > 1 = C_P(x, c)$ for each $x \in X \setminus \{a, b\}$, it follows that $f(P) = c$. Let $P_1' \in \mathcal{P}$ be such that $t(P_1') = b$, $t_{m-1}(P_1') = a$, and $t_1(P_1') = c$, and let $\widehat{P} = (P_1', P_{-1})$.

[16]Notice that, as $n \geq 3$, $t + s - 2 \geq 0$.

Since $C_{\widehat{P}}(a,b) = t+s-1 < t+1 = C_{\widehat{P}}(b,a)$, $C_{\widehat{P}}(b,c) = n-1 > 1 = C_{\widehat{P}}(c,b)$, and $C_{\widehat{P}}(b,x) > C_{\widehat{P}}(x,b)$ for each $x \in X \setminus \{a,c\}$, it follows that $f(P_1', P_{-1}) = b$. Therefore,

$$f(P_1', P_{-1}) P_1 f(P). \tag{54}$$

Furthermore, as $f(P) = t_1(P_1)$,

$$f(P_1', P_{-1}^{\star}) R_1 f(P_1, P_{-1}^{\star}) \tag{55}$$

for each $P_{-1}^{\star} \in \mathcal{P}^{n-1}$ such that $f(P_1, P_{-1}^{\star}) = f(P)$. By (54) and (55), f is not regret-free. □

A.9 Proof of Theorem 8

($\Longrightarrow$) Let $f : \mathcal{P}^2 \to \{a,b,c\}$ be a regret-free and neutral voting rule.
Claim: f is efficient. Assume f is not efficient. W.l.o.g., there are two cases to consider:

1. **$P \in \mathcal{P}^2$ is such that $f(P) = c$, $P_i : a,b,c$, and P_j is such that $x = t(P_j) \neq c$.** By regret-freeness, $f(P_i', P_j) = c$ for each $P_i' \in \mathcal{P}$. Let π be the permutation of X such that $\pi(c) = x$. By neutrality, $f(\pi P) = x$. Then, by regret-freenes, $f(P_i', \pi P_j) = x$ for each $P_i' \in \mathcal{P}$. This implies that, as $f(\pi P) = x P_j c = f(\pi P_i, P_j)$ and $f(P_i', \pi P_j) = x$ for each $P_i' \in \mathcal{P}$, agent j manipulates f and does not regret it.

2. **$P \in \mathcal{P}^2$ is such that $f(P) = b$ and $P_i = P_j : a,b,c$.** Let π be the permutation of X such that $\pi(a) = b$. By neutrality, $f(\pi P) = a$. By the previous case, $f(\pi P_i, P_j) \neq c$. We claim that $f(\pi P_i, P_j) = b$. Assume $f(\pi P_i, P_j) = a$. Let $P_j^{\star}$ be such that $f(P_i, P_j^{\star}) = b$. If $f(\pi P_i, P_j^{\star}) = c$, then agent i manipulates f at $(\pi P_i, P_j^{\star})$ via P_i and does not regret it. Therefore, $f(\pi P_i, P_j^{\star}) \neq c$. This implies that agent i manipulates f at P via πP_i and does not regret it. This proves the claim that $f(\pi P_i, P_j) = b$. By a similar reasoning to the one presented for agent i, we can see that agent j manipulates f at $(\pi P_i, P_j)$ via πP_j and does not regret it.

Since in both cases we reach a contradiction, f is efficient. This proves the claim.

Next, assume that f is not a dictatorship. We will prove that f is a N-maxmin rule. Let $\overline{P} \in \mathcal{P}^2$ be such that $\overline{P}_1 : a,b,c$ and $\overline{P}_2 : b,a,c$. By efficiency, $f(\overline{P}) \in \{a,b\}$. Assume, w.l.o.g., that $f(\overline{P}) = a$. We will prove that

$$f(P) = \max_{P_1} \ \mathcal{M}(P) \ \text{ for each } P \in \mathcal{P}^2.$$

Let $P \in \mathcal{P}^2$. There are three cases to consider:

1. $\boldsymbol{t(P_1) = t(P_2)}$. By efficiency, $f(P) = t(P_1) = \max_{P_1} \mathcal{M}(P)$.

2. $\boldsymbol{t(P_1) \neq t(P_2)}$ **and** $\boldsymbol{t_1(P_1) = t_1(P_2)}$**.** As $f(\overline{P}) = a$, by neutrality, $f(P) = t(P_1) = \max_{P_1} \mathcal{M}(P)$.

3. $\boldsymbol{t(P_1) \neq t(P_2)}$ **and** $\boldsymbol{t_1(P_1) \neq t_1(P_2)}$**.** Then,

$$\mathcal{M}(P) = X \setminus \{t_1(P_1), t_1(P_2)\}. \tag{56}$$

If $f(P) = t_1(P_i) = x$ for some $i \in \{1,2\}$, then $f(P) = t(P_j) = x$ with $j \neq i$ (because of efficiency). Then, by regret-freeness,

$$f(P_j, P_i') = x \text{ for all } P_i'.$$

Then, again by regret-freeness,

$$f(P_j', P_i') = x \text{ for all } P_i' \text{ and all } P_j' \text{ such that } t(P_j') = x.$$

Then, j is a dictator when he has top in x. Therefore, by neutrality, j is a dictator which is a contradiction. Thus,

$$f(P) \neq t_1(P_i) \text{ for all } i \in \{1,2\}. \tag{57}$$

Therefore, by (56) and (57), $\mathcal{M}(P) = \{f(P)\}$ and $f(P) = \max_{P_1} \mathcal{M}(P)$.

($\Longleftarrow$) Let f be a N-maxmin rule. It is clear that f is neutral and, furthermore, by Theorem 1 (ii), f is regret-free. If f is a dictatorship, it is trivial that it is neutral and regret-free. □

A.10 Proof of Theorem 9

($\Longleftarrow$) Let $f : \mathcal{P}^2 \to \{a,b,c\}$ be a successive elimination rule or an A-maxmin* rule. It is clear that f satisfies efficiency and anonymity. We will prove that f is regret-free. Assume there are $(P_1, P_2) \in \mathcal{P}^2$ and $P_1' \in \mathcal{P}$ such that

$$f(P_1', P_2) P_1 f(P_1, P_2). \tag{58}$$

We will prove that there exists $P_2^\star \in \mathcal{P}$ such that $f(P_1, P_2^\star) = f(P)$ and

$$f(P_1, P_2^\star) P_1 f(P_1', P_2^\star). \tag{59}$$

There are two cases to consider:

1. **f is a successive elimination rule with associated order $a \succ b \succ c$.** It is clear that
$$f(\widetilde{P})\widetilde{R}_i a \text{ for each } \widetilde{P} \in \mathcal{P}^2 \text{ and each } i \in \{1,2\}. \tag{60}$$
If $f(P_1,P_2)=a$, by (58) and efficiency, $aP_2 f(P_1',P_2)$, contradicting (60). Therefore, $f(P_1,P_2)P_1 a$ and, by (58), $t_1(P_1)=a$. There are two cases to consider:

   1.1. $\boldsymbol{bP_1cP_1a}$**.** Then, by (58), $f(P_1,P_2)=c$ and, by definition of f, cP_2aP_2b. Therefore, there is no $P_1' \in \mathcal{P}$ such that $f(P_1',P_2)=b$, contradicting (58).

   1.2. $\boldsymbol{cP_1bP_1a}$**.** Then, by (58), $f(P_1,P_2)=b$ and, by definition of f, $t(P_2)=b$. It follows from (58) that $f(P_1',P_2)=c$, implying that cP_2a and $cP_1'aP_1'b$. Now, let $P_2^\star \in \mathcal{P}$ be such that $bP_2^\star aP_2^\star c$. Then, $f(P_1,P_2^\star)=b$ and $f(P_1',P_2^\star)=a$. Since bP_1a, (59) holds and f is regret-free.

2. **f is a A-maxmin* rule with associated binary relation $\succ^\star$.** By definition of f, it is clear that
$$f(\widetilde{P}) \neq t_1(\widetilde{P}_i) \text{ for each } \widetilde{P} \in \mathcal{P}^2 \text{ and each } i \in \{1,2\}. \tag{61}$$
W.l.o.g, let $P_1 : a,b,c$. By (58), $t(P_2) \neq a$ and $f(P) \neq a$. Then, by (61), $f(P) = b \in \{t(P_2), t_2(P_2)\}$. By (58) and (61), $f(P_1',P_2)=a \in \{t(P_2),t_2(P_2)\}$. Therefore, as $t(P_2)\neq a$, $P_2 : b,a,c$. Then, by definition of f, $b \succ^\star a$. Therefore, as $f(P_1',P_2)=a$, $t_1(P_1')=b$. Now let $P_2^\star : b,c,a$. Then, by (61), $f(P_1,P_2)=b=f(P_1,P_2^\star)$ and $f(P_1',P_2^\star)=c$. Since bP_1c, (59) holds and f is regret-free.

($\Longrightarrow$) Assume that f is regret-free, efficient, and anonymous. We will prove that f is a successive elimination rule or an A-maxmin* rule. There are two cases to consider:

1. **there exist $a \in X$ and $\overline{P} \in \mathcal{P}^2$ such that $f(\overline{P}) = t_1(\overline{P}_i) = a$ for some $i \in \{1,2\}$.** By efficiency, $f(\overline{P}) = t(\overline{P}_j) = a$ for $j = N \setminus \{i\}$. It follows, by regret-freeness, that
$$f(P_i, \overline{P}_j) = a \text{ for each } P_i \in \mathcal{P}.$$
Then, again by regret-freeness,
$$f(P) = a \text{ for each } P \in \mathcal{P}^2 \text{ such that } t(P_j) = a.$$
Therefore, by anonymity,
$$f(P) = a \text{ for all } P \text{ such that } a \in \{t(P_1), t(P_2)\}.$$

This implies, by regret-freeness, that

$$f(P) R_i a \text{ for each } P \in \mathcal{P}^2 \text{ and each } i \in \{1,2\}. \tag{62}$$

Let $\widehat{P} \in \mathcal{P}^2$ be such that $\widehat{P}_1 : b, c, a$ and $\widehat{P}_2 : c, b, a$. By efficiency, $f(\widehat{P}) \in \{c, b\}$. W.l.o.g., assume that

$$f(\widehat{P}) = b. \tag{63}$$

Let $f^{\succ}$ be the successive elimination rule with associated order $a \succ b \succ c$ and let $P \in \mathcal{P}^2$. We will prove that $f = f^{\succ}$. There are two cases to consider:

1.1. **there exists $i \in \{1,2\}$ such that aP_ib or aP_ic.** Therefore, by (62), efficiency, and the definition of $f^{\succ}$, $f(P) = f^{\succ}(P)$.

1.2. **bP_ia and cP_ia for each $i \in \{1,2\}$.** If $t(P_1) = t(P_2)$, then by efficiency $f(P) = t(P_1) = f^{\succ}(P)$. Assume now that $t(P_1) \neq t(P_2)$. Then, by anonymity and (63), $f(P) = f(\widehat{P}) = b = f^{\succ}(P)$.

2. **$f(P) \neq t_1(P_i)$ for each $P \in \mathcal{P}^2$ and each $i \in \{1,2\}$.** First, let $\widehat{P} \in \mathcal{P}^2$ be such that $\widehat{P}_1 : b, c, a$ and $\widehat{P}_2 : c, b, a$. By efficiency, $f(\widehat{P}) \in \{b, c\}$. Assume, w.l.o.g., that $f(\widehat{P}) = b$. Second, let $\overline{P} \in \mathcal{P}^2$ be such that $\overline{P}_1 : b, a, c$ and $\overline{P}_2 : a, b, c$. By efficiency, $f(\overline{P}) \in \{b, a\}$. Assume, w.l.o.g., that $f(\widehat{P}) = a$. Third, let $\widetilde{P} \in \mathcal{P}^2$ be such that $\widetilde{P}_1 : c, a, b$ and $\widetilde{P}_2 : a, c, b$. By efficiency, $f(\widetilde{P}) \in \{c, a\}$. Assume, w.l.o.g., that $f(\widetilde{P}) = c$. We will prove that f is a A-maxmin rule* with associated binary relation $\succ^\star$ where $b \succ^\star c$, $a \succ^\star b$, and $c \succ^\star a$. This is, we need to show that

$$f(P) = \max_{\succ^\star} \mathcal{M}(P) \tag{64}$$

for each $P \in \mathcal{P}^2$. To do so, let $P \in \mathcal{P}^2$. If $P \in \{\widehat{P}, \overline{P}, \widetilde{P}\}$, it is clear that (64) holds. Assume $P \in \mathcal{P} \setminus \{\widehat{P}, \overline{P}, \widetilde{P}\}$. There are three cases to consider:

2.1. **$t(P_1) = t(P_2)$.** By efficiency, $f(P) = t(P_1) = \max_{\succ^\star} \mathcal{M}(P)$, so (64) holds.

2.2. **$t_1(P_1) \neq t_1(P_2)$.** As $|X| = 3$, there is $x \in X$ such that $\{x\} = X \setminus \{t_3(P_1), t_3(P_2)\}$. Therefore, as $f(P) \neq t_1(P_i)$ for each $i \in \{1,2\}$ (see hypothesis of Case 2), $f(P) = x$. Furthermore, as $t_3(P_1) \neq t_3(P_2)$, $\mathcal{M}(P) = \{x\}$ and then, (64) holds.

2.3. **$t(P_1) \neq t(P_2)$ and $t_1(P_1) = t_1(P_2)$.** Then, $(P_1, P_2) = (P'_1, P'_2)$ with $P' \in \{\widehat{P}, \overline{P}, \widetilde{P}\}$. By anonymity and the fact that (64) holds for P',

$$f(P) = f(P') = \max_{\succ^\star} \mathcal{M}(P') = \max_{\succ^\star} \mathcal{M}(P).$$

Therefore, f is a successive elimination rule or an A-maxmin* rule, as stated. □