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Bond risk premia, priced regime shifts, and macroeconomic fundamentals*

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Abstract

In this paper, we develop and estimate an arbitrage-free model of bond prices in which the evolution of the risk factors and the parameters of the stochastic discount factor are subject to occasional discrete changes in regimes. We show that the component of risk premia associated with regime shifts is related to the macroeconomic environment. In particular, the explicit pricing of regime shifts and the nonlinearities associated with the Markov switching model generates a strong connection between bond risk premia and the macroeconomy as summarized by variables such as inflation, industrial production, and unemployment.

Keywords: Yield Curve; Term structure of interest rates; Markov regime switching; Priced switching risk premia.
JEL classification: C13; C22; E43.

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1 Introduction

There is ample evidence that interest rates are subject to occasional discrete changes over time (Hamilton, 1988; Ang and Bekaert, 2002; Bansal and Zhou, 2002). Times of high and volatile interest rates, such as the period between the late 1970s and early 1980s, seem to be inherently different from normal times with low average interest rates and low volatility, such as the 1990s (Sims and Zha, 2006). Likewise, while the yield curve is upward sloping during normal times, we often observe inversions of the curve at the beginning of a tightening in monetary policy or economic contraction (Fama, 1986; Harvey, 1988; Estrella and Hardouvelis, 1991). Naturally, investors assessing the possibility of a sudden shift in the level and volatility of interest rates would seek compensation for the changing risks they face. Yet, most bond pricing models abstract from sudden regime-changes (Piazzesi, 2010) and, when considered, the models tend to neglect their direct contribution to bond prices, in that regime shifts are non-priced risk factors (Ang and Bekaert, 2002; Bansal and Zhou, 2002).[1] In this regard, a notable exception is the paper by Dai, Singleton and Yang (2007).

In this paper, we develop and estimate an arbitrage-free model of bond prices in which the evolution of the risk factors and the parameters of the stochastic discount factor are subject to occasional discrete changes in regimes. At each point in time, the economy can be in one of a finite set of possible regimes whose evolution is governed by a finite state Markov chain. In addition, there are three other risk factors summarized by the traditional level, slope, and curvature of the yield curve, which we call the continuous risk factors. The discrete Markov risk factor serves two purposes. First, it is used to model nonlinearities by assuming that the continuous risk factors evolve as a first order autoregression subject regime changes (as in Hamilton, 1988; Sola and Driffill, 1994). And second, the parameters of the stochastic discount factor used to price bonds depend on the discrete Markov regime in two ways: by directly discounting possible changes in regimes (or staying in the same regime), and by allowing for discrete changes in the way that fluctuations in the continuous risk factors affect bond prices. In this environment, the continuous risk factors and the discrete risk factor summarize the relevant information set that investors use to price bonds. Therefore, bond prices and expected excess returns reflect two sources of risk: the risks associated with fluctuations in the continuous risk factors and the risk of regime shifts.

Using this pricing model, we show that the component of risk premia associated with

[1]This omission implicitly assumes that systematic shifts in regimes can be regarded as an idiosyncratic risks that investors can diversify away.

regime shifts is related to the macroeconomic environment. In particular, the explicit pricing of regime shifts and the nonlinearities associated with the Markov switching model generates a strong connection between bond risk premia and the macroeconomy as summarized by variables such as inflation, industrial production, and unemployment.

We begin by documenting regime changes in the level and slope of the yield curve. The level factor is characterized by higher unconditional mean and variance during the early 1980s (as in Hamilton, 1988) and during the monetary policy tightening cycle of the late 1960s.[2] In addition, the yield curve tends to be inverted and the volatility of the slope increases around business cycle contractions. Consistent with the hypothesis that occasional regime shifts are relevant risks factors, we find that an indicator function that captures the evolution of the discrete regimes is a significant predictor of excess bond returns even after including the traditional level, slope, and curvature factors as additional regressors. This result implies that the regime indicator function is an unspanned factor in that it captures relevant information to predict expected excess returns other than that included in the usual level, slope, and curvature of the yield curve. Expected excess returns are on average lower in a regime associated with high interest rates and with an inverted and volatile slope of the yield curve.

We use that evidence to motivate the particular structure that we use in our arbitrage-free model of bond prices with regime shifts. In particular, we consider switches between two possible regimes in the level factor of the yield curve and switches between two possible regimes in the slope factor of the yield curve. For the level factor, we identify a regime of high unconditional mean and high volatility. As for the slope factor, we identify a period of low unconditional mean and high volatility. The combination of these two possibilities generates the Markov switching structure with four possible regimes that we use when estimating the arbitrage-free model.

Our model nests the single-regime model (as in Duffee, 2002) and a switching model without priced regime shifts (as in Bansal and Zhou, 2002, and Hevia et al., 2015).[3] We show that the data reject the restrictions implied by those specifications. The model with priced regime shifts generates bond risk premia that are substantially different from those of the restricted models. The single regime model generates risk premia that is subject to sudden and long swings while the model with priced regime shifts behaves

[2]See Bernanke (2022) for a discussion of monetary policy and its relation to the macroeconomy in those episodes.

[3]We note, however, that our model does not formally nest the model in Bansal and Zhou (2002) because they impose a regime switching structure into a single-factor "CIR-style" model in which the evolution of the risk factor is conditionally heteroskedastic while we assume that the distribution of the risk factors, conditional on the regime, is homoskedastic as in Dai, Singleton and Yang (2007).

less erratically with occasional jumps that reflect changes in regimes. The more erratic behavior of risk premia of the single regime model is due to ignoring the impact that changes in volatility have on bond prices, which the model with regime switching and priced regime shifts is able to capture. Interestingly, the estimated risk premia of the switching model without pricing regime shifts behaves closer to the risk premia of the single regime model than to the risk premia of the switching model with priced regime shifts. This observation emphasizes the importance of pricing regime risks. On average, expected excess returns of holding long maturity bonds are positive in normal times, in a regime with low volatility and positive slope of the yield curve, and tends to turn negative when the yield curve flattens or becomes inverted and the slope factor becomes more volatile.

In single regime, affine term structure models, the compensation of risk is proportional to the variance of the risk factors and, as such, variations in the bond premia are limited (Duffee, 2002). Moreover, traditional term structure models—when not taking an explicit stand on the relationship between the term structure and the macroeconomy—generate estimates of bond risk premia that are mostly disconnected from macroeconomic fundamentals (Duffee, 2011, 2013). Our model with priced regime shifts addresses both of those issues. Risk premia can be decomposed into a component related to fluctuations in the continuous risk factors and a component related to regime switching risk. We show that the component of the risk premium that captures the risk of switching regimes is significantly and strongly related to macroeconomic fundamentals.

Bond risk premia derived from the baseline model with priced regime shifts are highly correlated with inflation and with an indicator of economic activity, such as the cyclical component of industrial production or unemployment. Expected excess returns tend to be countercyclical, decreasing when both inflation and industrial production are above trend. These two macroeconomic indicators explain almost a third of the variation in one-month excess bond returns. This success is entirely related to the role of the price of regime switches. Ludvigson and Ng (2009) point out that business cycle variations are critical to explaining excess bond returns but they are not uncovered by factor models. In our model, the relationship between bond risk premia and macroeconomic fundamentals is revealed by the nonlinearity inherent in the price of regime switching. Defining the regimes by looking at the properties of the level and slope factors in isolation captures features of the data already subsumed by the factors. However, our regimes look at the combined properties of the level and slope factors, which cannot be recovered by a linear combination of the two factors. In our model, regimes in the level factor align well with the monetary policy tightening cycles, whereas the slope factor regime roughly identifies contractions in economic activity.

Our paper contributes to the extensive literature on affine term structure models (see Piazzesi, 2010, for a survey). In particular, in line with Hamilton (1988); Sola and Driffill (1994); Ang and Bekaert (2002); Bansal and Zhou (2002); Dai and Singleton (2003); Dai, Singleton and Yang (2007), we emphasize the importance of allowing for regime-switching when characterizing interest rates. In arbitrage-free models with Markov switching that allow for priced regime shifts developed in continuous time, the compensation for the risk of remaining in the current regime is necessarily zero (Dai and Singleton, 2003). Since we develop our model in discrete time, our model allows for a possible discounting (or premium) for staying in the current regime. Ang and Bekaert (2002), Bansal and Zhou (2002), and Hevia et al. (2015) estimate arbitrage-free dynamic term structure models but without pricing regime shifts.

The paper that is most related to ours is Dai, Singleton and Yang (2007). In both papers, the stochastic discount factor extends the usual log-linear discount factor with a term that adds discounting to future cash flows across different regimes given the current regime. However there are two important differences with that paper. First, Dai, Singleton and Yang (2007) consider a two-regime model, which they interpret as high and low volatility regimes. Instead, through a preliminary analysis of the data, we emphasize the importance of allowing for independent switches in the level and slope factors of the yield curve. The interaction between these independent changes in the level and slope of the yield curve imply a model with four possible regimes, which is a critical property to uncover a strong connection between bond risk premia and macroeconomic fundamentals. And second, we adopt the traditional timing protocol used in regime-switching models by specifying that the distribution of the continuous risk factors depend on the current regime, as in Hamilton (1989) and Bansal and Zhou (2002). In contrast, Dai, Singleton and Yang (2007) condition the distribution of the current continuous risk factor on the previous period's regime. While this timing protocol is convenient because it delivers exact closed-form solutions for bond prices, it also implies that changes in volatility are perfectly forecastable one period in advance and, therefore, investors know with certainty the particular node of the probability tree at which de economy will be in the following period. We choose the traditional timing convention for the model to be consistent with the view that turbulent times are inherently random. With our timing convention, investors assign probabilities to the possibility that volatility may change in the future and this inherent uncertainty affects bond prices and risk premia differently than if they knew the future volatility with certainty. We show that a general equilibrium consumption-based model in which consumption and the shocks to the marginal utility of consumption are indexed by a finite state Markov chain gives rise to a stochastic discount factor and timing protocol consistent with our model.

But using the traditional timing convention does not permit exact closed-form solutions for bond prices. As in Bansal and Zhou (2002) we look for approximate solution that, conditional on the current regime, are log-linear in the continuous risk factors. The difference, however, is that we approximate the non-linear term in the pricing equation using a first order Taylor approximation around the state-dependent long-run mean of the risk factors while Bansal and Zhou (2002) use the standard approximation $\exp(y) - 1 \approx y$ which implies approximating around zero. Given the approximate conditionally log-linear solution for bond prices, we obtain an analytic representation for the likelihood function that we use in our empirical analysis of U.S. Treasury zero-coupon bond yields.

Finally, while our empirical model follows the tradition of assuming that the relevant factors that determine the evolution of the yield curve can be recovered from a portfolio of bonds (as in Litterman and Scheinkman, 1991; Dai and Singleton, 2000; Joslin, Singleton and Zhu, 2011; and many others), it is well known that there is a strong relation between those factors and the macroeconomy (Bauer and Rudebusch, 2017). Therefore, while not directly referring to any specific macroeconomic regime, as in Sims and Zha (2006), our work relates to a more recent stream of the literature that focuses on the impact that changes in macroeconomic regimes have on the term structure of interest rates (see, for example, Bikbov and Chernov, 2013). The critical insight of our work is that regime switches and their evolution over time affect bond risk premia.

The rest of the paper is organized as follows. Section 2 presents a preliminary analysis of the data used to motivate the parametrization of our arbitrage-free model. Section 3 develops the arbitrage-free model with regime shifts and discusses a general equilibrium model for an endowment economy that generates a stochastic discount factor identical to that in the arbitrage-free model discussed previously. In Section 4 we discuss the specification and estimation of the model and Section 5 analyzes bond risk premia and its relation to the macroeconomy. Section 6 concludes.

2 Preliminary Analysis

This section documents regime shifts in the dynamics of interest rate factors and their relevance for bond risk premia. The analysis in this section is purely statistical in that we do not impose any restrictions derived from the lack of arbitrage across bonds. We use the insights of this section to impose structure and restrictions into the general arbitrage-free model of bond prices that we develop in Section 3.

As a first step, we analyze the dynamics of the yield of a 10-year government bond and the spread between the 10-year and the 3-month bonds. These variables represent the level and slope factors of the yield curve that are widely used to forecast returns and estimate risk premia. In this section, and in what follows, we use data on U.S. zero coupon bonds for the period Jan-1962 through Nov-2019 from Le and Singleton (2018) and updated by Anh Le. Specifically, we investigate if there are occasional discrete shifts in the parameters governing the evolution of the level and slope factors by estimating different specifications of an autoregressive model with dependent regime parameters,

$$x_t - \theta_{s_t} = \sum_{j=1}^{p} \phi_{j,s_t}(x_{t-j} - \theta_{s_{t-j}}) + \sigma_{s_t}\epsilon_t, \tag{1}$$

where s_t represents a two-state Markov chain; x_t is, either, the level or the slope factor; ϵ_t is a normal shock with zero mean and unit variance; and θ_{s_t}, σ_{s_t}, ϕ_{j,s_t} for $j = 1, 2, \ldots, p$ are parameters that depend on the Markov state s_t.

We consider three specifications of equation (1). In the first specification, the unconditional mean θ_{s_t} is the only parameter that can change. The second specification allows for simultaneous shifts in the unconditional mean θ_{s_t} and variance σ_{s_t}. The third specification allows for simultaneous changes in all the parameters of equation (1). For comparison purposes, we also estimate a linear (single regime) model.[4]

Table 1 reports summary statistics of the in-sample fit for the different specifications using as dependent variable the level factor (top panel) and the slope factor (bottom panel). In both cases, the three information criteria select models which allow for simultaneous changes in the unconditional mean and variance but constant autoregressive coefficients. Allowing for shifts in the autoregressive coefficients, however, does not lead to significant improvements in the log-likelihood and it produces worse scores in terms of information criteria relative to the model that allow for simultaneous shifts in the unconditional mean and variance. In addition, allowing for regime shifts substantially improves the fit relative to the linear model.

In light of the previous results, in what follows we focus on the model that allows for simultaneous changes only in the long-run mean and volatility parameters θ_{s_t} and σ_{s_t}.

Consider first the level factor (the 10-year yield). The model identifies a regime of high unconditional mean and high volatility of interest rates. The upper left panel of Figure 1 shows the evolution of the level factor along with the smoothed probability of

[4]In all cases, we set the number of lags in equation 1 to $p = 4$ to allow for sufficiently rich dynamics. The results are robust to different lag choices.

Table 1: **Alternative Regime Switching Specifications**

	Level Factor			
	Log-likelihood	Akaike	Hannan-Quinn	Bayesian
Single Regime	-187.319	0.556	0.571	0.596
RS LRM	-151.630	0.465	0.488	0.524
RS LRM+VOL	-107.778	0.341	0.366	0.407
RS LRM+AR+VOL	-106.400	0.348	0.384	0.440

	Slope Factor			
	Log-likelihood	Akaike	Hannan-Quinn	Bayesian
Single Regime	-385.364	1.126	1.141	1.165
RS LRM	-300.740	0.896	0.919	0.956
RS LRM+VOL	-214.853	0.651	0.676	0.716
RS LRM+AR+VOL	-211.368	0.652	0.688	0.744

Notes: This table reports different measures of model fit for alternative specifications of an AR(4) model with regime switching in the level factor (upper panel) and the slope factor (lower panel). RS LMR stands for with a model with regime switching only in the long-run mean μ_{s_t}; RS LRM+VOL allows for simultaneous shifts in the long-run mean and volatility parameters μ_{s_t} and σ_{s_t}; RS LRM+AR+VOL also includes switches in the autoregressive parameters ϕ_{j,s_t}. As a reference, we also report the results for a linear (single regime) model. The columns report the log-likelihood, Akaike, Hannan–Quinn, and Bayesian information criteria of the different models. Highlighted entries correspond to the model with the lowest value of the information criterion.

the high unconditional mean and high volatility regime. The upper right panel shows the monthly change in the 10-year yield and the smoothed probability of the regime. The filter identifies for the level factor a period of high mean and volatility from the early to the mid-1980s (see Hamilton (1988)) . Specifically, the high mean and volatility regime starts in the last quarter of 1979 and lasts over the entire Volcker disinflation period (see also Romer and Romer, 1989). It also selects observations from the mid-2003 and the 2008 recession as part of that regime, characterized by increased volatility in the level factor. Both of these events are associated with significant changes in monetary policy uncertainty and coincide with some form of forward guidance by the Fed (see, e.g., Bauer, Lakdawala and Mueller, 2021).

Consider now the slope factor (the 10-year yield minus 3-month yield). In this case, the model identifies a period of low unconditional mean and high volatility of the slope factor. The bottom panels of Figure 1 show the evolution of the slope factor (left panel) and its change (right panel) together with the smoothed probability of the low unconditional mean and high volatility regime. Changes in the regimes corresponding to the slope factor tend to be associated with periods of rapid monetary policy reversals, in the sense that term spreads are relatively low and highly volatile. A higher probability of this regime tends to be observed preceding all USA recessions except that in 1991.

Figure 1: **Level, slope, and estimated regime probabilites**

Note: The upper panels of the figure report the level factor and its change together with the estimated smoothed probability of a high unconditional mean and high volatility regime for the level factor computed by estimating equation (1) using $x_t = y_t^{10y}$. The lower panel reports the slope factor and its change along with the estimated smoothed probability of a low unconditional mean-high volatility regime for the slope factor computed by estimating equation (1) using $x_t = y_t^{10y} - y_t^{3m}$.

The previous evidence suggests that there are discrete changes in the economy that affect the dynamics of the level and slope of the yield curve. We next assess to what extent those discrete changes in regimes can help predict bond risk premia.

To that end, we start by constructing 1-month excess returns for bonds of different maturities. The 1-month excess return of an $n-$period bond is defined as $xr_{t+1}^n = p_{t+1}^{n-1} -$

$p_t^n - y_t^3$, where p_t^n is the log-price at time t of a bond come due in n months and y_t^3 is the 3-month interest rate. We perform predictive regressions of excess returns xr_{t+1}^n for different maturities n on the usual level, slope, and curvature factors along with regime indicators that we construct in different ways.

We first split the sample into periods of high and low unconditional mean for the level factor. To consider whether the level factor is in a regime of a high unconditional mean, we construct a moving average of the level factor using exponential weights. We label a period to be of high unconditional mean if this moving average is above the 80th percentile of the distribution of the level factor over the entire sample period. Likewise, we define periods with high or low volatility of the slope factor by constructing an exponentially weighted moving average of the square deviations of the slope factor relative to its unconditional mean. We define a period to be of high volatility when this moving average is above the 75th percentile of the distribution of the square deviations of the slope factor relative to its unconditional mean over the entire sample period. Using these definitions of the regimes, we obtain roughly the same separation of regimes as when using the Markov switching model reported in Figure 1.[5] An advantage of defining the regimes in this way instead of using the estimated probabilities in Figure 1 is that it is much simpler to replicate when bootstrapping yields to construct p-values, following the procedure suggested by Bauer and Hamilton (2017).

Having defined the regimes, we consider three cases depending on whether the level and slope regimes are each considered in isolation or combined. In the latter case, we define three dummy variables corresponding to periods of (i) high mean of the level factor and low volatility of the slope factor, (ii) low mean of the level factor and high volatility of the slope factor, and (iii) high mean of the level factor and high volatility of the slope factor. Next, we run predictive regressions of excess returns xr_{t+1}^n for $n = 12, 24, 36, \ldots, 120$ months on the time-t values of the level, slope, and curvature factors of the yield curve together with the dummy variables that captures the different regimes.

Table 2 reports bootstrapped p-values of the Wald test of significance of the dummy variables associated with the different regimes. If we construct the regimes using only the level factor or the slope factor in isolation, the regime dummy variables are not significant predictors of future excess returns. Intuitively, defining the regimes by looking at the properties of the level and slope factors in isolation captures features of the data that are already subsumed by the factors themselves, making the regime indicators

[5] Alternatively, we could define regimes by constructing moving averages of the volatility of the level factor and of the mean of the slope factor and separate regimes when these moving averages cross some threshold percentile of the distributions. This procedure selects roughly the same periods for each regime, with a concordance statistic greater than 80% for both cases.

redundant. Yet, when we combine the level and slope factors to define our regime indicator, we find that this combination is a significant predictor of future excess returns, particularly so at the short end of the yield curve. In particular, the dummy variable that simultaneously captures a high mean of the level factor and a high volatility of the slope factor predicts significantly lower excess returns. For instance, being in this particular regime is associated with excess returns of roughly 50 basis points below what it would otherwise be predicted by the level, slope, and curvature factors for $n = 12$ (p-value less than 1%) and up to 70 basis points lower for $n = 24$ (p-value of about 2%).

Table 2: **Excess returns and regime shifts**

	Level Only	Slope Only	Combined
Average (1y-5y)	0.704	0.428	0.066
Average (1y-10y)	0.892	0.598	0.124
1y	0.684	0.332	0.002
2y	0.530	0.627	0.010
3y	0.842	0.379	0.225
4y	0.987	0.606	0.037
5y	0.320	0.163	0.093
6y	0.667	0.436	0.171
7y	0.945	0.954	0.142
8y	0.158	0.770	0.042
9y	0.523	0.539	0.268
10y	0.554	0.885	0.167

Notes: Table 2 reports the p-values testing whether the regime dummies are unspanned factors. The first two columns consider the significance of a dummy for the periods associated with high unconditional mean for the level factor (Level Only) or high volatility of the slope factor (Slope Only). The last columns includes the results associated with the dummies that interact the level and slope factors regimes. The p-values are computed using the boostrap procedure discussed in Bauer and Hamilton (2017). In all cases, the predictive regressions include as independent variables time$-t$ level, slope, and curvature factors as additional regressors.

Summarizing, these results suggest that there are nonlinear interactions between the factors that appear to be relevant correlates of bond risk premia besides those captured by the traditional level, slope, and curvature factors of the yield curve. Moreover, these nonlinearities can be captured by occasional discrete changes in the economy. Using these insights, in the next section we develop an arbitrage-free model of the yield curve that explicitly prices the risks associated with occasional discrete shifts in regimes in US bond markets.

3 The model

In this section, we develop a statistical arbitrage-free model of bond prices that allows for priced regime shifts. As is common in the asset pricing literature, we specify processes for the short rate and the stochastic discount factor that are both functions of a set of risk factors. In addition, the parameters describing the evolution of the short rate and the risk factors are indexed by another stochastic process that evolves as a finite state Markov chain. Algebraic details not shown in the body of the paper are relegated to Appendix A.

To motivate the timing protocol and stochastic discount factor used in the statistical model of bond pricing, we next describe a general equilibrium model of a representative agent with time separable preferences that is subject to shocks to the marginal utility of consumption. Both, the parameters governing the evolution of consumption and the shocks to the marginal utility of consumption are indexed by a finite state Markov chain. While this particular consumption-based model is one way to motivate the statistical arbitrage-free model, there may be other equilibrium models that could lead to similar reduced form arbitrage-free prices.

3.1 The statistical arbitrage-free model

Time is discrete and denoted by $t = 0, 1, 2, \ldots$. Each time period in the model represents one month. A stochastic process that takes a finite number of values (regimes from now on) $s_t \in \{1, \ldots, S\}$ determines the distribution of an $M-$dimensional vector of risk factors X_t. The regime s_t evolves according to a Markov chain with transition probability of switching from regime $s_t = j$ to regime $s_{t+1} = k$ given by

$$\pi_{jk} = \Pr\left(s_{t+1} = k | s_t = j\right) \tag{2}$$

for $j, k = 1, ..., S$, with $\sum_{k=1}^{S} \pi_{jk} = 1$ for all j.

The conditional distribution of the risk factors X_{t+1} is a function of the form

$$\Pr\left(X_{t+1} | X_t, s_t = j, s_{t+1} = k\right) = \Pr\left(X_{t+1} | X_t, s_{t+1} = k\right)$$

that depends on the regime at $t + 1$, s_{t+1}, and the value of the risk factors at time t, X_t. In particular, given X_t and $s_{t+1} = k$, X_{t+1} follows the process

$$X_{t+1} = \mu_k + \Phi_k X_t + \Sigma_k \epsilon_{t+1}, \tag{3}$$

where $\epsilon_{t+1} \sim N(0, I)$ and Σ_k is lower triangular. The parameters of this conditionally linear process $(\mu_k, \Phi_k, \Sigma_k)$ depend on the value of the regime at time $t+1$, $s_{t+1} = k$.

This Markovian structure implies that, at time t, the state vector (s_t, X_t) is sufficient to characterize the entire probability distribution of (s_{t+1}, X_{t+1}). Throughout the paper, we assume that bond prices at time t are a function of the state vector (s_t, X_t).

In using the process (3), we adopt the timing protocol of Hamilton (1988) that specifies the distribution of X_{t+1} conditional on the contemporaneous regime s_{t+1}. This timing convention has been used in the bond pricing literature by Bansal and Zhou (2002) and Ang and Bekaert (2002). In contrast with those papers, however, we allow for priced regime shifts in the sense that the stochastic discount factor includes an explicit term discounting future cash flows depending on the realized values of the Markov chain in the following period. In the empirical section below we show that this additional regime-specific discounting is fundamental to understand the dynamics of risk premia and bond prices. On the other hand, Dai, Singleton and Yang (2007) impose that the the parameters of the conditional distribution of the risk factors X_{t+1} depends on the past regime s_t. This timing convention allows them to obtain exact closed-form solutions for bond prices but it also implies that investors know with certainty the particular node of the probability tree at which the economy will be in the following period. In that respect, the solution of their model is similar to that obtained when solving a single regime model.

Given the state vector (s_t, X_t), the continuously compounded yield on a one-period zero-coupon bond, denoted by $r_t \equiv r(s_t, X_t)$, is given by

$$r_t = \delta_0^{s_t} + \delta_1^{s_t\prime} X_t, \tag{4}$$

where $\delta_0^{s_t}$ is a scalar and $\delta_1^{s_t}$ is an $N-$dimensional vector.

Let $P^n_{t,s_t} = P^n(s_t, X_t)$ denote the price at time t of a zero-coupon bond with maturity of n periods when current regime is s_t and the risk factors are X_t. To complete the specification of the model, we impose the following stochastic discount factor to price future cash flows

$$M_{t,t+1} = e^{-r_t - \Gamma(s_t, s_{t+1}) - \frac{1}{2}\Lambda_t^{s_{t+1}\prime}\Lambda_t^{s_{t+1}} - \Lambda_t^{s_{t+1}\prime}\epsilon_{t+1}}, \tag{5}$$

$$\Lambda_t^{s_{t+1}} = \lambda_0^{s_{t+1}} + \lambda_1^{s_{t+1}} X_t, \tag{6}$$

where the $S \times S$ matrix $\Gamma(s_t, s_{t+1})$ is the market price of risk of switching from regime s_t to regime s_{t+1}, $\lambda_0^{s_{t+1}}$ is an $N-$dimensional vector, and $\lambda_1^{s_{t+1}}$ is an $N \times N$ matrix.

As in Dai, Singleton and Yang (2007), the stochastic discount factor (5) extends the usual log-linear stochastic discount factor with a term $\Gamma(s_t, s_{t+1})$ that adds an additional discounting to future cash flows across different regimes s_{t+1} given that the current regime is s_t. We do not impose that $\Gamma(s_t, s_t) = 0$, so there may be a discount (or premium) for staying in the current regime. To avoid clutter, we often use the more compact notation $\Gamma_{jk} \equiv \Gamma(s_t = j, s_{t+1} = k)$.

Given the current state $(s_t = j, X_t)$, the pricing equation for zero-coupon bonds is

$$\begin{aligned} P^{n+1}_{t,j} =& E\left[M_{t,t+1} P^n_{t+1,s_{t+1}} | s_t = j, X_t\right] \\ =& E\left[e^{-r_t - \Gamma_{s_t,s_{t+1}} - \frac{1}{2}\Lambda^{s_{t+1}\prime}_t \Lambda^{s_{t+1}}_t - \Lambda^{s_{t+1}\prime}_t \epsilon_{t+1}} P^n_{t+1,s_{t+1}} | s_t = j, X_t\right] \\ =& e^{-r_t} \sum_{k=1}^{S} \pi_{jk} e^{-\Gamma_{jk} - \frac{1}{2}\Lambda^{k\prime}_t \Lambda^k_t} E\left[e^{-\Lambda^{k\prime}_t \epsilon_{t+1}} P^n_{t+1,k} | s_t = j, s_{t+1} = k, X_t\right]. \end{aligned} \tag{7}$$

In general, the solution to this functional equation is nonlinear and requires numerical methods to be solved. To estimate the model using the method of maximum likelihood, however, we find it convenient to look for approximate log-linear solutions of the form

$$\log(P^n_{t,j}) = A^n_j + B^{n\prime}_j X_t, \tag{8}$$

for some scalar A^n_j and M-dimensional vector B^n_j.

Using the approximation (8) into equation (7) and taking logs gives

$$A^{n+1}_j + B^{n+1\prime}_j X_t = -r_t + \log\left(\sum_{k=1}^{S} \pi^Q_{jk} e^{F^n_k + B^{n\prime}_k \Phi^Q_k X_t}\right), \tag{9}$$

where

$$F^n_k = A^n_k + B^{n\prime}_k \mu^Q_k + \tfrac{1}{2} B^{n\prime}_k \Sigma_k \Sigma'_k B^n_k, \tag{10}$$

$$\mu^Q_k = \mu_k - \Sigma_k \lambda^k_0, \tag{11}$$

$$\Phi^Q_k = \Phi_k - \Sigma_k \lambda^k_1, \tag{12}$$

$$\pi^Q_{jk} = \pi_{jk} e^{-\Gamma_{jk}}. \tag{13}$$

To find the parameters of the approximate solution, we perform a first order Taylor approximation to the log-term in the right hand side of equation (9) around the (state-dependent) long-run value of the risk factors $X_t = \bar{\mu}_j \equiv (I - \Phi_j)^{-1} \mu_j$ given the current regime $s_t = j$. In Appendix A we show that the Taylor approximation is

$$\log\left(\sum_{k=1}^{S} \pi_{jk}^{Q} e^{F_k^n + B_k^{n\prime}\Phi_k^Q X_t}\right) \approx \log\left(\sum_{k=1}^{S} \pi_{jk}^{Q} e^{F_k^n + B_k^{n\prime}\Phi_k^Q \bar{\mu}_j}\right) + (X_t - \bar{\mu}_j)'\boldsymbol{h}_j^n, \tag{14}$$

where the $M \times 1$ vector $\boldsymbol{h}_j^n$, which depends on the bond maturity n and current state $s_t = j$, is given by

$$\boldsymbol{h}_j^n = \underbrace{\left[\begin{array}{cccc} \Phi_1^{Q\prime} B_1^n & \Phi_2^{Q\prime} B_2^n & \cdots & \Phi_S^{Q\prime} B_S^n \end{array}\right]}_{N\times S} \underbrace{\left[\begin{array}{c} \pi_{j1}^{Q} e^{F_1^n + B_1^{n\prime}\Phi_1^Q \bar{\mu}_j} \\ \pi_{j2}^{Q} e^{F_2^n + B_2^{n\prime}\Phi_2^Q \bar{\mu}_j} \\ \vdots \\ \pi_{jS}^{Q} e^{F_S^n + B_S^{n\prime}\Phi_S^Q \bar{\mu}_j} \end{array}\right]}_{S\times 1} \times \frac{1}{\sum_{h=1}^{S} \pi_{jh}^{Q} e^{F_h^n + B_h^{n\prime}\Phi_h^Q \bar{\mu}_j}}.$$

Using the approximation (14) and the short rate equation (4) into the pricing condition (9), and matching coefficients yields a recursion for the parameters in equation (8):

$$A_j^{n+1} = -\,\delta_0^j - (\boldsymbol{h}_j^n)'\bar{\mu}_j + \log\left(\sum_{k=1}^{S} \pi_{jk}^{Q} e^{F_k^n + B_k^{n\prime}\Phi_k^Q \bar{\mu}_j}\right) \tag{15}$$

$$B_j^{n+1} = \boldsymbol{h}_j^n - \delta_1^j, \tag{16}$$

with initial conditions $A_j^0 = 0$ and $B_j^0 = \mathbf{0}$ for $j = 1, 2, \cdots, S$.

In addition, for the model to correctly price the short term interest rate (4) we must impose the restriction

$$\sum_{k=1}^{S} \pi_{jk}^{Q} = \sum_{k=1}^{S} \pi_{jk} e^{-\Gamma_{jk}} = 1. \tag{17}$$

This condition is equivalent to imposing that the risk-neutral probabilities of regime shifts, which are given by $\pi_{jk}^{Q} = \pi_{jk} e^{-\Gamma_{jk}}$, add up to one for $j = 1, \ldots S$.

3.2 Yields and risk premia

Let $p_t^n = \log P_t^n(s_t, X_t)$ denote the log-price of an $n-$period zero coupon bond at time t when current the state vector is (s_t, X_t) and $y_t^n = -\frac{1}{n} p_t^n$ the corresponding log-yield. Given that bond prices satisfy equation (8), bond yields can be written as

$$y_t^n = a_{s_t}^n + b_{s_t}^{n\prime} X_t \tag{18}$$

for all maturities n, where $a^n_{s_t} = -A^n_{s_t}/n$, $b^n_{s_t} = -B^n_{s_t}/n$, and $A^n_{s_t}$ and $B^n_{s_t}$ satisfy the recursions (15) and (16).

While there are many possible definitions of risk premia, in this paper we focus on expected excess holding returns. In particular, the expected log holding return of buying an n-period zero coupon bond at time t in state $s_t = j$ and selling it as an $(n-1)$-period coupon bond at time $t+1$ is given by

$$E_t[p^{n-1}_{t+1}] - p^n_t = \sum_{k=1}^{S} \pi_{jk} \left[A^{n-1}_k + B^{n-1\prime}_k (\mu_k + \Phi_k X_t) \right] - (A^n_j - B^{n\prime}_j X_t).$$

The expected log-holding return in excess of the short rate r_t at time t is thus given by

$$\begin{aligned} E_t[p^{n-1}_{t+1}] - p^n_t - r_t = &\sum_{k=1}^{S} \pi_{jk} (A^{n-1}_k + B^{n-1\prime}_k \mu_k) - A^n_j - \delta^j_0 \\ &+ \left(\sum_{k=1}^{S} \pi_{jk} B^{n-1\prime}_k \Phi_k - B^{n\prime}_j - \delta^{j\prime}_1 \right) X_t. \end{aligned} \tag{19}$$

3.3 A general equilibrium model

In this subsection, we provide a general equilibrium model of an endowment economy that generates a stochastic discount factor and discrete regime shifts like those postulated in the arbitrage-free model of Section 3.1.

The state of the economy at time t is summarized by a vector $z_t = (s_t, X_t)$, where $s_t \in \{1, 2, \ldots, S\}$ is a discrete Markov chain with transition probabilities given by equation (2) and X_t is a vector of continuous random variables that evolve according to the process (3). We denote by $z^t = \{z_0, z_1, \ldots, z_t\}$ the partial history of shocks from time 0 up until time t, and by Z^t the set of all possible partial histories up to time t. The probability distribution over partial histories z^t conditional on the initial state z_0 is denoted by $F_t(z^t|z_0)$ and can be derived from the stochastic processes (2) and (3).

The representative household has preferences over contingent sequences of consumption $C_t(z^t)$ represented by an expected utility function of the form

$$U = \sum_{t=0}^{\infty} \int_{z^t \in Z^t} e^{-\delta t} \eta_t(z^t) \frac{C_t(z^t)^{1-\gamma} - 1}{1-\gamma} dF_t(z^t|z_0),$$

where $\delta > 0$ is the subjective discount rate, $\gamma > 0$ is the coefficient of risk aversion, and $\eta_t(z^t)$ is a preference shock whose evolution we describe below.

The stochastic discount factor is the intertemporal marginal rate of subtitution between consumption at dates $t+1$ and t,

$$M^c_{t,t+1} = e^{-\delta} \frac{\eta_{t+1}(z^{t+1})}{\eta_t(z^t)} \left(\frac{C_{t+1}(z^{t+1})}{C_t(z^t)} \right)^{-\gamma}. \tag{20}$$

where, in equilibrium, $C_t(z^t)$ equals the exogenous endowment of consumption.

We look for conditions under which the stochastic discount factor of the general equilibrium model coincides with that of the statistical arbitrage free model given by equation (5). To that end, suppose that the evolution of the preference shock $\eta_t(z^t)$ depends only on the discrete Markov chain process s_t and satisfies

$$\log(\eta_{t+1}(z^{t+1})) = \log(\eta_t(z^t)) - \Gamma(s_t, s_{t+1}), \tag{21}$$

with initial condition $\eta_0(z_0) = 1$.

In addition, if we let $c_t(z^t) = \log C_t(z^t)$, the stochastic discount factor (20) becomes

$$M^c_{t,t+1} = e^{-\delta - \Gamma(s_t,s_{t+1}) - \gamma\left[c_{t+1}(z^{t+1}) - c_t(z^t))\right]}.$$

For the general equilibrium model to map exactly into the statistical arbitrage-free model discussed above, we need that the two stochastic discount factors coincide: $M_{t,t+1} = M^c_{t,t+1}$. This coincidence happens whenever log-consumption growth follows the stochastic process

$$c_{t+1}(z^{t+1}) - c_t(z^t) = \frac{1}{\gamma}\left[r_t - \delta + \frac{1}{2}\Lambda_t^{s_{t+1}\prime}\Lambda_t^{s_{t+1}} + \Lambda_t^{s_{t+1}\prime}\epsilon_{t+1} \right],$$

where r_t and $\Lambda_t^{s_{t+1}}$ are given by equations (4) and (6), respectively.

4 Model specification and estimation

In this section, we discuss the details of the estimation of the dynamic model of bond pricing. We specify three observable factors constructed from bond yields. The factors are the usual level, (the negative of the) slope, and curvature of the yield curve. The level of the yield curve is defined as the 10-year yield, the slope is the ten-year yield minus the three-month yield, and the curvature is twice the two-year yield minus the sum of the ten-year and three-month yields. That is, the risk factors are

$$X_t = \left[y_t^{120},\ -(y_t^{120} - y_t^3),\ 2y_t^{24} - y_t^{120} - y_t^3 \right]'.$$

Bond yields y_t^n other than those used to construct the factors are observed for the maturities $n \in N = \{6, 12, 36, 60, 108\}$ months.

In our baseline model, the risks associated with changes in regimes are priced, in that transitioning between regimes (even remaining in the current one) have an associated discount $\Gamma(s_t, s_{t+1})$ that affects bond prices, as shown in equation (5). As we show below, allowing for priced regime-specific risks affect the estimated separation of the regimes through the impact that changes in the stochastic discount factor have on bond prices. To assess the contribution of each component of the model on bond prices and risk premia, we also estimate a version of the model assuming that regime changes are not priced (imposing $\Gamma(s_t, s_{t+1}) = 0$), and a single regime affine model.

We parameterize the model and evaluate the log-likelihood function in terms of the parameters of the risk neutral and physical measures. Given a value for these parameters, we recover λ_0, λ_1, and Γ using that the risk neutral and physical measures are related by the equations

$$\begin{aligned}
\lambda_0^k =& \Sigma_k^{-1} \left(\mu_k - \mu_k^Q \right) \\
\lambda_1^k =& \Sigma_k^{-1} \left(\Phi_k - \Phi_k^Q \right) \\
\Gamma_{jk} =& \log \left(\pi_{jk} / \pi_{jk}^Q \right).
\end{aligned}$$

Next, we use the evidence documented in Section 2 to discipline our preferred parametrization of the baseline model. As noted there, there are significant changes in the mean of the level factor, consistent with the findings in Hamilton (1988). In addition, periods of low levels of the slope factor tend to coincide with periods of high volatility. Accordingly, we impose that there are four different regimes derived from the interaction of two possible values for the mean of the level factor and two possible regimes for the slope factor charaterized, the first, by low mean and high volatility, and the second, by high mean and low volatility. For parsimony, we assume that changes in regimes of the level factor are independent from those of the slope factor. Therefore, we parameterize the 4×4 Markov transition matrix as

$$\Pi = \Pi_L \otimes \Pi_S,$$

where $\otimes$ denotes the Kronecker product, and Π_L and Π_S are 2×2 transition matrices for the discrete changes in the level and slope factors, respectively, given by

$$\Pi_L = \begin{bmatrix} q_L & 1-q_L \\ 1-p_L & p_L \end{bmatrix} \quad \text{and} \quad \Pi_S = \begin{bmatrix} q_S & 1-q_S \\ 1-p_S & p_S \end{bmatrix}.$$

We also assume that the transition matrix of the regimes under the risk-neutral measure inherits the same properties, so that $\Pi^Q = \Pi_L^Q \otimes \Pi_S^Q$, where Π_L^Q and Π_S^Q have the same structure as Π_L and Π_S.

In addition, as shown in Table 1, we do not find evidence that the coefficients on the lagged values of the factors X_t shift with changes in regimes. Therefore, we assume that the coefficient matrix Φ in the process (3) for the risk factors is the same for all regimes. Likewise, for parsimony, when estimating the model we also assume that the coefficient matrix Φ^Q under the risk-neutral measure is also state-independent but allow the risk-neutral drift μ_k^Q to depend on the regime $k = 1, 2, 3, 4$.[6] Dai, Singleton and Yang (2007) use a similar assumption.

Let $\theta_{s_t} = \left[\theta_{s_t}^L, \theta_{s_t}^S, \theta_{s_t}^C\right]'$ denote the vector of long-run means of X_t conditional on regime s_t. We assume that $\theta_{s_t}^L$ and $\theta_{s_t}^S$ can take two possible values each, and that $\theta_{s_t}^C = \theta^C$ is a constant. The parameters that we allow to change across regimes are, therefore, the long-run mean of the level factor, $\theta_{s_t}^L \in \left\{\theta_1^L, \theta_2^L\right\}$; and the long-run mean of the slope factor, $\theta_{s_t}^S \in \left\{\theta_1^S, \theta_2^S\right\}$, together with the conditional volatility matrix $\Sigma_{s_t} \in \{\Sigma_1, \Sigma_2\}$.[7] More specifically, the parameters associated with the four regimes are as follows,

$$\begin{aligned} \text{regime 1} &= \left\{\theta_1^L, \theta_1^S, \Sigma_1, \mu_1^Q, \Gamma_{1j}\right\} \\ \text{regime 2} &= \left\{\theta_1^L, \theta_2^S, \Sigma_2, \mu_2^Q, \Gamma_{2j}\right\} \\ \text{regime 3} &= \left\{\theta_2^L, \theta_1^S, \Sigma_1, \mu_3^Q, \Gamma_{3j}\right\} \\ \text{regime 4} &= \left\{\theta_2^L, \theta_2^S, \Sigma_2, \mu_4^Q, \Gamma_{4j}\right\}. \end{aligned}$$

The assumption about the separation of regimes—namely, that there are only 4 free parameters in the Markov transition matrix under, both, the physical and risk neutral measures Π and Π^Q—implies that there are only 4 free parameters in the 4×4 matrix of market prices of regime switch Γ since $\Gamma_{jk} = \log\left(\pi_{jk} / \pi_{jk}^Q\right)$.

To identify the model we also impose the following assumptions. First, since the term $B_k^{n\prime}\mu_k^Q$ that appears in equation (10) is a scalar for each k, we can only identify a single parameter in μ_k^Q. We thus set $\mu_k^Q = \left[\mu_k^{Q,L}, 0, 0\right]'$ for $k = 1, 2, 3, 4$. And second, we impose

[6] By a standard result, the volatility matrix Σ_k is the same in the physical and risk-neutral measures.

[7] To estimate the model, we find it convenient to parameterize the evolution of the risk factors X_t in terms of the long-run mean θ_{s_t} instead of the intercept μ_{s_t} as in equation (3). The mapping between these two parameters is $\mu_{s_t} = (I - \Phi)\theta_{s_t}$.

that the matrix Φ^Q has the following structure

$$\Phi^Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \phi_1^Q & \phi_2^Q \\ 0 & 0 & \phi_1^Q \end{bmatrix}.$$

Hamilton and Wu (2012) show that these restrictions imply that the model is identified with a well behaved likelihood function.

The previous assumptions imply that we estimate the model

$$y_t^n = a_{s_t}^n + b^{n\prime} X_t + \nu_t \text{ for } n \in N, \tag{22}$$

$$X_{t+1} = \mu_{s_{t+1}} + \Phi X_t + \Sigma_{s_{t+1}} \epsilon_{t+1}, \tag{23}$$

where $y_t^n \in \{y_t^6, y_t^{12}, y_t^{36}, y_t^{60}, y_t^{108}\}$ is the set of observed yields, X_t is the vector of observed factors, $\nu_t \in \Re^5$ is a normally distributed measurement error with mean zero and 5×5 diagonal covariance matrix H; $\mu_{s_t} \in \Re^3$ is a vector of regime-specific drifts, Φ is an 3×3 matrix, and $\epsilon_{t+1} \in \Re^3$ is normally distributed with mean zero and 3×3 covariance matrix I_3. Moreover, ν_t and ϵ_{t+1} are independent of each other at all leads, lags, and contemporaneously for all possible regimes s_t. In addition, the intercept and factor loadings of the observation equation are given by $a_{s_t}^n = -A_{s_t}^n/n$, $b^n = -B^n/n$, where $A_{s_t}^n$ and B^n satisfy the recursions (15) and (16).

Let $\mathcal{Y}^t = \{X_1, X_2, \ldots, X_t, y_1^n, y_2^n, \ldots y_t^n \text{ for } n \in N\}$ denote the history of observed data up until time t. Given data $\mathcal{Y}^T$, we estimate the model by the method of maximum likelihood and perform statistical inference using an algorithm and filter similar to that described in Hamilton (1994, pp. 692-694). As a by-product of the filter, we compute the filtered probability of the unobserved state s_t conditional on all the information up to time t, $\Pr\left(s_t|\mathcal{Y}^t\right)$. The consistency and asymptotic normality of the maximum likelihood estimator of the model with Markov switching are discussed in Francq and Roussignol (1998), Krishnamurthy and Ryden (1998), and Douc, Moulines and Rydén (2004).

4.1 Estimation results

Table 3 shows estimation results for the baseline model with priced regime shifts. The table shows the estimated parameters of the physical and risk neutral measures as well as the implied matrix of market prices of regime shifts parameters Γ. Table 4 shows the same results for the model with non-priced regime shifts, $\Gamma_{jk} = 0$, so that $\Pi = \Pi^Q$.[8]

[8] The results for the single-regime model are available upon request

Table 3: **Model with priced regime shifts**

Coefficient matrix on lagged values:

$$\Phi = \begin{bmatrix} 1.0020 & 0.0301 & -0.0172 \\ (0.0019) & (0.0082) & (0.0108) \\ -0.0162 & 0.8991 & 0.0863 \\ (0.0051) & (0.0118) & (0.0157) \\ -0.0008 & 0.0140 & 0.9118 \\ (0.0031) & (0.0121) & (0.0144) \end{bmatrix} \qquad \Phi^Q = \begin{bmatrix} 1 & 0 & 0 \\ & & \\ 0 & 0.9349 & 0.1055 \\ & (0.0004) & (0.0011) \\ 0 & 0 & 0.9349 \\ & & (0.0004) \end{bmatrix}$$

Volatility matrices:

$$\Sigma_1 = \begin{bmatrix} 0.0023 & 0 & 0 \\ (0.0000) & & \\ -0.0015 & 0.0020 & 0 \\ (0.0001) & (0.0001) & \\ 0.0004 & 0.0002 & 0.0027 \\ (0.0001) & (0.0001) & (0.0001) \end{bmatrix} \qquad \Sigma_2 = \begin{bmatrix} 0.0036 & 0 & 0 \\ (0.0001) & & \\ 0.0023 & 0.0183 & 0 \\ (0.0020) & (0.0008) & \\ -0.0011 & -0.0026 & 0.0078 \\ (0.0008) & (0.0008) & (0.0006) \end{bmatrix}$$

Long-run mean parameters:

$$\theta_1^L = 0.1354\,(0.0388) \qquad \theta_2^L = 0.0484\,(0.0300)$$
$$\theta_1^S = -0.0220\,(0.0056) \qquad \theta_2^S = -0.0136\,(0.0096)$$
$$\theta^C = -0.0044\,(0.0020)$$

Probabilities of regime shift:

$$p_L = 0.9561\,(0.0164) \qquad p_L^Q = 0.9725\,(0.0094)$$
$$q_L = 0.9565\,(0.0168) \qquad q_L^Q = 0.9924\,(0.0102)$$
$$p_S = 0.9740\,(0.0090) \qquad p_S^Q = 0.7971\,(0.0301)$$
$$q_S = 0.9649\,(0.0125) \qquad q_S^Q = 0.9587\,(0.0227)$$

Parameters of the drift μ_k^Q under the risk neutral measure:

$$\mu_1^{Q,L} = 0.0001\,(0.0000) \qquad \mu_2^{Q,L} = 0.0010\,(0.0001)$$
$$\mu_3^{Q,L} = -0.0007\,(0.0000) \qquad \mu_4^{Q,L} = 0.0016\,(0.0001)$$

Log-likelihood: -31844.2634.

Implied market prices of regime shifts and expected regime discount:

$$\Gamma = \begin{bmatrix} 0.1835 & -0.1818 & 1.9417 & 1.5764 \\ -2.0733 & -0.0104 & -0.3151 & 1.7477 \\ 0.6668 & 0.3015 & 0.1636 & -0.2017 \\ -1.5900 & 0.4729 & -2.0932 & -0.0303 \end{bmatrix} \qquad \begin{bmatrix} \sum_j \pi_{1j}\Gamma_{1j} \\ \sum_j \pi_{2j}\Gamma_{2j} \\ \sum_j \pi_{3j}\Gamma_{3j} \\ \sum_j \pi_{4j}\Gamma_{4j} \end{bmatrix} = \begin{bmatrix} 0.1461 \\ 0.0048 \\ 0.1825 \\ 0.0411 \end{bmatrix}$$

Table 4: **Model without priced regime shifts**

Coefficient matrix on lagged values:

$$\Phi = \begin{bmatrix} 0.9952 & 0.0238 & -0.0120 \\ (0.003) & (0.007) & (0.01) \\ -0.0280 & 0.9068 & 0.1212 \\ (0.007) & (0.012) & (0.08) \\ 0.0313 & 0.0579 & 0.8369 \\ (0.007) & (0.013) & (0.018) \end{bmatrix} \qquad \Phi^Q = \begin{bmatrix} 1 & 0 & 0 \\ & & \\ 0 & 0.9319 & 0.1062 \\ & (0.0005) & (0.0016) \\ 0 & 0 & 0.9319 \\ & & (0.0005) \end{bmatrix}$$

Volatility matrices:

$$\Sigma_1 = \begin{bmatrix} 0.002 & 0 & 0 \\ (0.0000) & & \\ -0.0015 & 0.0024 & 0 \\ (0.0001) & (0.0001) & \\ 0.0001 & 0.0007 & 0.0029 \\ (0.0001) & (0.0001) & (0.0001) \end{bmatrix} \quad \Sigma_2 = \begin{bmatrix} 0.0036 & 0 & 0 \\ (0.0001) & & \\ 0.0010 & 0.0139 & 0 \\ (0.0011) & (0.0009) & \\ -0.0002 & 0.0038 & 0.0074 \\ (0.0006) & (0.0007) & (0.0005) \end{bmatrix}$$

Long-run mean parameters:

$\theta_1^L = 0.1102\,(0.0204)$ $\quad$ $\theta_2^L = 0.0482\,(0.0113)$

$\theta_1^S = -0.0153\,(0.0044)$ $\quad$ $\theta_2^S = -0.0071\,(0.0075)$

$\theta^C = -0.0050\,(0.0041)$

Probabilities of regime switch:

$p_L = 0.9996\,(0.0003)$

$q_L = 0.9345\,(0.0043)$

$q_S = 0.9997\,(0.0002)$

$p_S = 0.9496\,(0.0087)$

Parameters of the drift μ_k^Q under the risk neutral measure:

$\mu_1^{Q,L} = 0.0004\,(0.0000)$ $\quad$ $\mu_2^{Q,L} = 0.0009\,(0.0001)$

$\mu_3^{Q,L} = -0.0002\,(0.0000)$ $\quad$ $\mu_4^{Q,L} = 0.0014\,(0.0001)$

Log-likelihood: -31539.8031.

Figure 2 shows, in the top panel, the evolution of the level, slope, and curvature of the yield curve and, in the middle and bottom panels, the separation of regimes implied by the models. The blue line in the middle panel represents the estimated smoothed probability of low slope and high volatility (regimes 2 or 4) while the blue line in the bottom panel is the smoothed probability of high long-run mean (regimes 1 or 2). Likewise, the orange lines in the middle and bottom panels represent the equivalent smoothed probabilities for the restricted model without priced regime shifts.

The combination of these two possibilities gives rise to the four different regimes: regime 1 represents periods with high average interest rates, low slope, and low volatility; regime 2 are periods with high average interest rates, high slope, and high volatility; regime 3 are periods with low average interest rates, low slope and low volatility; and regime 4 are periods with low average interest rates, high slope, and high volatility.

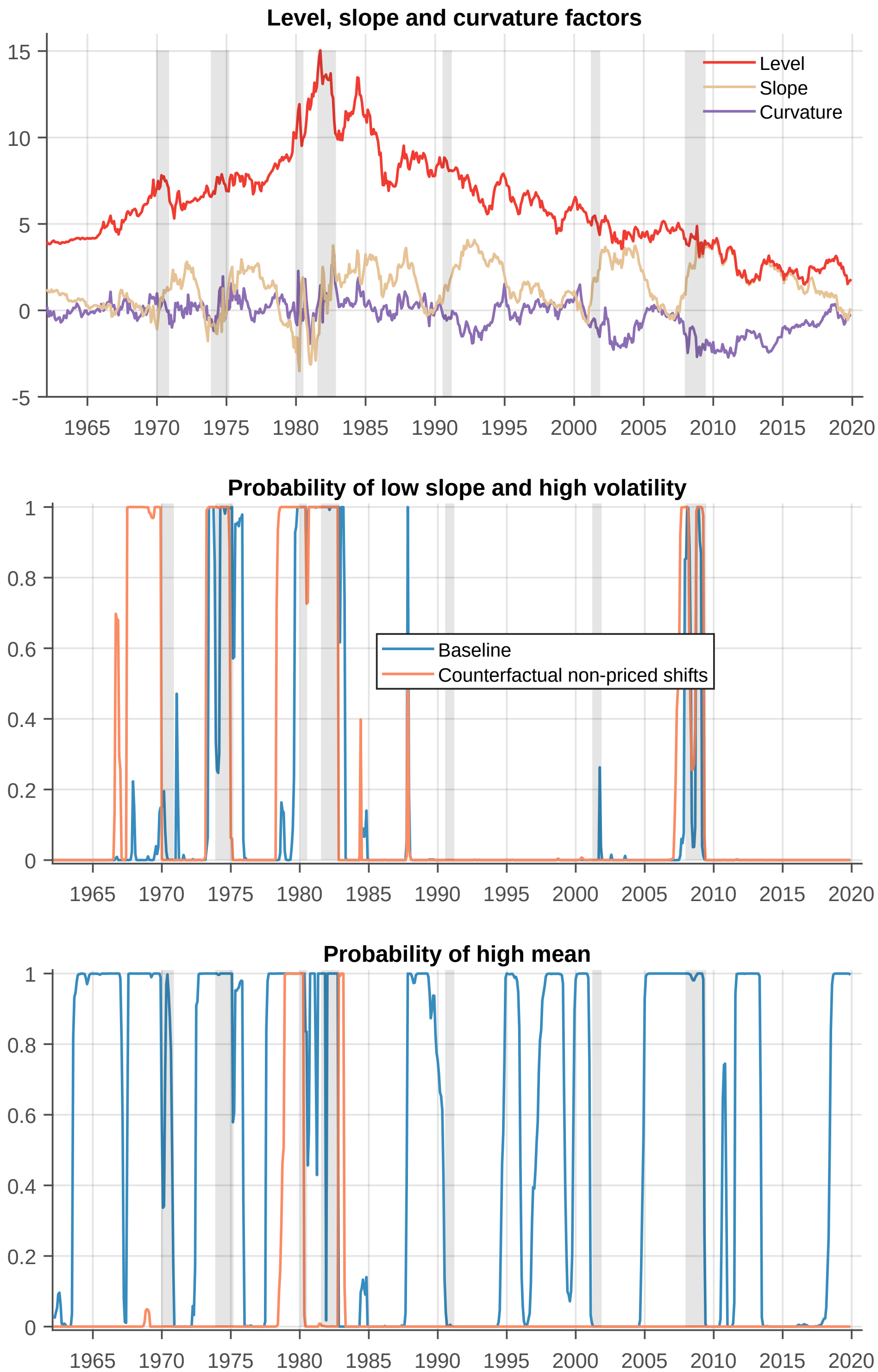
The separation of the model with priced regime shifts associates most of the high-volatility and low-slope regime with recessions, represented by the shaded gray areas in the plot.[9] Both models identify periods with high slope and high volatility (regimes 2 and 4). Moreover, the baseline model also identifies several periods over the entire sample of high mean for the level factor (regimes 1 and 2). In contrast, the model without priced regime shifts only associates periods of high interest rates with the Volcker disinflation period between the late 1970s and early 1980s.

As the figure shows, allowing for priced regime shifts ($\Gamma_{jk} \neq 0$) affects the estimated separation of the regimes through the impact that changes in the stochastic discount factor have on bond prices. To see this, the bottom part of Table 3 shows, in the left, the impliced matrix of market prices of regime shifts, Γ_{jk} and, in the right, the expected regime switching discount conditional on the current regime s_t. The table shows that periods of low slope and high volatility (regimes 1 and 3) have a higher expected discount than periods with high slope and low volatility (regimes 2 and 4) independently of the level of the yield curve. Therefore, on average, periods of low slope and high volatility of the yield curve (regimes 1 and 3) have lower bond prices and, therefore, higher returns, than periods with high slope and low volatility (regimes 2 and 4).

Since the model with non-priced regime shifts is a special case of the baseline model, we can use a likelihood ratio test to assess which model is favored by the data. The model without priced regime shifts has $\Gamma_{jk} = 0$ for all j, k, which implies that $\pi^{Q}_{jk} = \pi_{jk}$ for all j, k. This requirement imposes 4 restrictions into the baseline model. Hence, the likelihood ratio test is distributed as a chi-square with 4 degrees of freedom. Since the statistic value is $2(31844 - 31539) = 610$, we strongly reject the null hypothesis of

[9]Shaded areas are NBER recessions dates.

Figure 2: **Factors and smoothed probabilities from the baseline pricing model**



Note: The upper panel displays the evolution of the empirical level, slope, and curvature of the yield curve. The middle panel shows the smoothed probability of low slope and high volatility regimes for the baseline model and for the model without priced regime shifts. The bottom panel reports the probability of high long-run mean of the level factor for the baseline model and for the model without priced regime shifts.

non-priced regime shifts.

5 Risk premia

The measure of bond risk premia that we use is the expected one-month excess return of borrowing at the three-month rate to buy an n-period bond and selling it in one period, which we denote by

$$E_t[xr^n] = E_t[p_{t+1}^{n-1}] - p_t^n - y_t^3.$$

Here, $xr^n = p_{t+1}^{n-1} - p_t^n - y_t^3$ is the realized one-month holding return of an $n-$period bond in excess of the $3-$month interest rate.[10] Thus, to estimate bond risk premia, we need to compute the time-t expected bond price in period $t+1$. Computing this expectation depends critically on the properties of the model used to price the bonds.

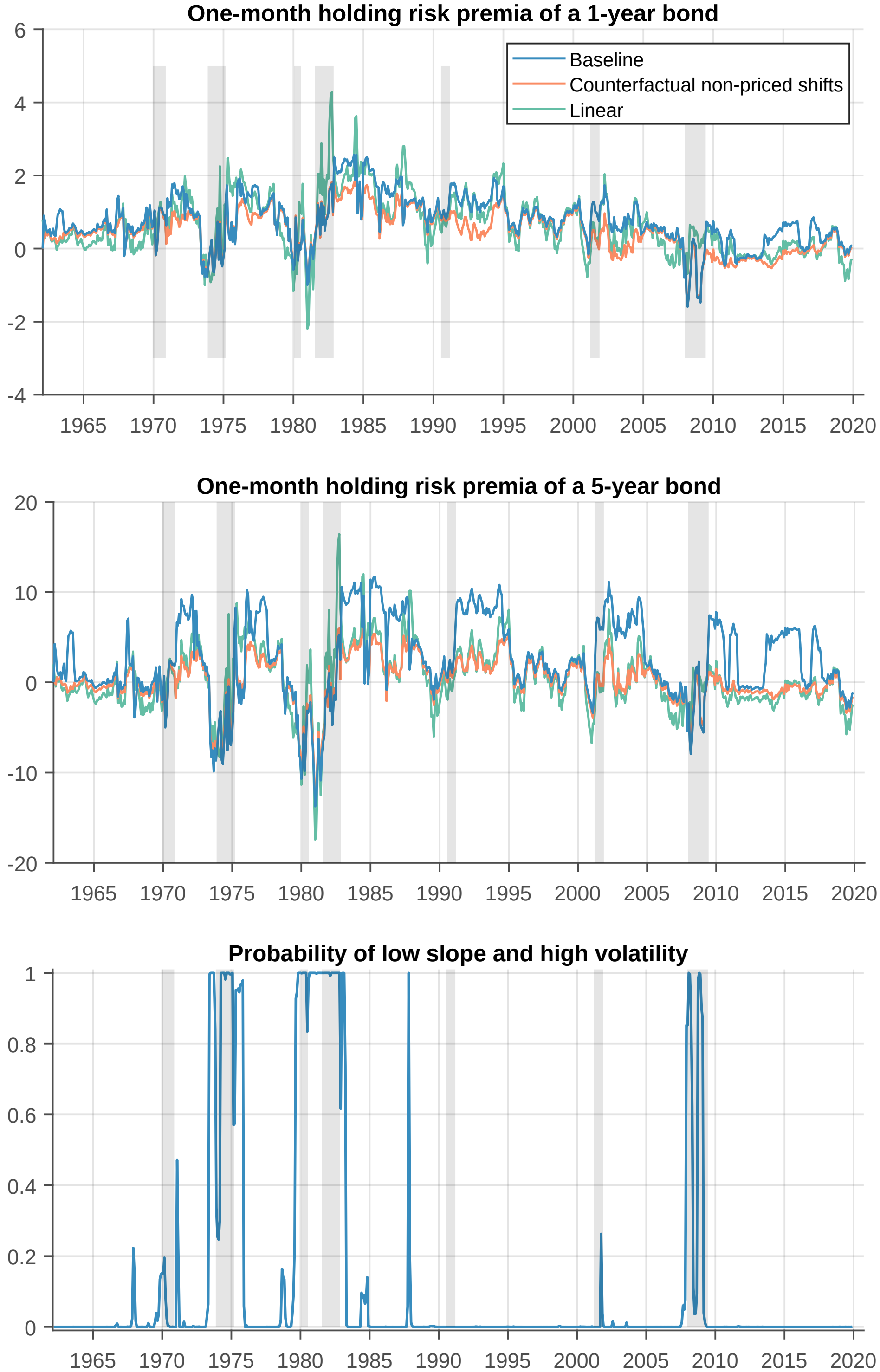
We showed previously the empirical relevance of the three main components of our baseline model: (a) the long-run mean of interest rates and the slope of the yield curve are subject to occasional discrete shifts, (b) the volatility of the level, slope, and curvature of the yield curve are also subject to discrete changes, and (c) these discrete changes in regimes are priced risk factors that affect bond prices. We next analyze how these components impact the evolution of bond risk premia. Since the estimated risk premia depends on the properties of the model under consideration, our evidence in favor of the model with priced regime shifts presumably translates into more reliable measure of the implied bond risk premium.

Figure 3 shows the estimated evolution of the risk premium of a 1-year bond (top panel) and of a 5-year bond (middle panel) for the baseline model, the single regime model, and for the model without priced regime shifts. In addition, the bottom panel of the figure displays the estimated smoothed probability of low slope and high volatility regimes from the baseline model together with shaded areas that represent NBER recessions dates.

Estimated risk premia in the linear model and in the switching model without priced regime shifts are similar across time and bonds. In fact, this similarity is more evident when considering long maturity bonds. This observation reflects the fact that, when computing the expected bond price several years into the future, the Markov regimes average out and the results resemble those of the single regime model. In contrast, the

[10]We define the expected excess returns relative to the three-month rate instead of the one-month rate because that is the shortest maturity bond that we have in our database.

Figure 3: **Factors and smoothed probabilities from the baseline pricing model**



Note: The upper panel displays the 1-month expected holding return of a 1-year bond in excess of the return of a 3-month bond. The middle panel the 1-month expected return of holding a 5-year bond in excess of that of a 3-month bond. The bottom panel displays the smoothed probability of low slope and high volatility regimes for the baseline model.

estimated risk premium in the baseline model with priced regime shifts is substantially different, both quantitatively and qualitatively, from those of the other two models, a difference that is more stricking for long-maturity bonds. In particular, we observe to important differences between the models. The first is that the estimated risk premium in the baseline economy is, on average, larger than that of the other models for long-maturity bonds. The second observation is that the estimated risk premia in the linear model and in the model without priced regime shifts are more volatile than that of the baseline economy during recessions. While the estimated risk premium in the first two models is subject to sudden and long swings, which seem to be particularly stricking at the end of recessions, the measure of risk premia in the baseline model moves less abruptly and in a tighter range. In effect, there are periods in which the risk premium barely moves in the baseline economy. Finally, note that periods of high volatility and low slope, which often coincide with recessions, are also periods in which the bond holding risk premium is low and typically negative, meaning that recessions tend to be periods in which holding short maturity bonds seems to be riskier than holding long maturity bonds.

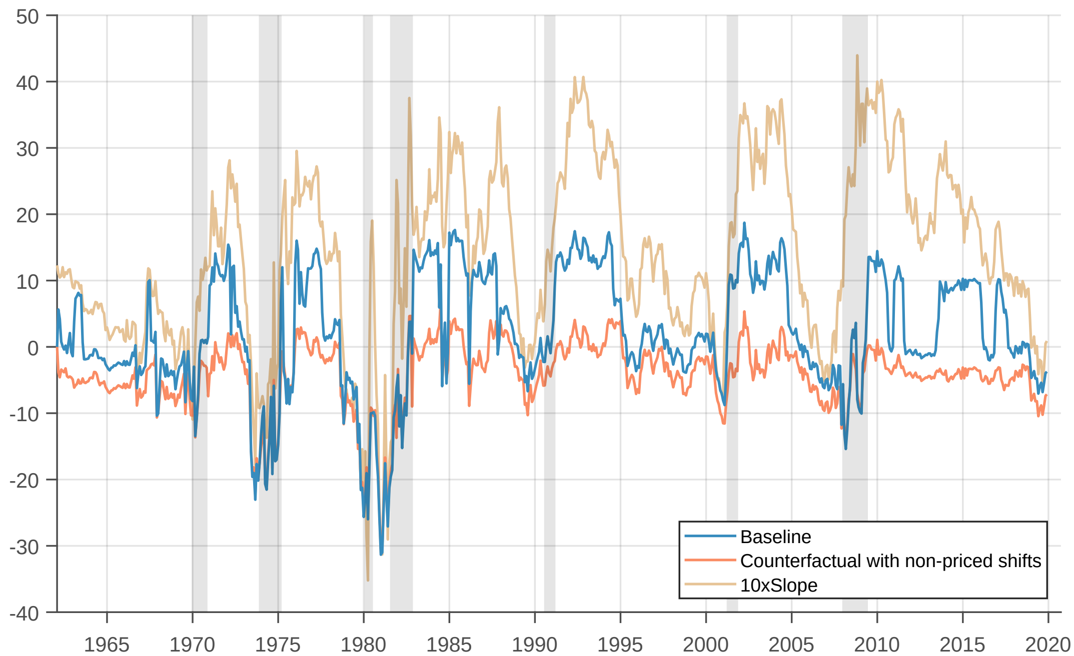
To isolate the contribution of the different components of the stochastic discount factor (5) to the expected risk premium, we construct a counterfactual pricing model in which we set to zero the price of regime switching risk (by setting $\Gamma_{s_t,s_{t+1}} = 0, \forall s_t, s_{t+1}$) but keep the other estimated parameters of the baseline model. We fix the other parameters of the baseline economy to compute the partial effect of the market price of regime shifts on bond risk premia.[11] We denote by $E_t[xr^n_{t+1,t}|\Gamma = 0]$ the expected excess return of an $n-$period bond in the counterfactual economy that does not price regime shifts. We interpret this expected excess return as capturing the contribution of the X_t risk factors on bond risk premia ignoring the contribution of the priced regime switching risks.

Figure 4 shows the spread between the risk premia of holding a ten-year bond and the risk premia of holding a six-month bond for the baseline economy and for the counterfactual economy without priced regime shifts, along with the slope of the yield curve. We draw two insights from this figure. First, movements of the spread in expected holding returns are much more prominent for the baseline model than for the counterfactual economy. In a risk-neutral world, the spread is zero. In the counterfactual economy only the risk factors X_t are priced while in the baseline economy we consider all the priced risks. And second, the slope mimics the evolution of the spread of the risk premia for

[11] Had we used the estimated model with $\Gamma_{ij} = 0$, we would be confounding the impact of the market price of regime shifts with the fact that all the other estimated parameters, such as those that determine the evolution of the stochastic process for X_t, and the separation of the regimes, would be different from those of the baseline economy.

the baseline model, which reinforces the insight that holding short-term bonds is riskier in recession when, the yield curve is inverted.

Figure 4: **Slope factor and spread of expected returns of a 10-year and a 6-month bond**



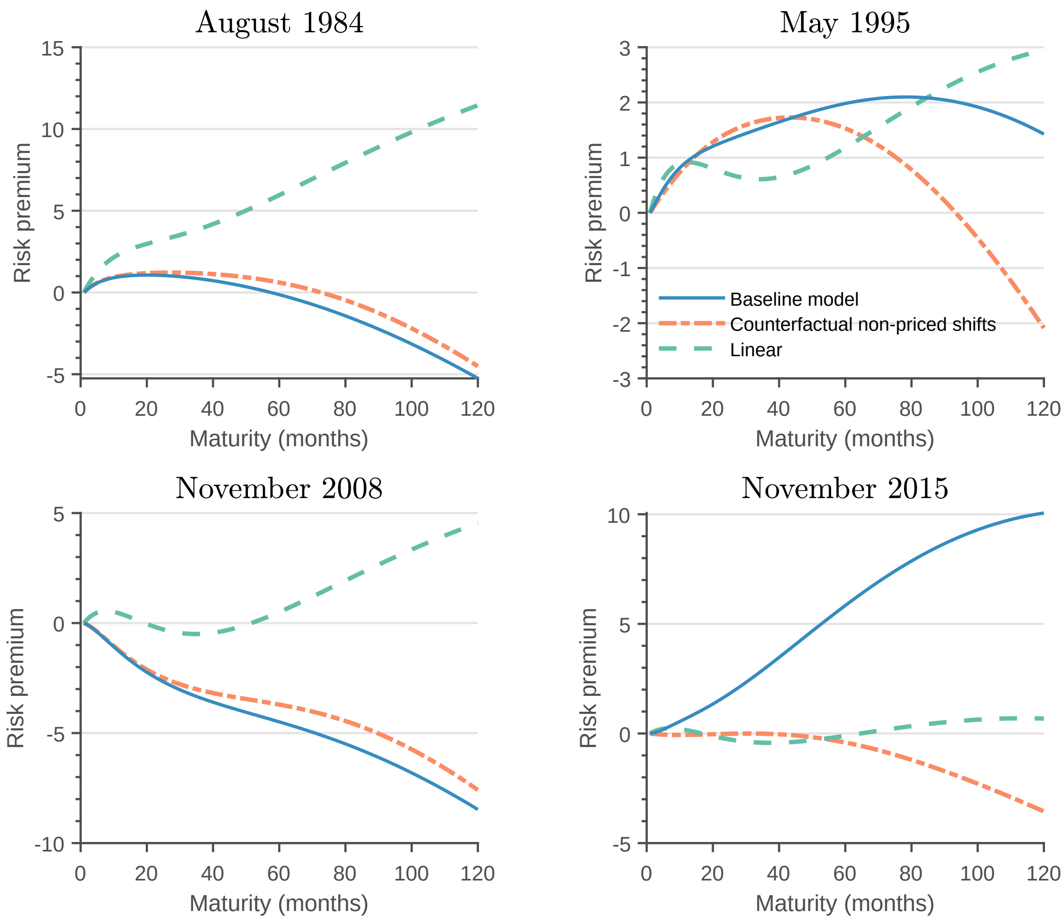
Note: The figure shows 10 times the slope of the yield curve (in blue) and the spread between the expected 1-month return of holding a 10-year bond in excess of the return of a 6-month bond both for the baseline model (in blue) and for the counterfactual economy with non-priced regime shifts (in orange).

To further analyze the difference in risk premia during volatile or calm times, we show in Figure 5 the term structure of the estimated bond holding risk premia on four different dates. The left panels show the term structure evaluated at dates of high volatility while the right panels show two periods of low volatility. In the baseline model, during volatile times, the premium is mostly negative or close to zero, an observation that does not hold in the linear model. In contrast, in periods of low volatility the premium tend to increase with the bond maturity while the shape of the curve using the counterfactual economy or the linear model is quite different.

5.1 Time varying risk premia and the macroeconomy

In this section, we investigate the relation between bond risk premia and macroeconomic fundamentals. Traditional term structure models usually do not take a stance on

Figure 5: **Cross section of bond holding risk premia**



Note: The figure shows the term structure of bond-holding risk premia as a function of the bond maturity for selected dates. The different lines represent the cross section of risk premia for the baseline economy (in blue), for the counterfactual economy without priced shifts (orange) and for the linear model (green).

the relationship between bond yields and macroeconomic variables (but see, for example, Ang and Piazzesi (2003) and Diebold, Rudebusch and Aruoba (2006) which include macro variables as risk factors). Yet, the ultimate goal of the literature is to relate yields and risk premia to macroeconomic fundamentals. As Duffee (2011) put it: *"the Holy Grail of the term structure liturature is a testable, intuitive model linking yields to fundamental macroeconomic forces."*

We find that measures of bond risk premia derived from the baseline model with priced regime shifts are highly correlated with inflation and with an indicator of economic activity, such as the cyclical component of industrial production or unemployment.[12] Predictive regressions for expected excess bond returns derived from the baseline model on these macroeconomic variables yield highly significant coefficients with R^2 as high as 0.30 for longer maturity bonds. In contrast, the same regressions for the single regime model and for the regime switching model without priced regime shifts yield much lower $R^2 s$ and less significant coefficients.

Table 5 show the results of the predictive regressions for expected one month excess bond holding returns on inflation and the cyclical component of industrial production.[13] Each column show the regression results for bonds of different maturities, measured in months. Inflation is the year-on-year log-difference in the consumer price index.[14] As for the cyclical component of industrial production, we subtract from each series its trend component following the procedure suggested by Hamilton (2018).[15] The table show results for the linear single-regime model, for the model with regime switching but in which regime shifts are non-priced, and for the baseline model with priced regime shifts. For the single regime model, the link between expected excess returns and the macroeconomic variables is tenuous, a well-known finding in the literature (Duffee, 2011, 2013). In all cases, macro variables explain at most 10% of the variation of expected excess returns across maturities and the only significant predictor for excess returns is industrial production for long-maturity bonds. In the model with regime switching but non-priced regime shifts none of the macroeconomic variables are statistically significant although the R^2 are somewhat larger than in the linear model. In contrast, the baseline model with priced regime shift show highly significant coefficients, especially for bonds of medium to long maturities. Increases in both, industrial production and inflation are negatively correlated with expected excess holding returns, with R^2s as high as 0.31 for one-month holding returns. Thus, expected excess returns tend to be countercyclical, decreasing when both, inflation and industrial production are above trend.

[12] Andreou et al. (2017) show that industrial production is strongly correlated with overall business cycle conditions.

[13] In Appendix B we show the results of using the unemployment rate as the measure of economic activity.

[14] Using the deflator for personal consumption expenditures gives similar results.

[15] The trend component is defined by a regression of the variable at date t on its four most recent values.

Table 5: Expected excess log-holding returns and macroeconomic fundamentals

	6	12	24	36	48	60	72	84	96	108	120
Single regime											
Industrial production	-0.012*	-0.024	-0.046	-0.070	-0.095*	-0.121*	-0.149**	-0.177**	-0.205**	-0.233**	-0.261***
	(0.006)	(0.015)	(0.031)	(0.044)	(0.055)	(0.063)	(0.071)	(0.078)	(0.085)	(0.092)	(0.099)
Inflation	0.033*	0.024	-0.051	-0.132	-0.200	-0.255	-0.302	-0.343	-0.383	-0.421	-0.458
	(0.019)	(0.043)	(0.093)	(0.136)	(0.171)	(0.202)	(0.229)	(0.255)	(0.280)	(0.305)	(0.330)
R^2	0.103	0.043	0.030	0.044	0.055	0.064	0.070	0.076	0.080	0.083	0.086
RS No Priced Risk											
Industrial production	-0.007	-0.019	-0.048	-0.077	-0.105	-0.131	-0.157	-0.183	-0.209	-0.236	-0.264
	(0.006)	(0.015)	(0.038)	(0.058)	(0.075)	(0.087)	(0.098)	(0.108)	(0.117)	(0.127)	(0.137)
Inflation	0.056	0.015	-0.194	-0.408	-0.576	-0.705	-0.810	-0.903	-0.992	-1.083	-1.179
	(0.017)	(0.041)	(0.104)	(0.163)	(0.212)	(0.251)	(0.286)	(0.319)	(0.351)	(0.384)	(0.419)
R^2	0.213	0.032	0.099	0.161	0.187	0.198	0.202	0.204	0.205	0.206	0.208
RS with Priced Risk											
Industrial production	0.000	-0.011	-0.048	-0.095**	-0.144**	-0.193***	-0.238***	-0.280***	-0.317***	-0.351***	-0.381***
	(0.006)	(0.014)	(0.030)	(0.045)	(0.059)	(0.073)	(0.086)	(0.099)	(0.111)	(0.123)	(0.134)
Inflation	0.022	-0.014	-0.156**	-0.348***	-0.562***	-0.784***	-1.008***	-1.230***	-1.447***	-1.658***	-1.864***
	(0.015)	(0.036)	(0.080)	(0.124)	(0.166)	(0.208)	(0.248)	(0.288)	(0.326)	(0.363)	(0.399)
R^2	0.043	0.010	0.089	0.157	0.204	0.237	0.261	0.279	0.294	0.305	0.315

Notes: This table reports linear regressions of the expected log-holding return in excess of the 3-month rate at different maturities for the single regime model (upper panel), the model with regime switching but non-priced regime shifts (middle panel), and the baseline model with priced regime shifts (bottom panel). For each regressor we report in parenthesis HAC standard errors following the bootstrap procedure proposed by Bauer and Hamilton (2017). $^{***}/^{**}/^{*}$ denote significant at the 1/5/10% level.

To investigate the source of the variation in expected excess returns, we next analyze the relationship between the macroeconomic variables and key objects in the model that determine bond risk premia, such as the expected market price of regime shifts, given by $E_t[\Gamma_{s_t,s_{t+1}}]$, and the expected market price of the X_t risk factors, captured by $E_t[\Lambda_t^{s_{t+1}}]$. In both cases, the expectations are conditional on the information available up until time t, which we compute using the estimated filtered probability distributions obtained as a by-product of the filtering step used to evaluate the likelihood function. In particular, given the estimates of the model, data up until time t, $\mathcal{Y}^t$, and the filtered probability of the unobserved state s_t, $\Pr\left(s_t|\mathcal{Y}^t\right)$, we compute

$$
\begin{aligned}
E_t[\Gamma_{s_t,s_{t+1}}] &= \sum_{s_t} \Pr\left(s_t|\mathcal{Y}^t\right) \sum_{s_{t+1}} \pi_{s_t s_{t+1}} \Gamma_{s_t,s_{t+1}}, \\
E_t[\Lambda_t^{s_{t+1}}] &= \sum_{s_t} \Pr\left(s_t|\mathcal{Y}^t\right) \sum_{s_{t+1}} \pi_{s_t s_{t+1}} \left(\lambda_0^{s_{t+1}} + \lambda_1^{s_{t+1}} X_t\right).
\end{aligned}
$$

To avoid clutter, in what follows we use the previous expressions but always noting that the parameters (π_{ij}, Γ_{ij} and so on) are all evaluated at their estimated values.

We analyze first the expected market price of regime shift $E_t[\Gamma_{s_t,s_{t+1}}]$. Table 6 show regression results of the expected price of regime shift on the cyclical component of industrial production and inflation, in the first column, and on the cyclical component of unemployment and inflation, in the second column. In both regressions, the macroeconomic variables are highly significant and explain a large fraction, of the order of 30%, of the variability in the expected price of regime shifts over time.

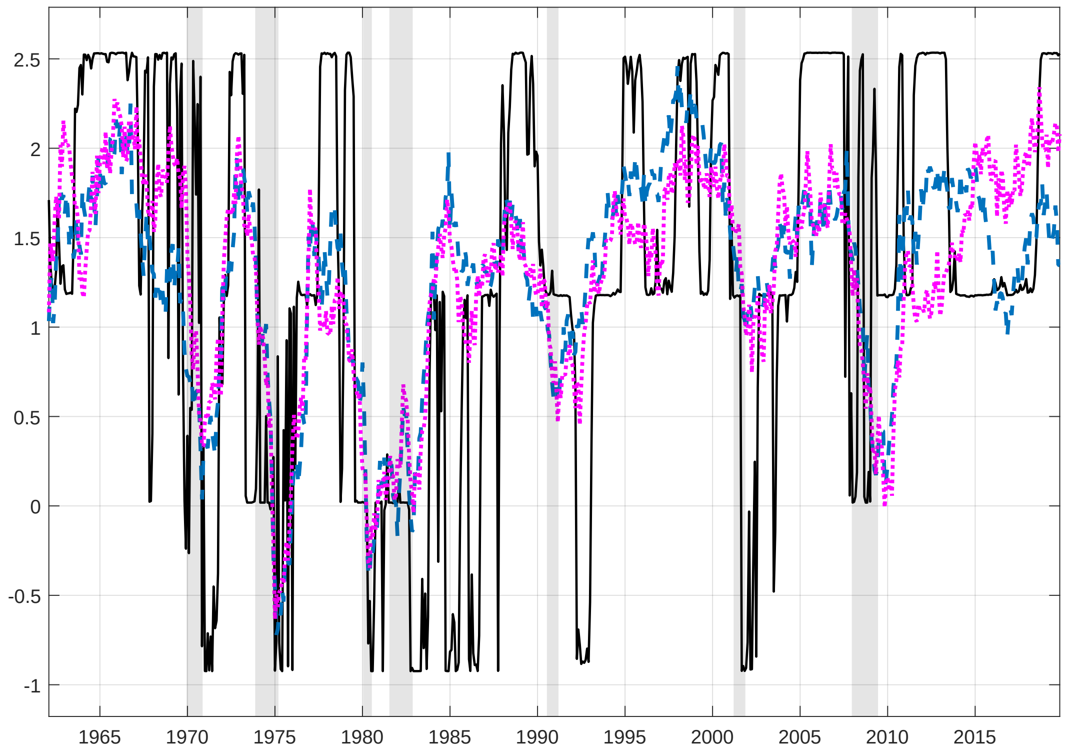
Table 6: Expected price of regime switch and macro fundamentals

	$E_t\Gamma(s_t,s_{t+1})$		$E_t\Gamma(s_t,s_{t+1})$
Ind. Prod.	0.081***	Unemp.	-0.023***
	(0.017)		(0.005)
Inflation	-0.094***	Inflation	-0.084**
	(0.030)		(0.034)
R^2	0.298	R^2	0.304

Notes: This table reports linear regressions of $E_t\Gamma(s_t,s_{t+1})$ and macroeconomic fundamentals. For each regressor we report (in parenthesis) HAC standard errors. ***/**/* denote significant at 1/5/10% level.

In Figure 6 we show the estimated evolution of $E_t[\Gamma_{s_t,s_{t+1}}]$ (solid line) together with the fitted values of the regression of the expected market price of regime shifts on the cyclical component of industrial production and inflation (dashed line) and the fitted value using as correlates the cyclical component of unemployment and inflation. The expected market price of regime shifts is strongly procyclical and a linear combination of the two macroeconomic fundamentals can explain almost a third of its variation. In addition, the series $E_t\Gamma(s_t, s_{t+1})$ tend to have its lowest values in the recessions or immediately after.

Figure 6: **Expected price of regime shifts and macro fundamentals**



Notes: $E_t\Gamma(s_t, s_{t+1})$ (continuous line) and fit associated with macroeconomic fundamentals, industrial production and inflation (broken line) and unemployment and inflation (dotted line). Shadow areas are NBER recessions.

These results show that the expected discounting due to possible changes in regimes are highly correlated with macroeconomic fundamentals. These regressions suggest that investors apply a higher discount due to switching risk when the economy is booming (as captured by the cyclical component of industrial production or unemployment) and a lower discount when inflation is higher. One possible explanation for this result is that, in periods when the return to real projects is low, the expected return of holding a bond is also low, which is reflected in a low value of the regime-specific risk. Those periods are, in turn, associated with high volatility and a low slope of the yield curve.

To isolate the contribution of the different components of the stochastic discount factor (5) to the expected risk premium, we not only consider the counterfactual pricing model $E_t[xr^n_{t+1,t}|\Gamma = 0]$ defined above but also the difference in expected excess returns between the baseline economy and the counterfactual economy, $E_t[xr^n_{t+1,t}] - E_t[xr^n_{t+1,t}|\Gamma = 0]$, as the contribution of discounting the priced regime switching risk on bond risk premia.

In the top panel of Table 7 we show the results of a regression of $E_t[xr^n_{t+1,t}|\Gamma = 0]$ for different maturities n on the macroeconomic fundamentals. We find that, for all the maturities that we consider, the macroeconomic variables are strongly correlated with the expected excess returns: higher inflation and a lower cyclical component of industrial production are both associated with a lower risk premium when setting $\Gamma_{ij} = 0$. Contrast these results with those in the middle panel of Table 5, which show no statistically significant coefficient on any regression. Since the results in the top panel of Table 7 only sets $\Gamma_{ij} = 0$ but keeps all the other parameters at their baseline values, we interpret the lack of statistically significance of the coefficients in the middle panel of Table 5 as reflecting the importance of including pricing regime shifts in the model. Once we include an explicit discounting for regime shift risks, we obtain that the variability in expected excess returns due to fluctuations in the risk factors X_t are highly correlated with macroeconomic fundamentals.

The bottom panel of Table 7, which shows that the component of expected excess returns that are partialed out from the market price of switching risk, show that the component $E_t[xr^n_{t+1,t}] - E_t[xr^n_{t+1,t}|\Gamma = 0]$ remain strongly related to inflation, particularly so for long maturity bonds, in which case macroeconomic fundamentals explain almost 25% of its variation. This result suggests that the discounting due to switching risk is mostly reflecting the impact that inflation has on the price of long maturity bonds besides that captured by fluctuations in the traditional level, slope, and curvature factors, as summarized by the results in the top panel of Table 7.

Table 7: Price of regime switch and macro fundamentals

	6	12	24	36	48	60	72	84	96	108	120
$E_t[rx^h_{t+1,t} \vert \Gamma(s_t, s_{t+1}) = 0]$											
Industrial production	-0.006***	-0.020***	-0.054***	-0.090***	-0.126***	-0.161***	-0.192***	-0.219***	-0.240***	-0.256***	-0.266***
	(0.002)	(0.005)	(0.013)	(0.022)	(0.033)	(0.043)	(0.053)	(0.062)	(0.070)	(0.076)	(0.081)
Inflation	-0.020***	-0.048***	-0.110***	-0.197***	-0.304***	-0.422***	-0.540***	-0.649***	-0.746***	-0.827***	-0.894***
	(0.004)	(0.010)	(0.026)	(0.045)	(0.066)	(0.088)	(0.109)	(0.128)	(0.145)	(0.158)	(0.168)
R^2	0.262	0.235	0.224	0.228	0.235	0.242	0.247	0.252	0.256	0.260	0.264
$E_t rx^h_{t+1,t} - E_t[rx^h_{t+1,t} \vert \Gamma(s_t, s_{t+1}) = 0]$											
Industrial production	0.006	0.009	0.006	-0.005	-0.018	-0.032	-0.046	-0.061	-0.077	-0.095	-0.115
	(0.006)	(0.012)	(0.024)	(0.033)	(0.040)	(0.046)	(0.052)	(0.058)	(0.064)	(0.071)	(0.078)
Inflation	0.043***	0.035	-0.047	-0.151	-0.258*	-0.363**	-0.469***	-0.581***	-0.701***	-0.831***	-0.970***
	(0.014)	(0.033)	(0.071)	(0.104)	(0.133)	(0.158)	(0.182)	(0.206)	(0.230)	(0.255)	(0.281)
R^2	0.161	0.029	0.013	0.055	0.095	0.128	0.156	0.181	0.204	0.225	0.244

Notes: This table reports linear regressions of the a decomposition of expected log-holding return in excess of the short rate at different maturities. For each regressor we report (in parenthesis) HAC standard errors following the boostrap procedure proposed by Bauer and Hamilton (2017). ***/**/* denote significant at the 1/5/10% level. ***/**/* denote significant at 1/5/10% level.

We now turn to the analysis of fluctuations in the variable $\Lambda_t^{s_{t+1}}$ of the stochastic discount factor (5). In the standard affine model of bond prices, the risk premium is a function of the state variables driving fluctuations in Λ_t, which is an affine function of the X_t factors (as in equation (6)) independent of the regime variable s_{t+1}. In contrast, in our baseline model changes in regime also affect the way that innovations to the risk factors X_t discount bond prices. In what follows we analyze to what extent the expected market price of X_t risk, $E_t\left[\Lambda_t^{s_{t+1}}\right]$, is related to the macroeconomic fundamentals. In particular, $\Lambda_t^{s_{t+1}}$ is a time-varying 3×1 vector that capture how innovations to the level, slope, and curvature factors $\epsilon_{t+1} = \left(\epsilon_{t+1}^L \ \epsilon_{t+1}^S \ \epsilon_{t+1}^C\right)'$ in equation (3) affect the stochastic discount factor (5) through the term

$$-\Lambda_t^{s_{t+1}\prime}\epsilon_{t+1} = -\begin{bmatrix}\Lambda_{L,t}^{s_{t+1}} & \Lambda_{S,t}^{s_{t+1}} & \Lambda_{C,t}^{s_{t+1}}\end{bmatrix}\begin{bmatrix}\epsilon_{t+1}^L \\ \epsilon_{t+1}^S \\ \epsilon_{t+1}^C\end{bmatrix}.$$

In the left panel of Table 8 we show regression results of the expected market prices of risk $E_t[\Lambda_{j,t}^{s_{t+1}}]$ for $j \in \{L, S, C\}$ on the cyclical component of industrial production and inflation.[16] The expected market prices of shocks to the level and slope factors are negatively correlated to inflation, whereas the expected market price of the curvature factor is positively related to the business cycle, displaying a marked procyclicality. Yet, these macroeconomic factors only account for a limited fraction of their variability as evidenced by the R^2s of the regressions, contrasting with the results in Table 5 and Table 7, pointing to macroeconomic factors as key drivers of expected excess returns.

Table 8: State dependent factors risk and macro fundamentals

	$E_t\Lambda_{L,t}^{s_{t+1}}$	$E_t\Lambda_{S,t}^{s_{t+1}}$	$E_t\Lambda_{C,t}^{s_{t+1}}$	$\bar{\Lambda}_{L,t}$	$\bar{\Lambda}_{S,t}$	$\bar{\Lambda}_{C,t}$	$E_t\Lambda_{L,t}^{s_{t+1}} - \bar{\Lambda}_{L,t}$	$E_t\Lambda_{S,t}^{s_{t+1}} - \bar{\Lambda}_{S,t}$	$E_t\Lambda_{C,t}^{s_{t+1}} - \bar{\Lambda}_{C,t}$
Ind. Prod.	-0.196	-0.112	0.285***	-0.906**	0.494	0.172***	0.710***	-0.607	0.113
	(0.353)	(0.541)	(0.095)	(0.375)	(0.375)	(0.039)	(0.165)	(0.380)	(0.098)
Inflation	-2.555***	-2.405*	0.190	-4.907***	3.354***	0.133	2.352***	-5.759***	0.058
	(0.710)	(1.348)	(0.246)	(0.854)	(1.119)	(0.110)	(0.621)	(0.769)	(0.283)
R^2	0.115	0.053	0.084	0.320	0.157	0.133	0.264	0.474	0.018

Notes: This table reports linear regressions of $E_t\Lambda_{j,t}$, $\bar{\Lambda}_{j,t}$ and $E_t\Lambda_{j,t}^{s_{t+1}} - \bar{\Lambda}_{j,t}$ where j = Level, Slope and Curvature. For each regressor we report (in parenthesis) HAC standard errors following the boostrap procedure proposed by Bauer and Hamilton (2017). $^{***}/^{**}/^{*}$ denote significant at the 1/5/10% level. $^{***}/^{**}/^{*}$ denote significant at 1/5/10% level.

To interpret these results, we decompose the variation in $E_t[\Lambda_{j,t}^{s_{t+1}}]$ in two components. A first component that represents an average time-varying market price of factor risk that

[16] Appendix B contain similar regressions using the cyclical component of unemployment instead.

is computed by averaging the estimated parameters $\lambda_0^{s_{t+1}}$ and $\lambda_1^{s_{t+1}}$ in equation (6) using the steady-state probabilities of the Markov chain (2).[17] We interpret this component, which we denote by $\bar{\Lambda}_{j,t}$, as isolating the impact of variations in X_t on $E_t[\Lambda_{j,t}^{s_{t+1}}]$. The second component is given by $E_t[\Lambda_{j,t}^{s_{t+1}}] - \bar{\Lambda}_{j,t}$, which represents the impact that changes in the Markov regimes has on the expected market prices of risk $E_t[\Lambda_{j,t}^{s_{t+1}}]$ over time. The results of the decomposition are shown in the middle and right panels of Table 8.[18]

The results of these regressions show that decomposing $E_t[\Lambda_{j,t}^{s_{t+1}}]$ into these two components is critical to uncover the link of the expected market prices of risk to the macroeconomic fundamentals. For instance, the macro variables explain roughly 10% of the variation in the expected market price of the level factor, $E_t[\Lambda_{L,t}^{s_{t+1}}]$, which is mainly associated with the contribution of inflation. Yet, this result masks a much stronger link between $\bar{\Lambda}_{L,t}$ and $E_t[\Lambda_{L,t}^{s_{t+1}}] - \bar{\Lambda}_{L,t}$ and the macroeconomic variables, which account for more than 30% and 25% of their variation, respectively. The macroeconomic variables tend to have an opposite effect on each component that partially offsets each other and weakens, or even eliminates, the link between the macro variables and the total variation in the expected market price of the level factor risk $E_t[\Lambda_{L,t}^{s_{t+1}}]$. We find that an increase in the cyclical component of industrial production reduces the sensitivity of the average market price of level risk, $\bar{\Lambda}_{L,t}$, but increase the sensitivity due to possible changes in regimes, represented by $E_t[\Lambda_{L,t}^{s_{t+1}}] - \bar{\Lambda}_{L,t}$. A similar result is found for an increase in inflation. As for the market price of the slope factor, there is no estatistical evidence that industrial production changes the sensitivity to slope shocks, but inflation does, again, with effects of opposite signs on $\bar{\Lambda}_{S,t}$ and $E_t[\Lambda_{S,t}^{s_{t+1}}] - \bar{\Lambda}_{S,t}$. Finally, the increase in the sensitivity to innovations in the curvature factor on $E_t[\Lambda_{C,t}^{s_{t+1}}]$ seems to come, for the most part, from its impact on the average component $\bar{\Lambda}_{C,t}$.

6 Concluding remarks

Interest rates are subject to occasional changes over time: periods of high interest rates and inverted yield curves are inherently different from other periods. Aware of the possibility of sudden changes in the environment, investors will seek compensation for the risks associated with those changes. We develop an arbitrage-free model of the term structure of interest rates with priced regime shifts and find strong evidence that regime shifts are an essential determinant of bond excess returns.

[17] That is, $\bar{\Lambda}_t = \sum_{s=1}^{4} \bar{\pi}_s(\lambda_0^s + \lambda_1^s X_t)$, where $\bar{\pi}_s$ is the steady-state probability of regime $s \in \{1,2,3,4\}$.

[18] Since $E_t[\Lambda_{j,t}^{s_{t+1}}] = \bar{\Lambda}_{j,t} + (E_t[\Lambda_{j,t}^{s_{t+1}}] - \bar{\Lambda}_{j,t})$, adding the coefficients in the middle and right panels of the table gives the coefficients of the regression in the left panel.

When estimating the arbitrage-free model, we allow for separate regimes characterizing the yield curve's level and slope factors. In our model, regimes in the level factor align well with the monetary policy tightening cycles, whereas the slope factor regime roughly identifies contractions in economic activity. As a result, the regime we uncover reflects the combined properties of the level and slope factors, which cannot be recovered by a linear combination of the yield curve factors. Therefore, the priced regime depends on the state of the business cycle and the monetary policy regime.

We decompose risk premia and investigate its sources of variation. We found that a measure of market price of regime-switching risk tracks the macroeconomic conditions. It has its lowest values in recessions and is strongly related to the macroeconomic fundamentals. We also found that the same macro variables are also related to the market price of risk of the traditional risk factors once we allow for priced regime shifts. The strong relationship between bond risk premia and macroeconomic fundamentals is inherently related to the nonlinearity arising from the price of regime switching. Allowing for a component of the risk premium that reflects the risk in shifting regimes increases the overall volatility of bond risk premia and uncovers the relationship to macroeconomic fundamentals. Expected excess returns in our model are countercyclical, decreasing when inflation and industrial production are above trend. Combining two macro indicators explains almost a third of the variation in one-month expected excess returns.

As Ludvigson and Ng (2009) point out, while business cycle variations are critical to explain excess bond returns, traditional factor models are unable to capture this relation. An essential aspect of our findings is that the business cycle variation in expected excess bond returns is revealed in the yield curve once correctly accounting for and pricing regime shifts. Our baseline model shows that the information in bond yields is enough to uncover countercyclical, business cycle variation in bond risk premia.

Appendix A Details of the model

This appendix fills in the details of the model presented in Section 3 and provides the proof of the approximate log-linear solution for bond prices.

As noted in the text, the pricing equation for zero-coupon bonds can be written as

$$P_{t,j}^{n+1} = e^{-r_t} \sum_{k=1}^{S} \pi_{jk} e^{-\Gamma_{jk} - \frac{1}{2}\Lambda_t^{k\prime}\Lambda_t^k} E\left[e^{-\Lambda_t^{k\prime}\epsilon_{t+1}} P_{t+1,k}^n | s_t = j, s_{t+1} = k, X_t\right]. \tag{A.1}$$

Using the log-linear guess for the bond price $P_{t,j}^n = e^{A_j^n + B_j^{n\prime} X_t}$ we can write the expectation on the right hand side of the previous expression as

$$\begin{aligned} E_t\left[e^{-\Lambda_t^{k\prime}\varepsilon_{t+1}} P_{t+1,k}^n | s_t = j, s_{t+1} = k\right] &= E_t\left[e^{-\Lambda_t^\prime \varepsilon_{t+1}} e^{A_k^n + B_k^{n\prime} X_{t+1}} | s_t = j, s_{t+1} = k\right] \\ &= e^{A_k^n + B_k^{n\prime}\left(\mu_k - \Sigma_k \lambda_0^k\right) + \frac{1}{2}\Lambda_t^\prime \Lambda_t + \frac{1}{2} B_k^{n\prime} \Sigma_k \Sigma_k^\prime B_k^n + B_k^{n\prime}\left(\Phi_k - \Sigma_k \lambda_1^k\right) X_t}. \end{aligned}$$

Replacing this expression into equation (A.1) and defining

$$\begin{aligned} F_k^n &\equiv A_k^n + B_k^{n\prime}\mu_k^Q + \frac{1}{2} B_k^{n\prime}\Sigma_k \Sigma_k^\prime B_k^n, \\ \mu_k^Q &\equiv \mu_k - \Sigma_k \lambda_0^k, \\ \Phi_k^Q &\equiv \Phi_k - \Sigma_k \lambda_1^k, \\ \pi_{jk}^Q &\equiv \pi_{jk} e^{-\Gamma_{jk}}, \end{aligned}$$

we can write the pricing equation as

$$P_{t,j}^{n+1} = e^{-r_t} \sum_{k=1}^{S} \pi_{jk}^Q e^{F_k^n + B_k^{n\prime}\Phi_k^Q X_t}.$$

Using the guess (8) implies

$$e^{A_j^{n+1} + B_j^{n+1\prime} X_t} = e^{-r_t} \sum_{k=1}^{S} \pi_{jk}^Q e^{F_k^n + B_k^{n\prime}\Phi_k^Q X_t},$$

and taking logs on both sides of the equation gives

$$A_j^{n+1} + B_j^{n+1\prime} X_t = -r_t + \log\left(\sum_{k=1}^{S} \pi_{jk}^Q e^{F_k^n + B_k^{n\prime}\Phi_k^Q X_t}\right). \tag{A.2}$$

To find the approximate solution, we perform a first order Taylor approximation to the last term on the right hand side of the previous equation. To that end, let

$$f_j^n\left(X_t\right) \equiv \log\left(\sum_{k=1}^{S} \pi_{jk}^{Q} e^{F_k^n + B_k^{n\prime}\Phi_k^Q X_t}\right). \tag{A.3}$$

This expression depends on the bond maturity n, the current regime $s_t = j$, and the risk factors X_t. We perform a first order Taylor approximation of $f_j^n\left(X_t\right)$ around the (state-dependent) long-run mean of the risk factors $X_t = \bar{\mu}_j \equiv (I - \Phi_j)^{-1}\mu_j$ given the current state $s_t = j$:

$$f_j^n\left(X_t\right) \approx f_j^n(\bar{\mu}_j) + (X_t - \bar{\mu}_j)'\nabla f_j^n(\bar{\mu}_j).$$

We need to compute the vector of derivatives $\nabla f_j^n(X_t)$ and evaluate it at $X_t = \bar{\mu}_j$. The Taylor approximation is

$$f_j^n\left(X_t\right) \approx \log\left(\sum_{k=1}^{S} \pi_{jk}^{Q} e^{F_k^n + B_k^{n\prime}\Phi_k^Q X_t}\right) + (X_t - \bar{\mu}_j)'\mathbf{h}_j^n, \tag{A.4}$$

where

$$\mathbf{h}_j^n = \underbrace{\left[\begin{array}{cccc} \Phi_1^{Q\prime}B_1^n & \Phi_2^{Q\prime}B_2^n & \cdots & \Phi_S^{Q\prime}B_S^n \end{array}\right]}_{N\times S} \underbrace{\left[\begin{array}{c} \pi_{j1}^{Q} e^{F_1^n + B_1^{n\prime}\Phi_{j1}^{*}\bar{\mu}_j} \\ \pi_{j2}^{Q} e^{F_2^n + B_2^{n\prime}\Phi_{j2}^{*}\bar{\mu}_j} \\ \vdots \\ \pi_{jS}^{Q} e^{F_S^n + B_S^{n\prime}\Phi_{jS}^{*}\bar{\mu}_j} \end{array}\right]}_{S\times 1} \times \frac{1}{\sum_{h=1}^{S}\pi_{jh} e^{F_h^n + B_h^{n\prime}\Phi_{jh}^{*}\bar{\mu}_j}}.$$

We now prove the previous claim.

Proof of the Taylor approximation.

Let I be a natural number, $\boldsymbol{x} = (x_1, x_2, ..., x_I)' \in \Re^I$ and $\boldsymbol{\beta}_k = (\beta_{k1}, \beta_{k2}, ..., \beta_{kI})' \in \Re^I$ for $k = 1, \ldots, S$. Consider a function $h : \Re^I \rightarrow \Re$ given by

$$h\left(\boldsymbol{x}\right) \equiv \log\left(\sum_{k=1}^{S} \alpha_k e^{\boldsymbol{\beta}_k^{\prime}\boldsymbol{x}}\right).$$

The partial derivative of $h(\boldsymbol{x})$ with respect to x_i is

$$\frac{\partial h\left(\boldsymbol{x}\right)}{\partial x_i} = \frac{\sum_{k=1}^{S}\alpha_k e^{\boldsymbol{\beta}_k^{\prime}\boldsymbol{x}}\beta_{ki}}{\sum_{h=1}^{S}\alpha_h e^{\boldsymbol{\beta}_h^{\prime}\boldsymbol{x}}} = \frac{\left[\begin{array}{cccc} \alpha_1 e^{\boldsymbol{\beta}_1^{\prime}\boldsymbol{x}} & \alpha_2 e^{\boldsymbol{\beta}_2^{\prime}\boldsymbol{x}} & \cdots & \alpha_S e^{\boldsymbol{\beta}_S^{\prime}\boldsymbol{x}} \end{array}\right]}{\sum_{h=1}^{S}\alpha_h e^{\boldsymbol{\beta}_h^{\prime}\boldsymbol{x}}} \left[\begin{array}{c} \beta_{1i} \\ \beta_{2i} \\ \vdots \\ \beta_{Si} \end{array}\right].$$

If we let

$$\eta_l \equiv \frac{\alpha_l e^{\boldsymbol{\beta}_l^{\prime}\boldsymbol{x}}}{\sum_{h=1}^{S}\alpha_h e^{\boldsymbol{\beta}_h^{\prime}\boldsymbol{x}}} \in (0, 1)$$

for $l = 1, 2, ..., S$, we can write the partial derivative as

$$\frac{\partial h(x)}{\partial x_i} = \begin{bmatrix} \eta_1 & \eta_2 & \cdots & \eta_S \end{bmatrix} \begin{bmatrix} \beta_{1i} \\ \beta_{2i} \\ \vdots \\ \beta_{Si} \end{bmatrix}$$
$$= \tilde{\boldsymbol{\beta}}_i' \boldsymbol{\eta}$$

Therefore, the gradient of $h(\boldsymbol{x})$ is given by

$$\nabla h(\boldsymbol{x}) = \begin{bmatrix} \tilde{\boldsymbol{\beta}}_1' \\ \tilde{\boldsymbol{\beta}}_2' \\ \vdots \\ \tilde{\boldsymbol{\beta}}_I' \end{bmatrix} \boldsymbol{\eta},$$

or

$$\nabla h(\boldsymbol{x}) = \begin{bmatrix} \beta_{11} & \beta_{21} & \cdots & \beta_{S1} \\ \beta_{12} & \beta_{22} & \cdots & \beta_{S2} \\ \vdots & & \ddots & \vdots \\ \beta_{1I} & \beta_{2I} & \cdots & \beta_{SI} \end{bmatrix} \boldsymbol{\eta}$$
$$= \boldsymbol{B}\boldsymbol{\eta}.$$

where

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \cdots & \boldsymbol{\beta}_S \end{bmatrix}.$$

We now put these expressions in terms of the notation of the bond pricing model. The function $h(\boldsymbol{x})$ is replaced by

$$f_j^n(X_t) = \log\left(\sum_{k=1}^{S} \pi_{jk} e^{F_{jk}^n + B_k^{n\prime} \Phi_{jk}^* X_t}\right)$$

for $j = 1, \ldots, S$. The mapping between the model's notation and that of the Taylor approximation above is

$$\alpha_{j,k} = \pi_{jk}^{Q} e^{F_k^n},$$
$$\boldsymbol{\beta}_k' = B_k^{n\prime} \Phi_k^{Q},$$
$$\boldsymbol{x} = X_t.$$

Note that we are also indexing the parameter $\alpha_{j,k}$ by the current state $j = 1, \ldots, S$. Then,

$$\eta_{jk,t} = \frac{\pi_{jk}^{Q} e^{F_k^n + B_k^{n\prime} \Phi_k^{Q} X_t}}{\sum_{h=1}^{S} \pi_{jh}^{Q} e^{F_h^n + B_h^{n\prime} \Phi_h^{Q} X_t}}$$

so that

$$\boldsymbol{\eta}_{jt} = \frac{1}{\sum_{h=1}^{S} \pi_{jh}^{Q} e^{F_h^n + B_h^{n\prime} \Phi_h^Q X_t}} \times \begin{bmatrix} \pi_{j1}^{Q} e^{F_1^n + B_1^{n\prime} \Phi_1^Q X_t} \\ \pi_{j2}^{Q} e^{F_2^n + B_2^{n\prime} \Phi_2^Q X_t} \\ \vdots \\ \pi_{jS}^{Q} e^{F_S^n + B_S^{n\prime} \Phi_S^Q X_t} \end{bmatrix}.$$

Also,

$$\boldsymbol{\beta}_k' = B_k^{n\prime} \Phi_k^Q \Rightarrow \boldsymbol{\beta}_k = \Phi_k^{Q\prime} B_k^n.$$

Then,

$$\begin{aligned} \boldsymbol{B} &= \begin{bmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \cdots & \boldsymbol{\beta}_S \end{bmatrix} \\ \boldsymbol{B}^n &= \begin{bmatrix} \Phi_1^{Q\prime} B_1^n & \Phi_2^{Q\prime} B_2^n & \cdots & \Phi_S^{Q\prime} B_S^n \end{bmatrix}. \end{aligned}$$

After all this algebra, we conclude that

$$\underbrace{\nabla f_j^n (X_t)}_{N \times 1} = \boldsymbol{B}_j^n \boldsymbol{\eta}_{jt} = \underbrace{\begin{bmatrix} \Phi_1^{Q\prime} B_1^n & \Phi_2^{Q\prime} B_2^n & \cdots & \Phi_S^{Q\prime} B_S^n \end{bmatrix}}_{N \times S} \underbrace{\begin{bmatrix} \pi_{j1}^{Q} e^{F_1^n + B_1^{n\prime} \Phi_1^Q X_t} \\ \pi_{j2}^{Q} e^{F_2^n + B_2^{n\prime} \Phi_2^Q X_t} \\ \vdots \\ \pi_{jS}^{Q} e^{F_S^n + B_S^{n\prime} \Phi_S^Q X_t} \end{bmatrix}}_{S \times 1} \times \frac{1}{\sum_{h=1}^{S} \pi_{jh}^{Q} e^{F_h^n + B_h^{n\prime} \Phi_h^Q X_t}}.$$

Evaluating this expression at the point $X_t = \bar{\mu}_j$ and letting $\nabla f_j^n(\bar{\mu}_j) \equiv \mathbf{h}_j^n$, we have

$$\mathbf{h}_j^n = \underbrace{\begin{bmatrix} \Phi_1^{Q\prime} B_1^n & \Phi_2^{Q\prime} B_2^n & \cdots & \Phi_S^{Q\prime} B_S^n \end{bmatrix}}_{N \times S} \underbrace{\begin{bmatrix} \pi_{j1}^{Q} e^{F_1^n + B_1^{n\prime} \Phi_1^Q \bar{\mu}_j} \\ \pi_{j2}^{Q} e^{F_2^n + B_2^{n\prime} \Phi_2^Q \bar{\mu}_j} \\ \vdots \\ \pi_{jS}^{Q} e^{F_S^n + B_S^{n\prime} \Phi_S^Q \bar{\mu}_j} \end{bmatrix}}_{S \times 1} \times \frac{1}{\sum_{h=1}^{S} \pi_{jh}^{Q} e^{F_h^n + B_h^{n\prime} \Phi_h^Q \bar{\mu}_j}}.$$

This concludes the proof of the Taylor approximation formula.□

Now, plugging the short-rate equation (4) and the Taylor approximation (A.4) into the pricing equation (A.2) yields

$$\begin{aligned} A_j^{n+1} + A_j^{n+1\prime} X_t &= - \delta_0^j - \delta_1^{j\prime} X_t + \log \left(\sum_{k=1}^{S} \pi_{jk} e^{F_{jk}^n + B_k^{n\prime} \Phi_{jk}^* \bar{\mu}_j} \right) + (X_t - \bar{\mu}_j)' \mathbf{h}_j^n \\ &= - \delta_0^j - \left(\mathbf{h}_j^n \right)' \bar{\mu}_j + \log \left(\sum_{k=1}^{S} \pi_{jk} e^{F_{jk}^n + B_k^{n\prime} \Phi_{jk}^* \bar{\mu}_j} \right) + (\mathbf{h}_j^n - \delta_1^j)' X_t. \end{aligned}$$

Matching coefficients, we have that the parameters A_j^{n+1} and B_j^{n+1} satisfy the recursions

$$\begin{aligned} A_j^{n+1} =& -\delta_0^j - \left(\mathbf{h}_j^n\right)' \bar{\mu}_j + \log\left(\sum_{k=1}^{S} \pi_{jk} e^{F_{jk}^n + B_k^{n\prime}\Phi_{jk}^* \bar{\mu}_j}\right) \\ B_j^{n+1} =& \mathbf{h}_j^n - \delta_1^j, \end{aligned} \tag{A.5}$$

for $j = 1, \ldots, S$. The recursion starts with $A_j^0 = 0$ and $B_j^0 = \mathbf{0}$ for all $j = 1, \ldots, S$.

Note that, in this case, $F_{jk}^0 = -\Gamma_{jk}$, which implies

$$log\left(\sum_{k=1}^{S} \pi_{jk} e^{F_{jk}^0 + B_0^{n\prime}\Phi_{jk}^* \bar{\mu}_j}\right) = \log\left(\sum_{k=1}^{S} \pi_{jk} e^{-\Gamma_{jk}}\right).$$

For the model to price correctly the short rate, we must have $\log\left(\sum_{k=1}^{S} \pi_{jk} e^{-\Gamma_{jk}}\right) = 0$, which imposes the restriction

$$\sum_{k=1}^{S} \pi_{jk} e^{-\Gamma_{jk}} = 1. \tag{A.6}$$

This restriction, of course, is that the risk-neutral probabilities of the Markov chain, defined as

$$\pi_{jk}^{\mathbb{Q}} \equiv \pi_{jk} e^{-\Gamma_{jk}},$$

add up to 1 for all $j = 1, \ldots, S$. In addition, $\mathbf{h}_j^0 = \mathbf{0}$. Hence, starting the recursion with $A_j^0 = 0$ and $B_j^0 = \mathbf{0}$ and imposing the restriction (A.6) implies that, for $n = 0$, we have $A_j^1 = 0$ and $B_j^1 = \delta_1^j$, which is consistent with the short-rate equation (4) for all j.

Appendix B Additional Results

Tables 9-11 replicate the analysis in Section 5.1 using the rate of unemployment as the business cycle indicator in the place of industrial production.

Table 9: Expected excess log-holding return and macroeconomic fundamentals

	6	12	24	36	48	60	72	84	96	108	120
1 Regime											
Unemp.	0.005**	0.009**	0.016**	0.024**	0.033**	0.043**	0.053***	0.064***	0.075***	0.086***	0.097***
	(0.002)	(0.004)	(0.009)	(0.013)	(0.015)	(0.018)	(0.020)	(0.022)	(0.024)	(0.026)	(0.029)
Inflation 0.029	0.018	-0.061	-0.147	-0.220	-0.282	-0.336	-0.386	-0.433	-0.479	-0.525	
	(0.020)	(0.046)	(0.097)	(0.141)	(0.178)	(0.210)	(0.239)	(0.266)	(0.293)	(0.319)	(0.346)
R^2	0.140	0.066	0.044	0.057	0.072	0.085	0.097	0.107	0.116	0.124	0.130
RS with Priced Risk											
Unemp.	0.000	0.003	0.016	0.031**	0.048**	0.064***	0.079***	0.093***	0.106***	0.118***	0.129***
	(0.002)	(0.005)	(0.010)	(0.015)	(0.019)	(0.024)	(0.028)	(0.032)	(0.036)	(0.039)	(0.043)
Inflation	0.023	-0.015	-0.165*	-0.365***	-0.588***	-0.820***	-1.053***	-1.284***	-1.509***	-1.727***	-1.940***
	(0.016)	(0.039)	(0.086)	(0.132)	(0.177)	(0.221)	(0.263)	(0.304)	(0.344)	(0.383)	(0.420)
R^2	0.043	0.011	0.099	0.173	0.225	0.260	0.286	0.305	0.320	0.332	0.342

Notes: This table reports linear regressions of the expected log-holding return in excess of the short rate at different maturities for a single regime model (1 Regime, upper panel) and the baseline model with regime switching and priced regime risk (RS with Priced Risk, lower panel). For each regressor we report (in parenthesis) HAC standard errors. ***/**/* denote significant at 1/5/10% level.

Table 10: Price of regime switch and macro fundamentals

	6	12	24	36	48	60	72	84	96	108	120
$E_t[rx^h_{t+1,t} \vert \Gamma(s_t, s_{t+1}) = 0]$											
Unemp.	0.002***	0.006***	0.016***	0.027***	0.038***	0.048***	0.057***	0.065***	0.072***	0.076	0.079***
	(0.001)	(0.002)	(0.004)	(0.007)	(0.010)	(0.013)	(0.016)	(0.019)	(0.021)	(0.023)	(0.024)
Inflation	-0.021***	-0.051***	-0.117***	-0.208***	-0.321***	-0.443***	-0.565***	-0.678***	-0.777***	-0.860***	-0.928***
	(0.004)	(0.011)	(0.027)	(0.047)	(0.069)	(0.092)	(0.114)	(0.133)	(0.150)	(0.164)	(0.175)
R^2	0.272	0.248	0.239	0.242	0.248	0.254	0.259	0.263	0.267	0.270	0.273
$E_t rx^h_{t+1,t} - E_t[rx^h_{t+1,t} \vert \Gamma(s_t, s_{t+1}) = 0]$											
Unemp.	-0.002	-0.003	-0.001	0.004	0.010	0.016	0.022	0.028*	0.035*	0.042**	0.050**
	(0.002)	(0.004)	(0.007)	(0.010)	(0.012)	(0.013)	(0.015)	(0.016)	(0.018)	(0.020)	(0.022)
Inflation	0.044***	0.036	-0.048	-0.157	-0.268**	-0.378**	-0.489***	-0.606***	-0.731***	-0.867***	-1.012***
	(0.015)	(0.034)	(0.073)	(0.107)	(0.137)	(0.163)	(0.187)	(0.211)	(0.235)	(0.261)	(0.287)
R^2	0.166	0.030	0.012	0.057	0.101	0.138	0.170	0.198	0.223	0.247	0.268

Notes: This table reports linear regressions of the a decomposition of expected log-holding return in excess of the short rate at different maturities. For each regressor we report (in parenthesis) HAC standard errors. $^{***}/^{**}/^{*}$ denote significant at 1/5/10% level.

Table 11: State dependents factors risk and macro fundamentals

	$E_t\Lambda_{1,t}^{s_{t+1}}$	$E_t\Lambda_{2,t}^{s_{t+1}}$	$E_t\Lambda_{3,t}^{s_{t+1}}$	$\bar{\Lambda}_{1,t}$	$E_t\bar{\Lambda}_{2,t}$	$\bar{\Lambda}_{3,t}$	$E_t\Lambda_{1,t}^{s_{t+1}} - \bar{\Lambda}_{1,t}$	$E_t\Lambda_{2,t}^{s_{t+1}} - \bar{\Lambda}_{2,t}$	$E_t\Lambda_{3,t}^{s_{t+1}} - \bar{\Lambda}_{3,t}$
Unemp.	-0.009	0.078	-0.092***	0.257**	-0.143	-0.055***	-0.266***	0.222*	-0.038
	(0.092)	(0.193)	(0.033)	(0.108)	(0.115)	(0.012)	(0.048)	(0.120)	(0.028)
Inflation	-2.466***	-2.496*	0.240	-5.004***	3.412***	0.161	2.538***	-5.908***	0.079
	(0.689)	(1.465)	(0.275)	(0.869)	(1.155)	(0.110)	(0.644)	(0.792)	(0.294)
R^2	0.112	0.056	0.108	0.320	0.157	0.166	0.336	0.492	0.025

Notes: This table reports linear regressions of $E_t\Lambda_{j,t}$, $\bar{\Lambda}_{j,t}$ and $E_t\Lambda_{j,t}^{s_{t+1}} - \bar{\Lambda}_{j,t}$ where j = Level, Slope and Curvature. For each regressor we report (in parenthesis) HAC standard errors following the bootstrap procedure proposed by Bauer and Hamilton (2017). ***/**/* denote significant at the 1/5/10% level. ***/**/* denote significant at 1/5/10% level.

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