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# Obvious Manipulations of tops-only Voting Rules

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# Obvious manipulations of tops-only voting rules*

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**Abstract**

In a voting problem with a finite set of alternatives to choose from, we study the manipulation of tops-only rules. Since all non-dictatorial (onto) voting rules are manipulable when there are more than two alternatives and all preferences are allowed, we look for rules in which manipulations are not obvious. First, we show that a rule does not have obvious manipulations if and only if when an agent vetoes an alternative it can do so with any preference that does not have such alternative in the top. Second, we focus on two classes of tops-only rules: (i) (generalized) median voter schemes, and (ii) voting by committees. For each class, we identify which rules do not have obvious manipulations on the universal domain of preferences.

*JEL classification:* D71, D72.

*Keywords:* obvious manipulations, tops-onlyness, (generalized) median voting schemes, voting by committees, voting by quota.

## 1 Introduction

Voting rules are systematic procedures that allow a group of agents to select an alternative, among many, according to their preferences. Within desirable properties a voting rule

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may satisfy, the concept of strategy-proofness has played a central role for studying the strategic behavior of the agents. A voting rule is strategy-proof if it is always in the best interest of the agents to reveal their true preferences. Unfortunately, Gibbard-Satterthwaite's celebrated theorem states that, outside of a dictatorship, there is no strategy-proof (onto) voting rule when more than two alternatives and all possible preferences over alternatives are considered (Gibbard, 1973; Satterthwaite, 1975). To circumvent this impossibility result, two main approaches have been taken. The first approach restricts the domain of preferences that agents can have over alternatives (see Barberà, 2011, and references therein). The second approach considers weakenings of strategy-proofness, and it has been an active field of research in recent years. The idea underlying this approach is that even though manipulations are pervasive, agents may not realize they can manipulate a rule because they lack information about others' behavior or they are cognitively limited.

Troyan and Morrill (2020) introduce the concept of obvious manipulation in the context of market design. They assume that an agent knows the possible outcomes of the mechanism conditional on his own declaration of preferences, and define a deviation from the truth to be an obvious manipulation if the best possible outcome under the deviation is strictly better than the best possible outcome under truth-telling, or the worst possible outcome under the deviation is strictly better than the worst possible outcome under truth-telling. A mechanism that does not allow any obvious manipulation is called not obviously manipulable.

In this paper we study (not) obvious manipulation of voting rules when a finite set of alternatives is involved. We focus on tops-only rules: rules that only consider agents' top alternatives in order to select a social choice. Obvious manipulations of non-tops-only rules have been thoroughly studied elsewhere (see Aziz and Lam, 2021). The importance of studying obvious manipulations of tops-only rules is three-fold. Firstly, because of their simplicity, tops-only voting rules are important rules on their own right and are both useful in practice and extensively studied in the literature. Secondly, assume we have a class of strategy-proof rules defined on a restricted domain of preferences and we want to study whether those rules are obviously manipulable in the universal domain of preferences. Then, it is natural to require tops-onlyness since, under mild assumptions, strategy-proofness implies tops-onlyness (see, for example, Barberà et al., 1991; Chatterji and Sen, 2011; Weymark, 2008). Thirdly, obvious manipulations allow us to discriminate among different tops-only rules, while other recently studied weakenings of strategy-proofness are incapable to do this because, under tops-onlyness, those concepts become equivalent to strategy-proofness (see, for example, Arribillaga et al., 2022).

Our main result gives a characterization of not obviously manipulable rules in terms of veto power of the agents. An agent vetoes an alternative if there is a preference report of the agent that forces the rule to never select such alternative. The veto is strong if the report of *any* preference with top different from the alternative forces the rule to never select it. Theorem 1 states that, within tops-only rules, not obvious manipulation is equivalent to each veto being a strong veto.

Next, we apply our main result to study well-known classes of tops-only voting rules when they are considered on the universal domain of preferences: (generalized) median voter schemes and voting by committees.

First, consider a problem where the set of alternatives $X$ is an ordered set. Without loss of generality, assume $X = \{a, a+1, a+2, \ldots, b\} \subseteq \mathbb{N}$ and $b = a + (m-1)$. For this problem, Moulin (1980) characterizes all strategy-proof and tops-only (onto) rules on the domain of single-peaked preferences[1] as the class of all generalized median voter schemes. Moulin (1980) also characterizes the subclass of median voter schemes as the set of all strategy-proof, tops-only, and anonymous (onto) rules on the domain of single-peaked preferences. A median voter scheme can be identified with a vector $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ of $n-1$ numbers in $X$, where $n$ is the cardinality of the set of agents $N$ and $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{n-1}$. Then, for each preference profile, the median voter scheme identified with $\alpha$ selects the alternative that is the median among the $n$ top alternatives of the agents and the $n-1$ fixed numbers $\alpha_1, \ldots, \alpha_{n-1}$. Since $2n-1$ is an odd number, this median always exists and belongs to $X$. Generalized median voter schemes constitute non-anonymous extensions of median voter schemes.

When the designer cannot guarantee that the domain restriction (single-peakedness) is met and the full domain of preferences has to be considered, then strategy-proofness no longer holds. For this reason, it is important to identify which rules within these families obey the less demanding property of non-obvious manipulability. In Theorem 2, we show that a median voter scheme is not obviously manipulable if and only if $\alpha_1 \in \{a, a+1\}$ and $\alpha_{n-1} \in \{b-1, b\}$. A similar condition applied to the extremal fixed ballots (for each

[1] An agent's preference is single-peaked if there is a top alternative that is strictly preferred to all other alternatives and at each of the two sides of the top alternative the preference is monotonic, increasing in the left, and decreasing in the right.

agent) in the monotonic family of fixed ballots associated with generalized median voter schemes characterizes those that are not obviously manipulable (Theorem 3).

Now, consider a problem in which agents have to choose a *subset of objects* from a set $K$ (with $|K| \geq 2$). Then, in this case, $X = 2^K$ and elements of $X$ are subsets of $K$. A generic element of $K$ is denoted by $k$. As an example, think of objects as candidates to become new members of a society that have to be elected by the current members of the society. Barberà et al. (1991) characterize, on the restricted domain of separable preferences,[2] the family of all strategy-proof (onto) rules as the class of voting by committees. Following Barberà et al. (1991), a voting by committees is defined by specifying for each object $k \in K$ a monotonic family of winning coalitions $\mathcal{W}_k$ (a committee). Then, the choice of the subset of objects made by a voting by committees at a preference profile is done object-by-object as follows. Fix a voting by committees $\mathcal{W} = \{\mathcal{W}_k\}_{k \in K}$ and a preference profile, and consider object $k$. Then, $k$ belongs to the chosen set (the one selected by $\mathcal{W}$ at the preference profile) if and only if the set of agents whose best subset of objects contains $k$ belongs to the committee $\mathcal{W}_k$.[3] Observe that voting by committees are in fact tops-only, so tops-onlyness is implied by strategy-proofness on the domain of separable preferences.

Again, if the domain restriction (separability, in this case) is not guaranteed to be met, identifying which rules are not obviously manipulable can be helpful. In Theorem 4, we show that a non-dictatorial voting by committees is not obviously manipulable if and only if no agent is a vetoer. In terms of the committees defining the rule, this is equivalent to say that, for each object: (i) no agent belongs to all the coalitions in the committee, and (ii) no singleton coalition belongs to the committee. When anonymity is added to the picture, voting by committees simplifies to voting by quota. In this case, for each $k \in K$, there is a number $q_k$ such that $\mathcal{W}_k$ is the set of all coalitions with cardinality at least $q_k$. We prove that non-obvious manipulability is equivalent to each committee having quota between 2 and $n-1$ (Corollary 3).

The paper of Aziz and Lam (2021) is the closest to ours and, to the best of our knowledge, is the first one that applies Troyan and Morrill (2020) notion of obvious manipulation to the context of voting. Aziz and Lam (2021) present a general sufficient condition for a

[2]An agent's preference is separable on the family of all subsets of objects if the division between good objects and bad objects guides the ordering of all subsets in the sense that adding a good object to any set leads to a better set, while adding a bad object leads to a worse set.

[3]Voting by committees can be seen as a family of extended majority voting (one for each object $k$), where the two alternatives at stake are whether or not $k$ belongs to the collectively chosen subset of objects.

voting rule to be not obviously manipulable. However, they focus on non-tops-only rules. They show that Condorcet consistent as well as some other strict scoring rules are not obviously manipulable. Furthermore, for the class of $k$-approval voting rules, they give necessary and sufficient conditions for obvious manipulability. Other recent papers that study obvious manipulations, in contexts other than voting, are Ortega and Segal-Halevi (2022) and Psomas and Verma (2022).

The rest of the paper is organized as follows. The model and the concept of obvious manipulations are introduced in Section 2. In Section 3, we present the main result of our paper that, under tops-onlyness, characterizes non-obvious manipulable voting rules. Section 4 deals with applications: in Subsection 4.1 we study (generalized) median voter schemes, and in Subsection 4.2 we study voting by committees. To conclude, some final remarks are gathered in Section 5.

# 2 Preliminaries

## 2.1 Model

A set of *agents* $N = \{1, \ldots, n\}$, with $n \geq 2$, has to choose an alternative from a finite and given set $X$ (with cardinality $|X| = m \geq 2$ ). Each agent $i \in N$ has a strict *preference* $P_i$ over $X$. Denote by $t(P_i)$ to the best alternative according to $P_i$, called the *top* of $P_i$. We denote by $R_i$ the weak preference over $X$ associated to $P_i$; *i.e.,* for all $x, y \in X$, $xR_iy$ if and only if either $x = y$ or $xP_iy$. Let $\mathcal{P}$ be the set of all strict preferences over $X$. A (preference) *profile* is a $n$-tuple $P = (P_1, \ldots, P_n) \in \mathcal{P}^n$, an ordered list of $n$ preferences, one for each agent. Given a profile $P$ and an agent $i$, $P_{-i}$ denotes the subprofile in $\mathcal{P}^{n-1}$ obtained by deleting $P_i$ from $P$.

A *(voting) rule* is a function $f : \mathcal{P}^n \longrightarrow X$ selecting an alternative for each preference profile in $\mathcal{P}^n$. We assume throughout that a voting rule is an *onto* function, i.e., for each $x \in X$ there is $P \in \mathcal{P}^n$ such that $f(P) = x$. A rule $f : \mathcal{P}^n \longrightarrow X$ is *tops-only* if for all $P, P' \in \mathcal{P}^n$ such that $t(P_i) = t(P'_i)$ for all $i \in N$, $f(P) = f(P')$. In this paper, we will focus on tops-only rules.

Given a rule $f : \mathcal{P}^n \longrightarrow X$ and $P_i \in \mathcal{P}$, an alternative report $P'_i \in \mathcal{P}$ is a *(profitable) manipulation of rule $f$ at $P_i$* if there is a preference sub-profile $P_{-i} \in \mathcal{P}^{n-1}$ such that

$$f(P'_i, P_{-i}) P_i f(P_i, P_{-i}).$$

A rule $f : \mathcal{P}^n \longrightarrow X$ is *strategy-proof* on $\mathcal{P}^n$ if no agent has a manipulation.

Other desirable properties we look at are the following. A rule $f : \mathcal{P}^n \longrightarrow X$ is *efficient* if for each $P \in \mathcal{P}^n$, there is no $x \in X$ such that $xP_i f(P)$ for each $i \in N$. A rule $f : \mathcal{P}^n \longrightarrow X$ is *anonymous* if it is invariant with respect to the agents' names; namely, for each one-to-one mapping $\sigma : N \longrightarrow N$ and each $P \in \mathcal{P}^n$, $f(P_1, \ldots, P_n) = f(P_{\sigma(1)}, \ldots, P_{\sigma(n)})$. A rule $f : \mathcal{P}^n \longrightarrow X$ is *dictatorial* if there exists $i \in N$ such that for each $P \in \mathcal{P}^n$, $f(P) = t(P_i)$.

The Gibbard-Satterthwaite Theorem states that a (onto) rule $f : \mathcal{P}^n \longrightarrow X$, with $m \geq 3$, is strategy-proof if and only if it is dictatorial (Gibbard, 1973; Satterthwaite, 1975). This negative result justifies the study of less demanding criteria of (lack of) manipulation when rules defined on the universal domain of preferences are considered. One such weakening of strategy-proofness is presented next.

### 2.2 Obvious manipulations

The notion of obvious manipulations has been introduced by Troyan and Morrill (2020) when applied to school choice models and later it has been studied by Aziz and Lam (2021) in the context of voting. They try to describe those manipulations that are easily identifiable by the agents. To do this, it is important to specify how much information each agent has about other agents' preferences. Troyan and Morrill (2020) assume that each agent has complete ignorance in this respect and, therefore, each agent focuses on the set of outcomes that can be chosen by the rule given its own report. Now, a manipulation is obvious if the best possible outcome under the manipulation is strictly better than the best possible outcome under truth-telling or the worst possible outcome under the manipulation is strictly better than the worst possible outcome under truth-telling.

Before we present the formal definition, we present some notation. Given a preference $P_i \in \mathcal{P}$, the *option set* left open by $P_i$ at $f$ is

$$O^f(P_i) = \{f(P_i, P_{-i}) \in X : P_{-i} \in \mathcal{P}^{n-1}\}.$$ [4]

Given $Y \subseteq X$, denote by $B(P_i, Y)$ to the best alternative in $Y$ according to preference $P_i$, and by $W(P_i, Y)$ to the worst alternative in $Y$ according to preference $P_i$.

[4] Barbera and Peleg (1990) were the first to use option sets in the context of preference aggregation.

**Definition 1** *(Troyan and Morrill, 2020) Let $f : \mathcal{P}^n \longrightarrow X$ be a rule, let $P_i \in \mathcal{P}$, and let $P'_i \in \mathcal{P}$ be a profitable manipulation of $f$ at $P_i$. A manipulation $P'_i$ is* ***obvious*** *if*

$$W(P_i, O^f(P'_i))\ P_i\ W(P_i, O^f(P_i)) \tag{1}$$

*or*

$$B(P_i, O^f(P'_i))\ P_i\ B(P_i, O^f(P_i)). \tag{2}$$

*The rule $f$ is* ***not obviously manipulable (NOM)*** *if it does not admit any obvious manipulation.*

## 3 Main theorem

As previously mentioned, in this paper we focus in obvious manipulations of tops-only rules. Because of their simplicity, tops-only voting rules are both useful in practice and extensively studied in the literature. Now, assume we have a class of strategy-proof rules defined on a restricted domain of preferences and we want to study whether those rules are obviously manipulable in the universal domain of preferences. Then, it is natural to require tops-onlyness since, under mild assumptions, strategy-proofness implies tops-onlyness (see, for example, Chatterji and Sen, 2011; Weymark, 2008). Furthermore, obvious manipulations allow us to discriminate among different tops-only rules, while other recently studied weakenings of strategy-proofness are incapable to do this because, under tops-onlyness, those concepts become equivalent to strategy-proofness (see, for example, Arribillaga et al., 2022).[5] Obvious manipulations of non-tops-only rules have been thoroughly studied elsewhere (see Aziz and Lam, 2021). In this section we provide a necessary and sufficient condition for a tops-only rule to be NOM.

In order to obtain our main result, we first need to define when an agent has veto power. An agent vetoes an alternative if there is a preference report of the agent that forces the rule to never select such alternative. Formally,

**Definition 2** *Let $f : \mathcal{P}^n \longrightarrow X$ be a rule and let $i \in N$, $x \in X$, and $P_i \in \mathcal{P}$. Agent* ***$i$ vetoes $x$ via $P_i$*** *if $x \notin O^f(P_i)$.*

[5] Arribillaga et al. (2022) assume that an agent knows a specific type of information about the preferences of the other individuals conditional on his own declaration of preferences, and focus on manipulations for which the worst possible outcome under the deviation consistent with the agent's information is strictly better than the outcome under truth-telling. It can be seen that, under tops-onlyness, the existence of a manipulation of the aforementioned type is equivalent to the existence of a classical manipulation.

Denote by $V_i$ the set of all alternatives that agent $i$ vetoes via some preference. Given $x \in V_i$, let $\mathcal{V}_i^x = \{P_i \in \mathcal{P} : i \text{ vetoes } x \text{ via } P_i\}$ be the set of all preferences by which $x$ is vetoed by agent $i$. As it is observed by Aziz and Lam (2021), if a rule $f$ has no vetoers then it is NOM, because $O^f(P_i) = X$ for each $i \in N$ and each $P_i \in \mathcal{P}$. However, as the next example shows, there are rules that satisfy NOM and admit (many) vetoers.[6]

**Example 1** *Let $a \in X$ and consider the* status quo rule at $a$, $f^a : \mathcal{P}^n \longrightarrow X$, *defined as*

$$f^a(P) = \begin{cases} x & \text{if } t(P_i) = x \text{ for each } i \in N \\ a & \text{otherwise.} \end{cases}$$

*Observe that, if $P_i \in \mathcal{P}$ is such that $t(P_i) = a$, then $O^{f^a}(P_i) = \{a\}$ and thus $V_i = X \setminus \{a\}$. Furthermore, for any $P_i \in \mathcal{P}$, $O^{f^a}(P_i) = \{a, t(P_i)\}$. Therefore, $W(P_i, O^{f^a}(P_i)) = aR_iW(P_i, O^{f^a}(P_i'))$ and $B(P_i, O^{f^a}(P_i)) = t(P_i)R_iB(P_i, O^{f^a}(P_i'))$ for each $P_i' \in \mathcal{P}$. Hence, $f^a$ is NOM.*

The previous example shows that the condition of a rule having no vetoers is far from being necessary for the rule to be NOM. What turns out to be important is what kind of vetoes are admissible by the rule. Next, we introduce a particular type of veto power that allows us to state a necessary and sufficient condition for NOM in tops-only rules. We say that the veto of an alternative by an agent is strong if the report of *any* preference with top different from the alternative forces the rule to never select it. Formally,

**Definition 3** *Let $i \in N$ and $x \in X$. Agent **i strongly vetoes x** if $\mathcal{V}_i^x = \{P_i \in \mathcal{P} : t(P_i) \neq x\}$.*

Denote by $SV_i$ the set of all alternatives strongly vetoed by agent $i$. Note that $SV_i \subseteq V_i$ for each $i \in N$. Clearly, the sets $V_i, SV_i$ and $\mathcal{V}_i^x$ depend on $f$ but we omit this reference to ease notation.

**Theorem 1** *A tops-only rule is NOM if and only if every veto is a strong veto, i.e., $SV_i = V_i$ for each agent $i \in N$.*[7]

[6] For an example of a non-tops-only rule that satisfies NOM and admits vetoers, see Lemma 5 in Aziz and Lam (2021).

[7] When $m = 2$ (two alternatives), the condition $SV_i = V_i$ is equivalent to the simple condition that agent $i$ does not veto $t(P_i)$ with $P_i$.

*Proof.* Let $f : \mathcal{P}^n \longrightarrow X$ be a tops-only rule.
($\Longrightarrow$) Assume there is $i \in N$ such that $V_i \neq SV_i$. Since $SV_i \subseteq V_i$, $V_i \neq \emptyset$ and there is $x \in X$ such that $x \in V_i \setminus SV_i$. Then,

$$\mathcal{V}_i^x \neq \{P_i \in \mathcal{P} : t(P_i) \neq x\}. \tag{3}$$

By (3), there are two cases to consider:

1. **There is $P_i' \in \mathcal{V}_i^x$ is such that $t(P_i') = x$.** By ontoness, there is $(P_i, P_{-i}) \in \mathcal{P}^n$ such that $f(P_i, P_{-i}) = x$. Furthermore, $t(P_i) \neq x$, since otherwise $t(P_i) = x$ and tops-onlyness would imply $P_i' \notin \mathcal{V}_i^x$. Let $\overline{P}_i \in \mathcal{P}$ be such that $t(\overline{P}_i) = t(P_i)$ and $b(\overline{P}_i) = x$. By tops-onlyness, $f(\overline{P}_i, P_{-i}) = x$. Since $P_i' \in \mathcal{V}_i^x$, $x \notin O^f(P_i')$. Then, $f(P_i', P_{-i}) \neq x$ and therefore $f(P_i', P_{-i})\overline{P}_i x = f(\overline{P}_i, P_{-i})$, implying that $P_i'$ is a profitable manipulation of $f$ at $P_i$. Furthermore, as $x \notin O^f(P_i')$,

   $$W(\overline{P}_i, O^f(P_i'))\ \overline{P}_i\ x = W(\overline{P}_i, O^f(\overline{P}_i)).$$

   Thus, $P_i'$ is an obvious manipulation of $f$.

2. **There is $P_i \in \mathcal{P}$ such that $t(P_i) \neq x$ and $P_i \notin \mathcal{V}_i^x$.** As $P_i \notin \mathcal{V}_i^x$, there is $P_{-i} \in \mathcal{P}^{n-1}$ such that $f(P_i, P_{-i}) = x$. Let $\overline{P}_i \in \mathcal{P}$ be such that $t(\overline{P}_i) = t(P_i)$ and $b(\overline{P}_i) = x$. By tops-onlyness, $f(\overline{P}_i, P_{-i}) = x$. Since $x \in V_i$, there is $P_i' \in \mathcal{P}$ such that $P_i' \in \mathcal{V}_i^x$, and the proof follows as in the previous case.

($\Longleftarrow$) Let $i \in N$ be such that $V_i = SV_i$. We will prove that agent $i$ has no obvious manipulations. If $V_i = \emptyset$, the proof is trivial. Assume that $V_i \neq \emptyset$. For each $P_i \in \mathcal{P}$, as $V_i = SV_i$,

$$O^f(P_i) = (X \setminus V_i) \cup \{t(P_i)\}.$$

Then, for each $P_i' \in \mathcal{P}$,

$$B(P_i, O^f(P_i)) = t(P_i)\ R_i\ B(P_i, O^f(P_i'))$$

and

$$W(P_i, O^f(P_i)) = W(P_i, (X \setminus V_i) \cup \{t(P_i)\})\ R_i\ W(P_i, (X \setminus V_i) \cup \{t(P_i')\}) = W(P_i, O^f(P_i')).$$

Hence, $i$ does not have an obvious manipulation. □

Corollary 1 shows that under efficiency and tops-onlyness NOM implies a very limited veto power: at most one agent can veto some alternatives or only one alternative can be vetoed by some agents.

**Corollary 1** *An efficient and tops-only rule is NOM if and only if some of the following statements hold:*

*(i) There is at most one $i \in N$ such that $V_i \neq \emptyset$ and, moreover, $SV_i = V_i$.*

*(ii) There is $y \in X$ such that $SV_i = V_i \subseteq \{y\}$, for each $i \in N$.*

*Proof.* Let $f : \mathcal{P}^n \longrightarrow X$ be an efficient and tops-only rule.
($\Longrightarrow$) Assume both conditions (i) and (ii) do not hold. Then, there are distinct $i, j \in N$ and distinct $x, y \in X$ such that $x \in V_i$ and $y \in V_j$. Now let $P \in \mathcal{P}^n$ be such that $P_i : y, x, \ldots$ and $P_k : x, y, \ldots$ for each $k \in N \setminus \{i\}$. By efficiency, $f(P) \in \{x, y\}$. Therefore, $P_i \notin \mathcal{V}_i^x$ or $P_j \notin \mathcal{V}_i^y$. So, by Theorem 1, $f$ is not NOM.
($\Longleftarrow$) By Theorem 1 it is clear that either condition is sufficient for $f$ to be NOM. □

Corollary 2 states that, under efficiency and anonymity, non-obvious manipulability is equivalent to having at most one alternative vetoed and that, if there is one such alternative, the veto is unanimous.

**Corollary 2** *An efficient, anonymous and tops-only rule is NOM if and only if either $V_i = \emptyset$ for each $i \in N$ or there is $y \in X$ such that $SV_i = V_i = \{y\}$ for each $i \in N$.*

*Proof.* It follows from Corollary 1 and anonymity. □

# 4 Applications

In this section, we apply Theorem 1 to study two classes of tops-only voting rules in two separate (but related) voting problems. Our results allow us to discriminate those rules in each class that are non-obviously manipulable in the universal domain of preferences.

In the first problem, presented in subsection 4.1, alternatives are endowed with a linear order structure. When preferences are single peaked over that order, the family of (generalized) median voting schemes encompass all tops-only and strategy-proof (onto) rules. In the second problem, presented in subsection 4.2, alternatives consist of *subsets* of objects chosen from a fixed finite set. When preferences are separable, the class of voting by committees encompass all strategy-proof (onto) rules.

## 4.1 Median Voter Schemes

In this subsection assume that $X$ is an ordered set. Without loss of generality, assume $X = \{a, a+1, a+2, \ldots, b\} \subseteq \mathbb{N}$ and $b = a + (m-1)$. A preference $P_i \in \mathcal{P}$ is *single-peaked* on $X$ if for all $x, y \in X$ such that $x < y < t(P_i)$ or $t(P_i) < y < x$, we have $t(P_i) P_i y P_i x$. We denote the domain of all single-peaked preferences on $X$ by $\mathcal{SP}$. Note that $\mathcal{SP} \subsetneq \mathcal{P}$.

Moulin (1980) characterizes the family of strategy-proof and tops-only (onto) rules on the domain of single-peaked preferences. This family contains many non-dictatorial rules. For example, when $n$ is odd, consider the rule $f : \mathcal{P}^n \longrightarrow X$ that selects, for each preference profile $P = (P_1, \ldots, P_n) \in \mathcal{P}^n$, the median among the top alternatives of the $n$ agents; namely, $f(P) = med\{t(P_1), \ldots, t(P_n)\}$.[8] This rule is anonymous, efficient, tops-only, and strategy-proof on $\mathcal{SP}$. All other rules in the family given by Moulin (1980) are extensions of $f$. Following Moulin (1980), and before presenting the general result, we first introduce the anonymous subclass and characterize those rules which are NOM. After that, we present the general class of all strategy-proof and tops-only rules on $\mathcal{SP}^n$ and characterize those that are NOM when operating on domain $\mathcal{P}^n$.

### 4.1.1 Anonymity

A rule $f : \mathcal{P}^n \longrightarrow X$ is a *median voter scheme* if there is a vector $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in X^{n-1}$ of $n-1$ fixed ballots such that, for each $P \in \mathcal{P}^n$,

$$f(P) = med\{t(P_1), \ldots, t(P_n), \alpha_1, \ldots, \alpha_{n-1}\}.$$

Without loss of generality, throughout the paper we assume that $\alpha_1 \leq \ldots \leq \alpha_{n-1}$. When we want to stress the dependence of the scheme on the vector $\alpha$ of fixed ballots, we write $f^\alpha$. Also, to simplify notation, we use $O^\alpha(P_i)$ instead of $O^{f^\alpha}(P_i)$. The characterization of median voter schemes is as follows:

**Fact 1** *(Moulin, 1980) A (onto) rule $f : \mathcal{SP}^n \longrightarrow X$ is strategy-proof, tops-only, and anonymous if and only if it is a median voter scheme.*[9]

[8] Given a set of real numbers $\{x_1, \ldots, x_K\}$, where $K$ is odd, define its *median* as $med\{x_1, \ldots, x_K\} = y$, where $y$ is such that $|\{1 \leq k \leq K : x_k \leq y\}| \geq \frac{K}{2}$ and $|\{1 \leq k \leq K : x_k \geq y\}| \geq \frac{K}{2}$. Since $K$ is odd the median is unique and belongs to the set $\{x_1, \ldots, x_K\}$.

[9] The definitions of strategy-proofness, tops-onlyness and anonymity on an arbitrary subdomain $\mathcal{U}^n \subsetneq \mathcal{P}^n$ are straightforward adaptations of the definitions given for the universal domain.

Median voter schemes are strategy-proof on the domain $\mathcal{SP}^n$ of single-peaked preferences. However, when they operate on the larger domain $\mathcal{P}^n$ they may become manipulable. Then, all median voter schemes are equivalent from the classical manipulability point of view. Next, we give a simple test to identify which median voter schemes are NOM.

**Theorem 2** *A median voter scheme $f^\alpha : \mathcal{P}^n \longrightarrow X$ is NOM if and only if*

$$\alpha_1 \in \{a, a+1\} \text{ and } \alpha_{n-1} \in \{b-1, b\}.$$

In order to prove Theorem 2, the following remark and lemma are useful.

**Remark 1** *By definition of option set and $f^\alpha$, for each $P_i \in \mathcal{P}$,*

$$O^\alpha(P_i) = \begin{cases} \{t(P_i), t(P_i)+1, \ldots, \alpha_{n-1}\} & \text{if } t(P_i) < \alpha_1 \\ \{\alpha_1, \alpha_1+1, \ldots, \alpha_{n-1}\} & \text{if } \alpha_1 \leq t(P_i) \leq \alpha_{n-1} \\ \{\alpha_1, \alpha_1+1, \ldots, t(P_i)\} & \text{if } \alpha_{n-1} < t(P_i) \end{cases}$$

*To see this when $t(P_i) < \alpha_1$, take $x \in \{t(P_i), t(P_i)+1, \ldots, \alpha_{n-1}\}$ and let $P_{-i} \in \mathcal{P}^{n-1}$ be such that $t(P_j) = x$ for each $j \in N \setminus \{i\}$. Then, $f^\alpha(P_i, P_{-i}) = x$ an so $x \in O^\alpha(P_i)$. Furthermore, if $x < t(P_i)$, let $P_{-i} \in \mathcal{P}^{n-1}$ be such that $t(P_j) = x$ for each $j \in N \setminus \{i\}$. Then, $f^\alpha(P_i, P_{-i}) = t(P_i)$ and, therefore, $f^\alpha(P_i, P'_{-i}) \neq x$ for each $P'_{-i} \in \mathcal{P}^{n-1}$. Thus, $x \notin O^\alpha(P_i)$. Symmetrically, it can be proven that $x \notin O^\alpha(P_i)$ when $x > \alpha_{n-1}$. The cases $t(P_i) \in \{\alpha_1, \ldots, \alpha_{n-1}\}$ and $t(P_i) > \alpha_{n-1}$ follow similar arguments and are omitted.*

**Lemma 1** *Let $f^\alpha : \mathcal{P}^n \longrightarrow X$ be a median voter scheme and let $i \in N$. Then, $x \in V_i$ if and only if either $x < \alpha_1$ or $x > \alpha_{n-1}$.*

*Proof.* ($\Longrightarrow$) Assume that $x \in V_i$. Then, there is $P_i \in \mathcal{P}$ such that $x \notin O^\alpha(P_i)$. Thus, by Remark 1, either $x < \alpha_1$ or $x > \alpha_{n-1}$.
($\Longleftarrow$) First, assume $x < \alpha_1$ and let $P_i \in \mathcal{P}$ be such that $t(P_i) = \alpha_1$. Then, $x \notin \{\alpha_1, \alpha_1 + 1, \ldots, \alpha_{n-1}\} = O^\alpha(P_i)$. Thus, $x \in V_i$. Now, let $x > \alpha_{n-1}$ and let $P_i \in \mathcal{P}$ be such that $t(P_i) = \alpha_{n-1}$. Then, $x \notin \{\alpha_1, \alpha_1+1, \ldots, \alpha_{n-1}\} = O^\alpha(P_i)$. Thus, $x \in V_i$. □

*Proof of Theorem 2.* Let $f^\alpha : \mathcal{P}^n \longrightarrow X$ be a median voter scheme and let $i \in N$. The proof relies in the following two facts:

$$x < \alpha_1 \text{ and } x \in SV_i \text{ if and only if } x = a, \tag{4}$$

and

$$\alpha_{n-1} < x \text{ and } x \in SV_i \text{ if and only if } x = b. \tag{5}$$

To see (4), assume first that $x \in SV_i$ and $a < x < \alpha_1$. Let $P_i \in \mathcal{P}$ be such that $t(P_i) = a$. Then, $x \in O^\alpha(P_i) = \{a, a+1, \ldots, \alpha_{n-1}\}$, contradicting that $x \in SV_i$. Next, assume that $x = a$. Let $P_i \in \mathcal{P}$ be such that $t(P_i) \neq a$. Then, by Remark 1, $x \notin O^\alpha(P_i)$. Hence, $i \in SV_i$. Thus, (4) holds. The proof of (5) is symmetric and therefore it is omitted. To complete the proof of the theorem, assume $f^\alpha$ is NOM. By Theorem 1 and Lemma 1, $SV_i = \{x \in X : x < \alpha_1 \text{ or } x > \alpha_{n-1}\}$. By (4) and (5), $\{x \in X : x < \alpha_1 \text{ or } x > \alpha_{n-1}\} \subseteq \{a, b\}$. Therefore, $\alpha_1 \in \{a, a+1\}$ and $\alpha_{n-1} \in \{b-1, b\}$. Now assume that $\alpha_1 \in \{a, a+1\}$ and $\alpha_{n-1} \in \{b-1, b\}$. Then, $\{x \in X : x < \alpha_1 \text{ or } x > \alpha_{n-1}\} \subseteq \{a, b\}$. By Lemma 1, $V_i \subseteq \{a, b\}$ and, hence, by (4) and (5), $SV_i = V_i$. Therefore, by Theorem 1, $f^\alpha$ is NOM. □

#### 4.1.2 General Case

Now we present the characterization of all strategy-proof, tops-only (onto) rules on the domain of single-peaked preferences for all $n \geq 2$. A generalized median voter scheme can be identified with a set of fixed ballots $\{p_S\}_{S \in 2^N}$ on $X = \{a, a+1, \ldots, b\}$, one for each subset of agents $S$. Then, for each preference profile, the generalized median voter scheme identified with $\{p_S\}_{S \in 2^N}$ selects the alternative $x \in X$ that is the smallest one with the following two properties: (i) there is a subset of agents $S$ whose top alternatives are smaller than or equal to $x$, and (ii) the fixed ballot $p_S$ associated to $S$ is also smaller than or equal to $x$. Formally, we say that a collection $p = \{p_S\}_{S \in 2^N}$ is a *monotonic family of fixed ballots* if: (i) $p_S \in X$ for all $S \in 2^N$ with $p_N = a$ and $p_\varnothing = b$, and (ii) $T \subseteq Q$ implies $p_Q \leq p_T$. A rule $f : \mathcal{P}^n \longrightarrow X$ is a *generalized median voter scheme* if there exits a monotonic family of fixed ballots $p = \{p_S\}_{S \in 2^N}$ such that, for each $P \in \mathcal{P}^n$,

$$f(P) = \min_{S \in 2^N} \max_{j \in S} \{t(P_j), p_S\}.$$

When we want to stress the dependence of the scheme on the collection $p$ of fixed ballots, we write $f^p$. Also, to simplify notation, we use $O^p(P_i)$ instead of $O^{f^p}(P_i)$. The characterization of generalized median voter schemes is as follows:

**Fact 2** *(Moulin, 1980) A (onto) rule $f : \mathcal{SP}^n \longrightarrow X$ is strategy-proof and tops-only if and only if it is a generalized median voter scheme.*

The dictatorial rules are strategy-proof and tops-only, therefore they are generalized median voting schemes. It is easy to see that if the agents $i$ is the dictator, then $p_{\{i\}} = a$ and $p_{N\setminus\{i\}} = b$. Trivially these rules are NOM. Next, we give a simple test to identify which non-dictatorial generalized median voter schemes are NOM.

**Theorem 3** *A non-dictatorial generalized median voter scheme $f^p : \mathcal{P}^n \longrightarrow X$ is NOM if and only if, for each $i \in N$,*

$$p_{N\setminus\{i\}} \in \{a, a+1\} \text{ and } p_{\{i\}} \in \{b-1, b\}. \tag{6}$$

In order to prove Theorem 3, the following remark and lemma are useful.

**Remark 2** *By monotonicity of $p$, $p_{N\setminus\{i\}} \leq p_S$ for each $S$ such that $i \notin S$ and $p_T \leq p_{\{i\}}$ for each $T$ such that $i \in T$. Assume that $p_{N\setminus\{i\}} \leq p_{\{i\}}$. By definition of option set and $f^p$, for each $P_i \in \mathcal{P}$,*

$$O^p(P_i) = \begin{cases} \{t(P_i), t(P_i)+1, \ldots, p_{\{i\}}\} & \text{if } t(P_i) < p_{N\setminus\{i\}} \\ \{p_{N\setminus\{i\}}, p_{N\setminus\{i\}}+1, \ldots, p_{\{i\}}\} & \text{if } p_{N\setminus\{i\}} \leq t(P_i) \leq p_{\{i\}} \\ \{p_{N\setminus\{i\}}, p_{N\setminus\{i\}}+1, \ldots, t(P_i)\} & \text{if } p_{\{i\}} < t(P_i) \end{cases}$$

*The proof follows an argument similar to the one used in Remark 1, with $p_{N\setminus\{i\}}$ playing the role of $\alpha_1$ and $p_{\{i\}}$ playing the role of $\alpha_{n-1}$.*

**Lemma 2** *Let $f^p : \mathcal{P}^n \longrightarrow X$ be a generalized median voter scheme and let $i \in N$.*

*(i) If $p_{N\setminus\{i\}} \leq p_{\{i\}}$, then $x \in V_i$ if and only if either $x < p_{N\setminus\{i\}}$ or $x > p_{\{i\}}$.*

*(ii) If $p_{\{i\}} < p_{N\setminus\{i\}}$, then $V_i = X$.*

*Proof.* Let $f^p : \mathcal{P}^n \longrightarrow X$ be a generalized median voter scheme and let $i \in N$.
(i) The proof follows an argument similar to the one used in the proof of Lemma 1 (invoking Remark 2 instead of Remark 1).
(ii) Let $x \in X$. There are two cases to consider:

1. $\boldsymbol{p_{\{i\}} < x}$. Let $P_i \in \mathcal{P}$ be such that $t(P_i) = p_{\{i\}}$. Then, $f^p(P_i, P_{-i}) \leq p_{\{i\}}$ for each $P_{-i} \in \mathcal{P}^{n-1}$. This implies that $x \in V_i$.

2. $\boldsymbol{x \leq p_{\{i\}} < p_{N\setminus\{i\}}}$. Let $P_i \in \mathcal{P}$ such that $t(P_i) = p_{N\setminus\{i\}}$. Then, $f^p(P_i, P_{-i}) \geq p_{N\setminus\{i\}}$ for each $P_{-i} \in \mathcal{P}^{n-1}$ (because $p_{N\setminus\{i\}} \leq p_S$ for each $S$ such that $i \notin S$). This implies that $x \in V_i$.

In both cases $x \in V_i$. Therefore, $V_i = X$. □

*Proof of Theorem 3.* Let $f^p$ be a non-dictatorial generalized median voter scheme.
($\Longleftarrow$) The proof that condition (6) implies that $f^p$ is NOM follows a similar argument to that of the proof of Theorem 2, with $p_{N\setminus\{i\}}$ and $p_i$ playing the role of $\alpha_1$ and $\alpha_{n-1}$, respectively. Therefore it is omitted.
($\Longrightarrow$) Assume that $f^p$ is NOM. First, assume there is an agent $i^\star \in N$ such that $p_{\{i^\star\}} < p_{N\setminus\{i^\star\}}$. Then, by Lemma 2 , $V_{i^\star} = X$. By Theorem 1, $SV_{i^\star} = X$. Thus, agent $i^\star$ is a dictator, contradicting that $f^p$ is non-dictatorial. Therefore $p_{N\setminus\{i\}} \leq p_{\{i\}}$ for each $i \in N$. Now, the proof follows a similar argument to the proof of Theorem 2 and, therefore, it is omitted. □

## 4.2 Voting by Committees

Now assume that agents have to choose a *subset of objects* from a set $K$ (with $|K| \geq 2$). Then, in this case, $X = 2^K$ and elements of $X$ are subsets of $K$. A generic element of $K$ is denoted by $k$ and a generic element of $X$ is denoted by $S$ . As an example, think of objects as candidates to become new members of a society that have to be elected by the current members of the society. Barberà et al. (1991) characterize, on the restricted domain of separable preferences, the family of all strategy-proof (onto) rules as the class of voting by committees. A preference $P_i$ of agent $i$ is separable if the division between good objects (those $k \in K$ such that $\{k\}P_i\emptyset$) and bad objects (those $k \in K$ such that $\emptyset P_i\{k\}$) guides the ordering of subsets in the sense that adding a good object leads to a better set, while adding a bad object leads to a worse set. Formally, agent $i$'s preference $P_i \in \mathcal{P}$ on $2^K$ is *separable* if for all $S \in 2^K$ and $k \notin S$,

$$S \cup \{k\}P_i x \text{ if and only if } \{k\}P_i\emptyset.$$

Let $\mathcal{S}$ be the set of all separable preferences on $2^K$. Observe that for any separable preference its top is the subset consisting of all good objects. That is, for any separable preference $P_i \in \mathcal{S}$,

$$t(P_i) = \{k \in K : \{k\}P_i\emptyset\}.$$

We now define the class of rules known as voting by committees. Let $N$ be a set of agents and $k \in K$ be an object. A *committee $\mathcal{W}_k$ for $k$* is a non-empty set of non-empty coalitions (subsets) of $N$, that satisfies the following monotonicity condition:

$$M \in \mathcal{W}_k \text{ and } M \subseteq M' \text{ imply } M' \in \mathcal{W}_k.$$

A rule $f : \mathcal{P}^n \longrightarrow 2^K$ is a *voting by committees* if for each $k \in K$ there is a committee $\mathcal{W}_k$ such that, for each $P \in \mathcal{P}^n$,

$$k \in f(P) \text{ if and only if } \{i \in N : k \in t(P_i)\} \in \mathcal{W}_k.$$

By definition, these rules are tops-only and the selected subset of objects at each preference profile is obtained by analyzing one object at a time. Given a committee $\mathcal{W} = \{W_k\}_{k \in K}$, let $f^{\mathcal{W}}$ be its associated voting by committees. Furthermore, and to ease notation, we write $O^{\mathcal{W}}(P_i)$ instead of $O^{f^{\mathcal{W}}}(P_i)$. Barberà et al. (1991) characterize this class when it operates on the separable domain as follows.

**Fact 3** *(Barberà et al., 1991) A (onto) rule $f : \mathcal{S}^n \longrightarrow 2^K$ is strategy-proof if and only if it is voting by committees.*

Observe that, since voting by committees are tops-only, tops-onlyness is implied by strategy-proofness on the domain of separable preferences.

It is clear that dictatorial voting by committees are NOM. In these rules, the dictator $i \in N$ is such that $i \in M$ for all $M \in \mathcal{W}_k$ and $\{i\} \in \mathcal{W}_k$ for all $k \in K$. Next, we give a simple test to identify which non-dictatorial voting by committees are NOM.

**Theorem 4** *A non-dictatorial voting by committees $f^{\mathcal{W}} : \mathcal{P}^n \longrightarrow 2^K$ is NOM if and only if for each $k \in K$:*

*(i) $\bigcap_{M \in \mathcal{W}_k} M = \emptyset$, and*

*(ii) $|M| \geq 2$ for each $M \in \mathcal{W}_k$.*

In order to prove Theorem 4, the following remark and lemma are useful.

**Remark 3** *Let $i \in N$. If $S \in 2^K$ is such that*

*(i) for each $k \in S$, $i \notin \bigcap_{M \in \mathcal{W}_k} M$ and*

*(ii) for each $k \notin S$, $\{i\} \notin \mathcal{W}_k$,*

*then*

$$S \in O^{\mathcal{W}}(P_i) \text{ for each } P_i \in \mathcal{P}. \tag{7}$$

*To see this, let $P_i \in \mathcal{P}$ and let $P_{-i} \in \mathcal{P}^{n-1}$ be such that $t(P_j) = S$ for each $j \in N \setminus \{i\}$. Observe that condition (i) for $S$ implies that $N \setminus \{i\} \in \mathcal{W}_k$ for each $k \in S$ and, thus, $S \subseteq f^{\mathcal{W}}(P_i, P_{-i})$. Moreover, $k \notin S$ implies, by condition (ii) for $S$, that $k \notin f^{\mathcal{W}}(P_i, P_{-i})$. Therefore, $f^{\mathcal{W}}(P_i, P_{-i}) = S$ and (7) holds.*

**Lemma 3** *Let $f^{\mathcal{W}} : \mathcal{P}^n \longrightarrow 2^K$ be a voting by committees, and let $i \in N$. Then, $S \in V_i$ if and only if there is $k^\star \in K$ such that either:*

*(i)* $k^\star \in S$ *and* $i \in \cap_{M \in \mathcal{W}_{k^\star}} M$, *or*

*(ii)* $k^\star \notin S$ *and* $\{i\} \in \mathcal{W}_{k^\star}$.[10]

*Proof.* Let $f^{\mathcal{W}} : \mathcal{P}^n \longrightarrow 2^K$ be a voting by committees and let $i \in N$.
($\Longrightarrow$) Assume that $S \in V_i$. Then, there is $P_i \in \mathcal{P}$ such that $S \notin O^{\mathcal{W}}(P_i)$. Thus, by Remark 3, there is $k^\star \in K$ such that either $k^\star \in S$ and $i \in \cap_{M \in \mathcal{W}_{k^\star}} M$, or $k^\star \notin S$ and $\{i\} \in \mathcal{W}_{k^*}$.
($\Longleftarrow$) There are two cases to consider:

1. **There is $k^\star \in K$ such that $k^\star \in S$ and $i \in \cap_{M \in \mathcal{W}_{k^\star}} M$.** Let $P_i \in \mathcal{P}$ be such that $k^\star \notin t(P_i)$. Then, $k^\star \notin f(P_i, P_{-i})$ and, therefore, $f(P_i, P_{-i}) \neq S$ for each $P_{-i} \in \mathcal{P}^{n-1}$. Thus, agent $i$ vetoes $S$ with $P_i$. Hence, $S \in V_i$.

2. **There is $k^\star \in K$ such that $k^\star \notin S$ and $\{i\} \in \mathcal{W}_{k^\star}$.** Let $P_i \in \mathcal{P}$ be such that $k^\star \in t(P_i)$. Then, $k^\star \in f(P_i, P_{-i})$ and, therefore, $f(P_i, P_{-i}) \neq S$ for each $P_{-i} \in \mathcal{P}^{n-1}$. Thus, agent $i$ vetoes $S$ with $P_i$. Hence, $S \in V_i$.

□

*Proof of Theorem 4.* Let $f^{\mathcal{W}} : \mathcal{P}^n \longrightarrow 2^K$ be a non-dictatorial voting by committees.
($\Longleftarrow$) By Lemma 3, (i) and (ii) in Theorem 4 imply that $V_i = \emptyset$ for each $i \in N$. Then, $SV_i = V_i$ for each $i \in N$ and, by Theorem 1, $f^{\mathcal{W}}$ is NOM.
($\Longrightarrow$) Let $f^{\mathcal{W}}$ be NOM. Then, by Theorem 1, $SV_i = V_i$ for each $i \in N$. The next claim states that $V_i = \emptyset$ for each $i \in N$.
**Claim: $V_i = \emptyset$ for each $i \in N$.** Assume, by contradiction, that there are $i \in N$ and $S \in X$ such that $S \in V_i$. We will show that such $i$ is a dictator, i.e., for each $k \in K$,

$$\{i\} \in \mathcal{W}_k, \tag{8}$$

[10] In the context of voting by committees, when $i \in M$ for each $M \in \mathcal{W}_{k^\star}$, it is usual to say that agent $i$ is a *vetoer* of $k^\star$ in the sense that alternative $k^\star$ must be in the top of agents $i$'s preference to be included in the outcome of the rule. Such veto notion is different from the veto condition in the present paper. Our notion is used to describe when an agent vetoes a subset $S \subseteq 2^K$ in the sense that there is a preference $P_i$ for agent $i$ such that $S$ is never chosen when $i$ declares $P_i$.

and

$$i \in \bigcap_{M \in \mathcal{W}_k} M. \tag{9}$$

By Lemma 3, there are two cases to consider:

1. **There is $k^\star \in S$ and $i \in \bigcap_{M \in \mathcal{W}_{k^\star}} M$.** By Lemma 3, $\{k^\star\} \in V_i$. Assume (8) does not hold. Then, there is $k \in K$ such that $\{i\} \notin \mathcal{W}_k$. Let $P_i \in \mathcal{P}$ be such that $t(P_i) = \{k, k^\star\}$. Then, $\{k^\star\} \in O^{\mathcal{W}}(P_i)$ and, therefore, $\{k^\star\} \notin SV_i$. This contradicts $SV_i = V_i$, so (8) holds. Now, assume that (9) does not hold. Then, there is $k \in K$ and $M \in \mathcal{W}_k$ such that $i \notin M$. By Lemma 3, $\{k^\star, k\} \in V_i$. Let $P_i \in \mathcal{P}$ be such that $t(P_i) = \{k^\star\}$. Then, $\{k, k^\star\} \in O^{\mathcal{W}}(P_i)$. Thus, $\{k, k^\star\} \notin SV_i$. This contradicts $SV_i = V_i$, so (9) holds. Since both (8) and (9) hold, $i$ is a dictator.

2. **There is $k^\star \notin S$ and $\{i\} \in \mathcal{W}_{k^\star}$.** By Lemma 3, $\emptyset \in V_i$. Assume (8) does not hold. Then, there is $k \in K$ such that $\{i\} \notin \mathcal{W}_k$. Let $P_i \in \mathcal{P}$ be such that $t(P_i) = \{k\}$. Then, $\emptyset \in O^{\mathcal{W}}(P_i)$ and, therefore, $\emptyset \notin SV_i$. This contradicts $SV_i = V_i$, so (8) holds. Now, assume that (9) does not hold. Then, there is $k \in K$ and $M \in \mathcal{W}_k$ such that $i \notin M$. By Lemma 3, $\{k\} \in V_i$. Let $P_i \in \mathcal{P}$ be such that $t(P_i) = \emptyset$. Then, $\{k\} \in O^{\mathcal{W}}(P_i)$. Thus, $\{k\} \notin SV_i$. This contradicts $SV_i = V_i$, so (9) holds. Since both (8) and (9) hold, $i$ is a dictator.

The fact that $i$ is a dictator contradicts that $f^{\mathcal{W}}$ is non-dictatorial. Therefore, $V_i = \emptyset$ for each $i \in N$. This finishes the proof of the Claim.

In order to complete the proof of the theorem, observe that the Claim and Lemma 3 imply (i) and (ii) in Theorem 4. □

If we add anonymity to Fact 3 the class of voting by committees must be reduced to a relevant subclass of rules which are called voting by quota. A voting by committees is a *voting by quota* if, for each $k \in K$, there is $q_k$, with $1 \leq q_k \leq n$, such that the associated committee $\mathcal{W}_k$ satisfies that

$$M \in \mathcal{W}_k \text{ if and only if } |M| \geq q_k.$$

Given $q = \{q_k\}_{k \in K}$, let $f^q$ be its associated voting by quota. Next, we give a simple test to identify which voting by quota are NOM.

**Corollary 3** *A voting by quota $f^q : \mathcal{P}^n \longrightarrow 2^K$ is NOM if and only if $2 \leq q_k \leq n-1$ for each $k \in K$.*

*Proof.* It follows easily by specifying Conditions (i) and (ii) in Theorem 4 to voting by quota. □

Corollary 3 is rather surprising. In general, rules in which quotas are either 1 or $n$ are the most robust to manipulation within all voting by quota from several standpoints (for example, see Arribillaga and Massó, 2017; Fioravanti and Massó, 2022). Our result, in contrast, includes them within obviously manipulable ones.

## 5 Final Remarks

Table 1 summarizes our main findings about tops-only, median voter (MV), generalized median voter (GMV), voting by committees (VbC) and voting by quota (VbQ) rules.

| | | |
|---|---|---|
| Tops-only | $f$ NOM $\iff \forall i \in N: \; SV_i = V_i$ | Th. 1 |
| MV | $f^\alpha$ NOM $\iff \alpha_1 \in \{a, a+1\}$ and $\alpha_{n-1} \in \{b-1, b\}$ | Th. 2 |
| GMV† | $f^p$ NOM $\iff \forall i \in N: \; p_{N \setminus \{i\}} \in \{a, a+1\}$ and $p_{\{i\}} \in \{b-1, b\}$ | Th. 3 |
| VbC† | $f^{\mathcal{W}}$ NOM $\iff \forall k \in K: \; \bigcap_{M \in \mathcal{W}_k} M = \emptyset$ and $\|M\| \geq 2 \; \forall M \in \mathcal{W}_k$ | Th. 4 |
| VbQ | $f^q$ NOM $\iff \forall k \in K: 2 \leq q_k \leq n-1$ | Cor. 3 |

† The characterization applies to non-dictatorial rules.

Table 1: *Summary of results.*

Three final remarks are in order. First, following the proof of Theorem 1, it can be seen that the condition $SV_i = V_i$ for each $i \in N$ implies NOM even when tops-onlyness is removed. Although clearly it is not a necessary condition of NOM (see the proof of Lemma 5 in Aziz and Lam, 2021).

Second, a manipulation is called *worst-case obvious* if Condition (1) in Definition 1 holds and *best-case obvious* if Condition (2) in Definition 1 holds. In the proof of Theorem 1, we show that $SV_i \neq V_i$ implies that the rule has a worst-case obvious manipulation. Therefore, by Theorem 1, if a rule does not have a worst-case obvious manipulation then it does not have a best-case obvious manipulation either. Therefore, for top-only rules, a manipulation is obvious if and only if it is a worst-case obvious manipulation.

Finally, we present some observations for the case in which $X$ is infinite. If adequate assumptions over the set of preferences are done in order that (1) and (2) in Definition 1 are well-defined, Theorem 1 is also valid in such context. For example, let $X = [a, b] \subseteq \mathbb{R}$ and let $\mathcal{U}$ be the set of all continuous preferences on $[a, b]$ with a unique top (indifferences between non-top alternatives are admitted). If $f^{\alpha} : \mathcal{U}^n \longrightarrow X$ is a median voter scheme and $P_i \in \mathcal{U}$, then the option set is given by:

$$O^{\alpha}(P_i) = \begin{cases} [t(P_i), \alpha_{n-1}] & \text{if } t(P_i) < \alpha_1 \\ [\alpha_1, \alpha_{n-1}] & \text{if } t(P_i) \in [\alpha_1, \alpha_1] \\ [\alpha_1, t(P_i)] & \text{if } t(P_i) > \alpha_{n-1} \end{cases}$$

and, therefore, $O^{\alpha}(P_i)$ is a closed interval. Hence, (1) and (2) in Definition 1 are well-defined. For these rules, if $a < \alpha_1$, for any $x \in X$ such that $a < x < \alpha_1$ an argument similar to the one used in the proof of Lemma 1 shows that $x \in V_i$; and an argument similar to the one used in the proof of Theorem 2 shows that $x \notin SV_i$. The same is true if $\alpha_{n-1} < b$ for any $x \in X$ such that $\alpha_{n-1} < x < b$. Therefore, we have the following simple characterization of median voter schemes when $X = [a, b] \subseteq \mathbb{R}$ .

**Theorem 5**

*(i) A median voter scheme $f^{\alpha} : \mathcal{U}^n \longrightarrow [a, b]$ is NOM if and only if $\alpha_1 = a$ and $\alpha_{n-1} = b$.*

*(ii) A non-dictatorial generalized median voter scheme $f^{p} : \mathcal{U}^n \longrightarrow [a, b]$ is NOM if and only if $p_{N\setminus\{i\}} = a$ and $p_{\{i\}} = b$ for each $i \in N$.*

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