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Luciano de Castro
Antonio F. Galvao
Gabriel Montes Rojas
Jose Olmo

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Portfolio Selection in Quantile Decision Models*

Luciano de Castro† Antonio F. Galvao‡ Gabriel Montes-Rojas§ Jose Olmo¶

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Abstract

This paper develops an optimal portfolio allocation model for an investor with quantile preferences, i.e., who maximizes the τ-quantile of the portfolio return. Quantile preferences allow to study heterogeneity in individuals' portfolio choice and have a solid axiomatic foundation. We derive conditions under which the optimal portfolio allocation problem has an interior solution guaranteeing diversification and conditions under which the portfolio allocation is characterized by two regions: full diversification for quantiles below the median and no diversification for upper quantiles. These results are illustrated via simulation and empirically with a portfolio of cash, a stock index and a bond index.

Keywords: Optimal Asset Allocation, Quantile Preferences, Portfolio Theory, Risk Attitude.

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†Department of Economics, University of Iowa, Iowa City, USA. decastro.luciano@gmail.com

‡Corresponding author: Department of Economics, University of Arizona, Tucson, USA. agalvao@email.arizona.edu

§CONICET and Instituto Interdisciplinario de Economía Polítuca, Universidad de Buenos Aires, Ciudad Autónoma de Buenos Aires, Argentina. gabriel.montes@fce.uba.ar

¶Department of Economic Analysis, Universidad de Zaragoza, Gran Via 2, 50005, Zaragoza, Spain. joseolmo@unizar.es, and Department of Economics, University of Southampton, University Rd., Southampton, SO17 1BJ, United Kingdom.

1 Introduction

Portfolio selection is a fundamental topic in economics and finance and one of the leading applications of decision theory under uncertainty. Modern portfolio theory derives its main results on diversification and risk under the important paradigm of the expected utility (EU) theory; see, for instance, Cochrane (2005) and Campbell (2018). Nevertheless, the EU has been subjected to many criticisms. From a theoretical perspective, Rabin (2000) shows that risk aversion with respect to low stakes gambles imply unrealistic behavior with respect to large sums. Empirical evidence suggests that investors do not always act as risk averters (see, e.g., Rabin and Thaler (2001) and Kahneman et al. (1982)). They behave differently on gains and losses, and they are more sensitive to losses than to gains (loss aversion).[1] Investors may also exhibit a preference for positive skewness of returns. In these cases the optimal portfolio allocation may result in underdiversification compared to standard mean-variance efficient allocations; see Mitton and Vorkink (2007) and references therein. Experimental evidence also shows that individuals do not allocate their resources as predicted by EU theory; see Simon (1979), Tversky and Kahneman (1981), Payne et al. (1992) and Baltussen and Post (2011) as seminal examples.

In this paper we depart from the EU framework and investigate the optimal portfolio allocation of individuals concerned with maximizing a specific quantile of the distribution of portfolio returns. This individual's behavior is motivated from practical and scientific points of view. Regarding the former, quantiles have been used in decision making in banking and investment (in the form of Value at Risk and goal-reaching problems) and in mining, oil and gas industries (in the form of "probabilities of exceeding" a certain level of production). On the latter, there has been increasing theoretical, empirical and experimental interest in decision under uncertainty under quantile preferences (QP). This alternative preference has been characterized in early work by Manski (1988), who studied properties of a quantile model for individual's behavior. More recently, QP have been formally axiomatized by Chambers (2009), Rostek (2010), and de Castro and Galvao (2019b). Mendelson (1987) introduced the concept of quantile-preserving spread, which is a notion of risk aversion for the quantile model that establishes a parallelism with mean-preserving spreads in the standard EU framework. de Castro and Galvao (2019a) developed a dynamic model of rational behavior under uncertainty, in which the agent maximizes a stream of the future quantile utilities. From an experimental point of view, de Castro et al. (2020) found out that the behavior of between 30 and 50% of the individuals in their experiment can be better described with quantile preferences rather than expected utility.

Overall, QP have several attractive features.[2] An individual's decision is independent of

[1] Examples of risk orderings that reflect such findings are Friedman and Savage (1948), Savage (1954), Kahneman and Tversky (1979), Edwards (1996), Baucells and Heukamp (2006), among several others.

[2] Rostek (2010) discusses other advantages of the quantile preference, such as robustness, ability to deal with categorical (instead of continuous) variables, and the flexibility of offering a family of preferences indexed by

the form of her utility function and thus an optimal choice is relatively easy to compute.[3] The measure of risk aversion is simple, intuitive, and determined by the quantile $\tau \in (0, 1)$. The increasing interest on this recent approach to modeling individuals' behavior under uncertainty suggests that it is important to understand portfolio choice in this context, and this paper fulfills this gap.

Our study of portfolio choice begins with the observation that the individuals' risk attitude under QP is captured by a single-dimensional parameter, the quantile $\tau \in (0, 1)$. The lower the τ, the more averse to risk the τ-quantile-maximizer decision maker (τ-DM) is. Next we establish properties of the quantile model. First we focus on a simple portfolio given by a risk-free and a risky asset. In contrast to the capital market line characterizing the mutual fund separation theorem in a mean-variance setting (Tobin, 1958), the optimal portfolio allocation under QP is to fully invest on the risk-free asset for quantile preferences given by τ below the magnitude of the risk-free rate, and on the risky asset, otherwise. The extension of the portfolio to considering two risky assets and a risk-free asset provides similar insights indicating an optimal binary response with respect to the risk-free asset. However, in this case, we find that diversification between the two risky assets may also be an optimal outcome for middle quantiles even if the allocation to the risk-free asset is null.

The optimal portfolio choice problem is then extended to consider a portfolio of two risky assets. We formally show that a τ-DM always diversifies (invest in both assets) if the distribution functions of the assets in the portfolio have same lower end (worst-case scenario) and τ is sufficiently low. In contrast, if τ is sufficiently high, we find that there is no diversification at all: the τ-DM only invests on the riskier asset. We illustrate these rich heterogeneous behaviors and theoretical insights with examples of two uniform random variables, a case in which we are able to obtain an analytical solution of the portfolio selection problem. We then provide further conditions under which the optimal portfolio decision has an interior solution with diversification vis-à-vis no diversification. The intuition behind the obtained characterization is that, in general, there will be diversification for investors concerned with low quantiles and downside risk. These diversification insights are illustrated in numerical simulation exercises that cover several canonical cases under different scenarios. For the particular case of two independent and identically distributed (iid) random variables, full diversification is optimal for $\tau \leqslant \tau_0$ but not for values of $\tau > \tau_0$. The optimal strategy in the upper part of the distribution is investing fully in one risky asset. Therefore, the quantile model is flexible and allows for the possibility of underdiversification in the sense that, in some scenarios, the optimal portfolio choice may be no diversification.

The insights of the QP model for optimal portfolio allocation are applied to illustrate the

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quantiles.

[3]Intuitively, the monotonicity of quantiles allows one to avoid modeling individuals' utility function. This is because the maximization problem is invariant to monotonic transformations of the distribution of portfolio returns.

similarities and differences between the optimal portfolio choices of EU and QP individuals. To do this, we consider a simple portfolio selection of stocks, bonds and cash, and compare the optimal asset allocation of QP individuals with that of EU individuals with mean-variance and power utility functions. We consider monthly data collected from three assets: the risk-free one-month nominal yield on the U.S. Treasury bill rate, the S&P 500 and the G0Q0 Bond Index, for the period January 1980 to December 2016. We compute the optimal portfolio allocation for the full range of $\tau \in (0, 1)$. For low enough quantiles, the optimal strategy is to invest fully in the risk-free asset, and for high enough quantiles, the optimal solution is fully invest in S&P 500 index. There is a rich diversification pattern among the three assets for middle quantile indexes. Overall, the results show portfolio diversification heterogeneity across risk attitudes, with no diversification for very low and large quantiles. These empirical findings contrast with the results obtained from two standard EU cases, the mean-variance and CRRA utility cases that exhibit full diversification under risk aversion.

We remark that the initial reaction to the consideration of quantile preferences might be of doubt, since quantile maximization is different from the familiar and well known EU model. Quantile maximization implies, indeed, some choices that might seem unusual at first glance, but can be considered reasonable after we overcome the influence of the EU over our intuition, as we discuss in Section 1.1 below. More than that, as we also argue in that section, there are situations where the maximization of a quantile seems very natural and by varying τ they encompass the whole range of risk aversion attitudes encountered in EU. It is not our contention, nevertheless, that quantile maximization is a decision method to be prescribed in all cases and problems, but rather that it is a complement analysis to EU. As such, it is important to document its properties and implications on relevant economic settings, such as the portfolio selection problem that is the focus of this paper. Those properties could then be tested in laboratories or confronted with data.

The remainder of the paper is laid out as follows. Section 1.1 has a brief review of the literature on optimal portfolio decision under uncertainty. Section 2 discusses the risk attitudes in QP models. Section 3 contains the main results of the paper. This section derives conditions under which there is full or null diversification in the tails, and conditions that provide focal optimal portfolios for risk averse and risk loving individuals. In addition, this section presents a numerical simulation exercise that illustrates the theoretical insights of the paper. Section 4 presents a simple portfolio allocation exercise among stocks, bonds and a risk-free asset, and Section 5 concludes. All mathematical derivations and proofs are collected in the Appendix.

1.1 Literature review

This paper relates to a number of streams of literature in portfolio selection and economic theory. First, the paper relates to the extensive literature on optimal decision theory under uncertainty and portfolio selection. Optimal portfolio decision based on the EU has been the basis of asset pricing equilibrium models such as the Sharpe-Lintner CAPM (Sharpe, 1964; Lintner, 1965) and more recent alternatives based on factor models. In this context, the investor's optimal portfolio decision relies heavily on the specification of the utility function for modeling individuals' preferences. Thus the theoretical optimal portfolio allocation of individuals with constant absolute risk aversion and constant relative risk aversion preferences may be very different although, in practice, it may be difficult to differentiate between both attitudes towards risk from real data. A robust approach within the EU paradigm is stochastic dominance. This theory allows one to rank risky alternatives without relying on specific forms of the individuals' utility function. Early work by Porter (1974) and Fishburn (1977) characterize the optimal portfolio decisions of EU individuals using stochastic dominance criteria of different orders. However, the equivalence between EU maximization and stochastic dominance is only satisfied, under risk aversion, for well-behaved (increasing and concave) utility functions.

Second, this paper is related to a branch of the literature on models for portfolio selection with alternative preferences to the EU. Many alternative preference measures to the EU have been put forward in the portfolio choice literature. Most of these approaches replace the utility function, which is essentially a distortion in wealth, by a distortion in the probability distribution of wealth. This probability distortion function, as Yaari (1987) shows, represents the risk preference in a different way. Similar approaches involving subjective probability distributions include, most significantly, Kahneman and Tversky (1979)'s prospect theory. Garlappi et al. (2007) develop a model for an investor with multiple priors and aversion to ambiguity. We extend the last two literatures by replacing EU and its variations with QP.

Third, this paper is related to a few works on economic models using the quantile preferences. QP were first studied by Manski (1988) and subsequently axiomatized by Chambers (2009), Rostek (2010) and de Castro and Galvao (2019b). The optimization of quantile measures as target variables in economic problems is not new. Recently, QP models have been employed in modeling economic behavior in dynamic frameworks, see de Castro and Galvao (2019a). The risk attitude under QP is based on the concept of quantile-preserving spreads introduced in Mendelson (1987) and reformulated in Manski (1988) in terms of a single-crossing criterion between distribution functions. Bhattacharya (2009) studies the problem of optimally dividing individuals into peer groups to maximize a quantile of social gains from heterogeneous peer effects. In the asset pricing literature the use of QP has been hardly explored though. Giovannetti (2013) presents a two-period standard economy with one risky and one risk-free asset, where the agent has QP instead of the standard EU. Also, as mentioned previously, de Castro et al. (2020) show experimental evidence on the use of quantile preferences.

Fourth, there is a small literature on optimal portfolio allocation using a quantile target variable. Kulldorff (1993) and Föllmer and Leukert (1999) study the goal-reaching problem where the target variable is a specific quantile, and He and Zhou (2011) propose a portfolio choice model in continuous time, where the quantile function of the terminal cash flow is the decision variable. In a similar context, Brown and Sim (2009) provide a framework for measuring the quality of risky positions with respect to their ability to achieve some aspiration level, that can be interpreted with a quantile probability. In the mutual fund industry, quantiles have been used as alternative performance measures, see, for example, Kempf and Ruenzi (2007). We contribute to these two last lines of research by taking the QP together with the quantile maximization to a portfolio selection model and deriving its properties.

We conclude this section by discussing a related literature that shares some of the insights of QP theory but is different in scope. This discussion may serve as further general motivation for use of QP. Quantile measures have been used as risk measures in optimal portfolio allocation. In particular, Value-at-Risk (VaR) and expected shortfall models are closely linked to a low quantile selection, see Duffie and Pan (1997) and Jorion (2007) for a comprehensive review of VaR models. In an optimal asset allocation context, the VaR quantiles act as constraints in the asset allocation optimization exercise rather than as target variables to be optimized. These mean-risk models discussed in Fishburn (1977) can be considered as an extension of standard mean-variance formulations, see Markowitz (1952), rather than as QP models for optimal portfolio allocation. The relevant literature includes Basak and Shapiro (2001), Krokhmal et al. (2001), Campbell et al. (2001), Wu and Xiao (2002), Bassett et al. (2004), Engle and Manganelli (2004) and Ibragimov and Walden (2007), among others, and sheds an interesting light on the properties of VaR-optimal portfolios while acknowledging considerable computational difficulties (Gaivoronski and Pflug, 2005; Rachev et al., 2007).

2 Quantile preferences and the risk attitude

The concept of risk is central in economic and financial analyses. This section briefly reviews the definition of risk under quantile preferences (QP).

2.1 Preliminaries

We first introduce the notation and definition of QP. Given any random variable $X : \Omega \to \mathbb{R}$, we denote by $F_X : \mathbb{R} \to [0, 1]$ the cumulative distribution function (CDF) of X. Given $\tau \in (0, 1)$, the τ-quantile of X is defined as

$$Q_\tau[X] \equiv \inf\{x \in X : F_X(x) \geqslant \tau\}.$$

A well-known and important property of quantiles, used below, is its invariance with respect to monotonic transformations. More formally, if $\psi : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing, then

$$Q_\tau[\psi(X)] = \psi\left(Q_\tau[X]\right). \tag{1}$$

A preference $\succeq$ over random variables is a τ-quantile preference for some fixed $\tau \in (0,1)$ if

$$X \succeq Y \iff Q_\tau[u(X)] \geqslant Q_\tau[u(Y)], \tag{2}$$

where $u(\cdot)$ is the utility function over the possible outcomes of the random variables X and Y. Note that $u(X)$ and $u(Y)$ are also random variables.

It is important to notice that the QP defined by (2) are in fact independent of the utility function. Indeed, for any continuous and strictly increasing $u : \mathbb{R} \to \mathbb{R}$, from (1),

$$X \succeq Y \iff Q_\tau[u(X)] \geqslant Q_\tau[u(Y)] \iff u(Q_\tau[X]) \geqslant u(Q_\tau[Y]) \iff Q_\tau[X] \geqslant Q_\tau[Y]. \tag{3}$$

This result shows that the utility function plays absolutely no role in defining the preference. We can use (1) to make any transformation of u; therefore, we could transform a concave utility function into a convex one without changing the preference. In particular, this implies that the concavity of the utility function has absolutely no implication for the risk attitude (nor any property) of QP.

Manski (1988) was the first to study QP as in (2). Chambers (2009) shows that these preferences satisfy the properties of monotonicity, ordinal covariance, and continuity. In contrast, Rostek (2010) axiomatized the QP in the context of Savage (1954)'s subjective framework. Recently, de Castro and Galvao (2019b) provide an alternative axiomatization for the static case using an uncertainty setting and finite state space.

2.2 Quantile-preserving spreads

The study of the risk attitude of quantile maximizing decision makers begins with the concept of quantile-preserving spreads introduced by Mendelson (1987), who formalizes four other conditions and shows that they are all equivalent. The concept is inspired by the familiar mean-preserving spreads of Rothschild and Stiglitz (1970) and similarly captures the notion of "added noise." To wit, we can formalize "Y is equal to X plus noise" with either the statement that "Y is a mean-preserving spread of X" or that "Y is a quantile-preserving spread of X," in sense that Y and X share the same quantile, but Y is more wide-spread. The choice of either formalization is a subjective matter. However, while mean-preserving spreads help to study the risk attitude in expected utility models, quantile-preserving spreads support the same task for quantile preferences. This result was first observed by Manski (1988), although he used a different terminology (minmax spreads). Its formal definition is as follows:

Definition 2.1 (Quantile-preserving spread). *We say that Y is a τ-quantile-preserving spread of X if for some $q \in \mathbb{R}$, $Q_\tau[Y] = Q_\tau[X] = q$ and the following holds:*

(i) $t < q \implies F_Y(t) \geqslant F_X(t)$;

(ii) $t > q \implies F_Y(t) \leqslant F_X(t)$.

Y *is a quantile-preserving spread of* X *if it is a* $\bar{\tau}$*-quantile-preserving spread of* X *for some* $\bar{\tau} \in (0, 1)$.

Figure 1 below illustrates the CDFs of random variables Y and X when Y is a $\bar{\tau}$-quantile-preserving spread of X.[4] Note that Definition 2.1 captures the notion that Y is riskier than X, since Y puts weight in more extreme values than X. Manski (1988) uses a different terminology for the same concept referring to the property of "single crossing from below": F_X crosses F_Y from below when Y is a quantile-preserving spread of X.

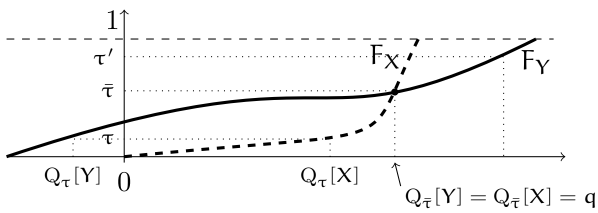


Figure 1: Y is a $\bar{\tau}$-quantile-preserving spread of X.

Notice that if $Q_\tau[Y] = q$ and X is equal to q with probability 1, then Y is a τ-quantile-preserving spread of X. In other words, any risky asset Y with τ-quantile q is a quantile-preserving spread of any risk-free asset X with value q.

Figure 1 suggests that the choice of a τ-quantile maximizer or τ-decision maker (τ-DM) depends on whether τ is below or above the quantile $\bar{\tau}$ where the two CDFs cross. That is, when $\tau < \bar{\tau}$ as in Figure 1, a τ-DM prefers the safer asset X: $Q_\tau[X] \geqslant Q_\tau[Y]$. On the other hand, if $\tau' > \bar{\tau}$, a τ-DM prefers the riskier asset Y: $Q_\tau[X] \leqslant Q_\tau[Y]$. The following result formalizes this intuition.

Proposition 2.2 (Manski)**.** *Let* Y *be a* $\bar{\tau}$*-quantile-preserving spread of* X *for* $\bar{\tau} \in (0, 1)$. *Then:*

(i) $\tau \leqslant \bar{\tau} \implies Q_\tau[X] \geqslant Q_\tau[Y]$, *that is, a* τ*-DM prefers the less risky asset* X *if* τ *is low;*

(ii) $\tau \geqslant \bar{\tau} \implies Q_\tau[X] \leqslant Q_\tau[Y]$, *that is, a* τ*-DM prefers the riskier asset* Y *if* τ *is high.*

It is useful to observe that the above implications are not equivalences: the opposite direction is false. Indeed, the definition of $\bar{\tau}$-quantile-preserving spreads does not preclude the case where $Q_\tau[X] = Q_\tau[Y]$ for τ below or above $\bar{\tau}$. However, the above implications can obviously be written in the reverse order, that is,

(i') $Q_\tau[X] < Q_\tau[Y] \implies \tau > \bar{\tau}$;

(ii') $Q_\tau[X] > Q_\tau[Y] \implies \tau < \bar{\tau}$.

[4] Mendelson (1987) suggests four other conditions and shows that they are all equivalent to the above definition.

2.3 Risk attitudes are determined by the quantile

We now consider the problem of comparing the risk attitude of τ-DM with different τ. Ghirardato and Marinacci (2002, Definition 4, p. 263) suggest the following definition:

Definition 2.3 (Ghirardato-Marinacci)**.** *A preference* $\succeq'$ *is more uncertainty averse than preference* $\succeq$ *if for any* $q \in \mathbb{R}$*, and random variable* X*,* $q \succeq X \Rightarrow q \succeq' X$ *and* $q \succ X \Rightarrow q \succ' X$*.*

The intuition for Definition 2.3 is that if a DM with preference $\succeq$ would rather have the certain outcome $q \in \mathbb{R}$ than the risky prospect X, then the more uncertainty averse $\succeq'$ DM prefers it as well. Ghirardato and Marinacci (2002)'s definition is a generalization of the standard notion of risk aversion in the context of risk under expected utility. This notion allows us to provide a suitable characterization for risk attitude for quantile preferences, as t is possible to construct a similar characterization of risk attitude for quantile preferences, as the following result establishes.[5]

Proposition 2.4. *Consider quantile maximizing preferences* $\succeq_\tau$ *and* $\succeq_{\tau'}$*. The following statements are equivalent:*

(1) $\tau \geqslant \tau'$*;*

(2) $\succeq_{\tau'}$ *is more uncertainty averse than* $\succeq_\tau$*;*

(3) *If* Y *is a quantile-preserving spread of* X *and* $X \succ_\tau Y$*, then* $X \succeq_{\tau'} Y$*.*[6]

(4) *If* Y *is a quantile-preserving spread of* X *and* $Y \succ_{\tau'} X$*, then* $Y \succeq_\tau X$*.*

Proposition 2.4 shows that $\succeq^\tau$ is more risk averse than $\succeq^{\tau'}$ if and only if $\tau < \tau'$. This property implies that an agent with quantile given by τ_1 is more risk preferring than another agent with quantile given by τ_2 if $\tau_1 > \tau_2$, independently of the functional form of the utility function. Thus, a decision maker that maximizes a lower quantile is more risk averse than one who maximizes a higher quantile. In other words, the risk attitude can be related to the quantile rather than to the concavity of the utility function. Moreover, Proposition 2.4 shows that in the QP framework individuals' risk attitude is related to the quantile rather than to the concavity of the utility function.

3 Optimal portfolio choice problem

In this section we first study the case of a portfolio given by a risk-free and a risky asset. This simple portfolio problem serves to establish the intuition about the optimal behavior of

[5] Elements of Proposition 2.4 can be found in Rostek (2010, Section 6.1) and Manski (1988, Section 5), but not in the form presented here. In particular, they do not use the language of quantile-preserving spreads introduced by Mendelson (1987) nor the notion of "more uncertainty averse than" used by Ghirardato and Marinacci (2002).

[6] Notice that we are not specifying what is the quantile $\bar{\tau}$ for which Y is a $\bar{\tau}$-quantile-preserving spread of X. The same observation is valid for the other item.

individuals with quantile preferences and motivate the problem of interest, which is the study of the optimal portfolio allocation between two or more risky assets. We focus on the analysis of two risky assets. Prior to this, we set the foundations of the optimal portfolio allocation problem.

3.1 Quantile portfolio selection under quantile preferences

We now formally describe the portfolio selection problem under quantile preferences (QP). The portfolio manager has a budget $b > 0$ to invest in n assets for a given fixed period of time. She will end up devoting $a_i \in [0, b]$ to asset i, satisfying $\sum_{i=1}^{n} a_i = b$. The initial price of asset i is $p_i > 0$, so that $a_i = p_i q_i$, where q_i denotes the number of units of asset i that the portfolio manager buys. After the investment period, asset i's price will be $\tilde{p}_i \geqslant 0$, which is random if asset i is not a risk-free asset. Therefore, the net return on asset i after the investment period is $\tilde{r}_i = \frac{\tilde{p}_i}{p_i} - 1$, which is a random variable. Consider the following portfolio

$$S_w = \sum_{i=1}^{n} w_i \tilde{r}_i,$$

where $w \equiv (w_1, ..., w_n) \in [0, 1]^n$, with $\sum_{i=1}^{n} w_i = 1$. The weights $w_i = \frac{a_i}{b} \geqslant 0$ denote the fraction of wealth invested on asset i. Implicitly, we are assuming that the portfolio manager does not short assets.[7]

To be consistent with the literature on optimal portfolio theory under EU preferences, we assume that individuals are endowed with a utility function $u(S_w)$, where $u : \mathbb{R} \to \mathbb{R}$, for describing individual's preferences on wealth. Then, for a given risk attitude $\tau \in (0, 1)$, the portfolio choice problem under QP is

$$\max_{w \in [0,1]^n} Q_\tau \left[u \left(S_w \right) \right], \text{ s.t. } \sum_{i=1}^{n} w_i = 1. \tag{4}$$

Importantly, we show that the choice of utility function is irrelevant under QP. This is due to the invariance of this approach with respect to the utility function. Hence the quantile optimization problem (4) using a given utility function is equivalent to maximizing the quantile obtained directly from the distribution of the random variable.

Lemma 3.1. *Let $u(\cdot)$ be a continuous and increasing utility function defined over the domain of the random variable S_w for $w \equiv (w_1, ..., w_n) \in [0, 1]^n$. The maximization argument w^**

[7] ~~In principle our model could encompass this possibility~~ Our model can deal with short sale, but we leave this to future work.

solves (4) *if and only if it solves the following:*

$$\max_{w \in [0,1]^n} Q_\tau [S_w], \ \ s.t. \ \sum_{i=1}^{n} w_i = 1. \tag{5}$$

Equations (4) and (5) show an important result of QP theory relative to EU, which is that the optimal choice of the portfolio under QP does *not* depend on any particular choice of utility function. Therefore, for the remaining of the paper, we focus the main analyses on the problem in (5). It is also worth noting that the above portfolio choice problem is also different, and more general, than the goal-reaching problem proposed by Kulldorff (1993) in a quantile setting. This author maximizes the cumulative probability of $\sum_{i=1}^{n} w_i \tilde{r}_i$ subject to achieving some target return r_0. More formally, the objective function is $\max_{(w_1,...,w_n)} P\{\sum_{i=1}^{n} w_i \tilde{r}_i \geqslant r_0\}$ subject to $\sum_{i=1}^{n} w_i = 1$.

The result in Lemma 3.1 is important in the context of portfolio allocation. Theoretically, it implies that the quantile choice rule is able to separate beliefs from tastes. The relevance of this separation criterion was put forward by Ghirardato et al. (2005) in the context of decision theory under uncertainty. These authors offered a result with this separation, but they did not insist on a complete separation of tastes and beliefs, because such a separation would rule out most of the choice rules commonly considered by decision theorists.[8] In contrast, as shown in Lemma 3.1, the quantile preferences deliver a complete separation of tastes and beliefs. Empirically, this separation is very important as well. In particular, it allows portfolio managers to make choices on a particular portfolio without the knowledge of any specified utility function. For instance, a manager only needs to learn about the quantile τ of an agent to choose the portfolio weights from a given selection of returns $\tilde{r}$.

3.2 Optimal portfolio allocation when there is a risk-free asset

Manski (1988) derives the preferences of a QP maximizer between two outcomes X and Y when one of the outcome measures is degenerate. In particular, this author finds a complete separation in preferences between the degenerate and risky outcome. The deterministic choice is the preferred strategy for low quantiles. In contrast, for high quantiles, the risky outcome is the preferred strategy.

In what follows, we provide further formality to the example in Manski (1988) and frame it in an optimal asset allocation context. We assume there is a riskless security that pays a rate of return equal to $R_f = \bar{r}$, and just one risky security that pays a stochastic rate of return equal

[8] For instance, if the preference is given by EU, the belief is captured by the probability while the tastes by the utility function over outcomes or consequences (such as monetary payoffs). Beliefs and tastes are not completely separated, however, because if we take a monotonic transformation of the utility function, which maintains the same tastes over consequences, we may end up with a *different* preference. That is, the pair beliefs and tastes come together and are stable only under affine transformations of the utility function. In other words, the EU preferences, as many other preferences, do not allow a separation of tastes and beliefs.

to R with distribution function F_R. The portfolio return is defined by the convex combination

$$R_p = w\bar{r} + (1-w)R = \bar{r} + (1-w)(R-\bar{r}),$$

and the investor's maximization problem (5) for a specific quantile τ is $\arg\max_w Q_\tau[u(\bar{r}+(1-w)(R-\bar{r}))]$. Using the monotonicity of the quantile process, for a continuous and increasing utility function, the investor's problem simplifies to

$$\arg\max_w (1-w)Q_\tau[R] + w\bar{r}.$$

Simple algebra shows that the individual portfolio choice w is then given by the following:

$$w^* = \begin{cases} 1 & \text{when } Q_\tau[R] < \bar{r} \\ 0 & \text{when } Q_\tau[R] > \bar{r} \\ \text{any } w \in [0,1] & \text{when } Q_\tau[R] = \bar{r}. \end{cases}$$

The intuition of this solution is simple. For small values of τ the individual's optimal portfolio choice is $w^* = 1$ and corresponds to full investment on the risk-free asset. This is so because $\bar{r} > Q_\tau[R_p]$ for any combination R_p characterized by $0 < w < 1$. For larger values of τ, such that $Q_\tau[R] > \bar{r}$, the optimal portfolio decision reverses and yields $w^* = 0$. For $Q_\tau[R] = \bar{r}$, the QP maximizer is indifferent between the risk-free and the risky asset for any $w \in [0,1]$ defining the portfolio return.

In contrast to the capital market line characterizing the mutual fund separation theorem in a mean-variance setting, Tobin (1958), the optimal portfolio allocation under QP specializes in the risk-free asset for individuals with τ below the magnitude of the risk-free rate and on the risky asset, otherwise. In Appendix C.4, we extend the analysis of the risk-free asset by adding a second risky asset to the portfolio. We obtain the same findings indicating an optimal binary response to the risk-free asset. We notice that in this case, however, diversification between the two risky assets may be optimal for middle quantiles even if the allocation to the risk-free asset is null.

3.3 The case of two risky assets

This section considers the optimal portfolio allocation problem for an economy with two risky assets and a decision maker endowed with QP.[9] Consider two risky assets represented, respectively, by the continuous random variables X and Y. Let the portfolio be defined as

$$S_w \equiv wX + (1-w)Y, \tag{6}$$

[9]See, e.g., Damodaran (2010).

with $0 \leqslant w \leqslant 1$ a scalar portfolio weight. The portfolio selection problem in this context will be:

$$\max_{w \in [0,1]} Q_\tau [S_w] \,. \tag{7}$$

Define the solution to this problem by $w^*(\tau) : (0, 1) \to [0, 1]$. Whenever there is no confusion we will use simply w^*. We will also assume that X and Y have joint distribution function given by a continuous probability density function (p.d.f.) f defined over intervals $\mathcal{I}_X$ and $\mathcal{I}_Y$. More formally:

Assumption 1. *X and Y have joint distribution function given by a continuous p.d.f. $f : \mathcal{I}_X \times \mathcal{I}_Y \to \mathbb{R}$, where $\mathcal{I}_X = [\underline{x}, \overline{x}]$, $\mathcal{I}_Y = [\underline{y}, \overline{y}]$, $-\infty \leqslant \underline{x} < \overline{x} \leqslant \infty$, $-\infty \leqslant \underline{y} < \overline{y} \leqslant \infty$ and $0 < f(x, y) < \infty$ for all $(x, y) \in \mathcal{I}_X \times \mathcal{I}_Y$.*

It is important to emphasize that the above assumption does not exclude distributions with support in the whole real line. In particular, X and Y can be normal variables, for instance. In fact, almost all distributions studied in finance satisfy Assumption 1. It shall be understood that if $\overline{x} = \infty$ or $\overline{y} = \infty$, the intervals are, respectively, $[\underline{x}, \infty)$ and $[\underline{y}, \infty)$. An analogous observation holds when $\underline{x} = -\infty$ or $\underline{y} = -\infty$. We will maintain Assumption 1 in all results of this section and will not repeat it.

In the remaining of the section we establish the existence of the optimal portfolio choice under the QP theory as well as derive conditions that determine the existence of diversification.

Lemma 3.2. *The optimization problem (7) has at least one solution.*

Lemma 3.2 shows that the QP problem has at least one optimal vector, $w^*(\tau)$, that solves the problem for a given quantile τ. Deriving an explicit expression for w^* is in general a difficult task, but in Appendix B, Proposition B.1, we deduce the expression of w^* for the case in which X is uniform on (a, b) and Y is uniform on $(0, 1)$. This explicit expression is used to plot figures illustrating some of the results below, that we derive for general random variables X and Y. We organize those results according to their message in the subsections below.

3.3.1 Diversification for low quantiles

Our first main result is that "in general" for τ sufficiently small, diversification is optimal, that is, there exists an interior solution $w^* \in (0, 1)$. As we will see in a moment, this requires some assumptions. Perhaps the most important setting is the one described in the following:

Theorem 1. *Assume that $\underline{x}, \underline{y} > -\infty$. If $\underline{x} = \underline{y}$ and $\tau \in (0, 1)$ is sufficiently small, then any optimal solution must be interior. More formally, there exists τ_0 such that if $\tau < \tau_0$ then $w \in \{0, 1\}$ does not solve (7) for that τ, or yet: if w^* solves (7), $w^* \in (0, 1)$.*

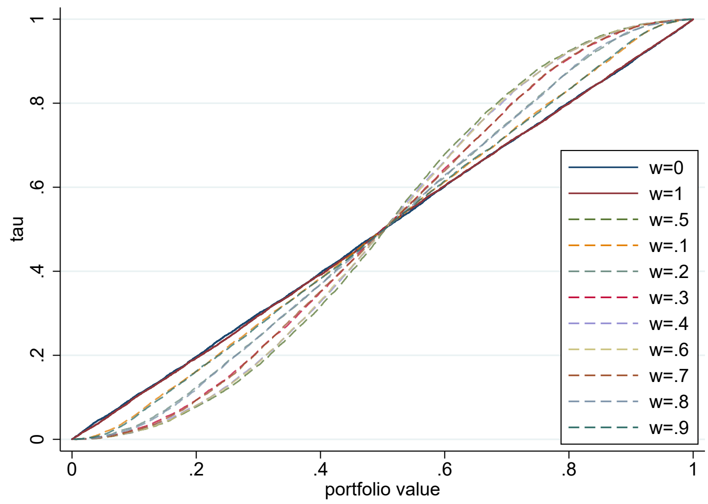


Figure 2: CDF of $S_w = wX + (1-w)Y$ indexed by $w \in [0,1]$ when $X \sim U(0,1)$ and $Y \sim U(0,1)$.

Intuitively, for a fixed small τ, the convex combination of the two assets is able to generate a larger quantile. Notice that this result is *surprisingly general*: it requires no further assumption on the distributions other than the restriction of the same lower bound. Although this condition excludes normal distributions, it includes important distributions, as for example, the lognormal. Nevertheless, Theorem 1 can be extended to symmetric normal distributions, although we omit a formal statement for space considerations.

It is useful to illustrate Theorem 1 for the case of two standard uniform distributions, $X \sim U(0,1)$ and $Y \sim U(0,1)$; see Example 3.3. Figure 2 shows the CDFs of the random variable S_w for different w. One can see that for low τ (in this case, $\tau \leqslant 0.5$), the CDF curve most to the right corresponds to $w^*(\tau) = 0.5$. If $\tau > 0.5$, the optimal $w^*(\tau)$ is either 0 or 1, that is, there is no diversification (both 0 and 1 are solutions because the two assets are identical in this case).

Example 3.3. *Consider* $X \sim U(0,1)$ *and* $Y \sim U(0,1)$*, independent. From Proposition B.1 in Appendix B, in this case we have:*

$$w^*(\tau) = \begin{cases} 0.5, & \text{if } \tau \in \left(0, \frac{1}{2}\right] \\ 1, & \text{if } \tau \in \left(\frac{1}{2}, 1\right). \end{cases}$$

Notice that there can be diversification even when one of the distribution functions stochastically dominates the other. See, for instance, Example 3.4 for the case of $X \sim U(0,2)$ and $Y \sim U(0,1)$. In this case, even though $U(0,2)$ stochastically dominates $U(0,1)$, there is diversification for $\tau \leqslant 1/4$: the optimal w is interior, $w^* = 0.5$.

Example 3.4. *Consider* $X \sim U(0,2)$ *and* $Y \sim U(0,1)$*. From Proposition B.1 in Appendix B, in this case we have:*

$$w^*(\tau) = \begin{cases} 0.5, & \text{if } \tau \in \left(0, \frac{1}{4}\right] \\ 1, & \text{if } \tau \in \left(\frac{1}{4}, 1\right). \end{cases}$$

This is an interesting result, because despite the fact that X first order stochastically dominates Y, there exists a convex combination S_w that dominates both random variables X and Y for low quantiles. Notice, however, that this feature is desirable, because the independence of X and Y makes a convex combination of the two less risky than any of them.

3.3.2 No diversification with different lower ends of the distributions

Given the result in Theorem 1, a natural question is what would happen if the assumption that $\underline{x} = \underline{y}$ of the theorem does not hold, that is, if $\underline{x} \neq \underline{y}$. The naïve intuition may be that for low quantiles diversification is always optimal. Theorem 1 shows that this occurs if τ is sufficiently small and there is no obvious difference in the lower limits given by $\underline{x}$ and $\underline{y}$. However, when the tail behavior of the assets in the portfolio is very different, the next result shows that diversification is not optimal for low quantiles. More formally, Theorem 2 shows that for $\underline{x} - \underline{y} \geqslant \frac{M}{2m}$, with m and M suitable constants, the quantile of X is larger than the quantile of any convex combination of X and Y for low values of τ. A natural interpretation of this result in a risk management context is to say that the VaR of X is larger than the VaR of any diversified combination of the assets.[10] The investor allocates all the portfolio weight in the variable that dominates the other in the left tail of the distribution. This is the message of the next result, where we denote X's τ-quantile by x_τ.

Theorem 2. *Assume that* $\underline{x} > \underline{y} > -\infty$*. Fix* $\overline{\tau} \in [0,1]$*. Let* M *and* m *be such that* $m \leqslant f(x,y) \leqslant M$ *for all* $(x,y) \in [\underline{x}, x_{\overline{\tau}}] \times [\underline{y}, \overline{y}] \cup [\underline{x}, \overline{x}] \times [\underline{y}, x_{\overline{\tau}}]$*.*[11] *If* $\underline{x} - \underline{y} \geqslant \frac{M}{2m}$*,* $w^* = 1$ *is the unique solution to* (7) *for all* $\tau \in (0, \overline{\tau})$*.*[12]

Theorem 2 shows that the optimal choice is $w^* = 1$ for all τ small, provided that the difference between the two distributions at the left end point is sufficiently large. Example 3.5 illustrates this result for the case of $X \sim U(0.5, 1)$ and $Y \sim U(0,1)$. Since X and Y are uniform and $\underline{y} = 0$, we can take $m = M$ and the assumption of Theorem 2 simplifies to $\underline{x} \geqslant \frac{1}{2}$, which is precisely the condition satisfied in this example (with $\underline{x} = \frac{1}{2}$).

Example 3.5. *Consider* $X \sim U(0.5, 1)$ *and* $Y \sim U(0,1)$*. Then,* $w^* = 1$ *for all* $\tau \in (0,1)$*.*

Notice that in this example, we have the choice $w^* = 1$ for all $\tau \in (0,1)$, which is stronger than the result stated in Theorem 2. We can in fact establish this stronger condition if the

[10] In the context of risk management, Theorem 2 provides theoretical support to the lack of subadditivity of VaR measures in general settings, see Artzner et al. (1999).

[11] The inferior limit m may be taken over a limited region, not over the whole support. That is, we can accommodate cases in which the support is infinite so that $f(x,y) \to 0$ when $y \to \infty$.

[12] Of course, $w^* = 0$, when $\underline{y} > \underline{x} > -\infty$ and $\underline{y} - \underline{x} \geqslant \frac{M}{2m}$.

bounds m and M hold for the entire interval. More precisely, we can fix $\bar{\tau} = 1$ in Theorem 2 and conclude the following:

Corollary 3.6. *Assume that* $\underline{x} > \underline{y} > -\infty$*. Let* M *and* m *be such that* $m \leqslant f(x, y) \leqslant M$ *for all* $(x, y) \in [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}]$*. If* $\underline{x} - \underline{y} \geqslant \frac{M}{2m}$*,* $w^* = 1$ *is the unique solution to (7) for all* $\tau \in (0, 1)$*.*

The latter result shows the absence of diversification, across $\tau \in (0, 1)$, for individuals with quantile preferences under the conditions of the corollary. This result suggests that in many cases the efforts of portfolio managers may be futile under QP theory. A major implication of Corollary 3.6 is to show that, under QP, if there is no diversification in the lower left tail of the distribution it is very likely that there is no diversification for higher quantiles.

Nevertheless, the behavior with different lower end points can be complex. For instance, it may be the case that the optimal choice is $w^* \in \{0, 1\}$ for small τ, it becomes interior for intermediate values of τ and then becomes $w^* \in \{0, 1\}$ again for large τ's. The following example illustrates a case when the conditions in the corollary are not satisfied. In this case we find the existence of diversification in the middle quantiles of the distribution despite the fact that there is no diversification in the lower left tail.

Example 3.7. *Consider* (X, Y) *a bivariate random vector. Let* $f_X(x) = M$ *be the marginal density function of* X *if* $0 < \underline{x} \leqslant x \leqslant d$*,* $f_X(x) = m$ *if* $d < x \leqslant \bar{x} < \infty$*, and* $f_X(x) = 0$ *otherwise. The support of* X *is* $[\frac{1}{2}, \frac{3}{2}]$*, i.e.* $\underline{x} = \frac{1}{2}$ *and* $\bar{x} = \frac{3}{2}$*. Let* $Y \sim U(0, 1)$ *such that* $f_Y(y) = 1$ *for* $y \in [\underline{y}, \bar{y}]$*, with* $\underline{y} = 0$ *and* $\bar{y} = 1$*, and* $f_Y(y) = 0$*, otherwise. Then, the following conditions of Corollary 3.6 are satisfied: (i)* $\underline{x} > \underline{y} > -\infty$*, (ii)* M *and* m *are such that* $m \leqslant f(x, y) \leqslant M$ *for all* $(x, y) \in [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}]$*. However, for* $M = 4$*,* $m = \frac{2}{3}$ *and* $d = 0.6$*, we have* $\underline{x} - \underline{y} = 0.5 < 3 = \frac{M}{2m}$*, such that the remaining condition of the corollary is not satisfied.*

Figure 3 reports the CDFs for different combinations S_w indexed by $w \in [0, 1]$ for the density functions in Example 3.7. For small and large values of τ the optimal allocation is $w^* = 1$, however, there is a middle interval of τ for which diversification is optimal. This can be seen by noting that some CDF for $w \in (0, 1)$ crosses from below that of $w = 1$ that corresponds to X.

Another interesting case is posed by Theorem 2. One might think that the conclusion of this theorem could hold more generally, whenever $\underline{x} > \underline{y}$ and τ is sufficiently small. This is false, however: the difference between $\underline{x}$ and $\underline{y}$ must be bounded away from zero to ensure the conclusion. The following examples illustrate this observation from two applications of Proposition B.1 in Appendix B.

Example 3.8. *Consider* $X \sim U(0.25, 0.75)$ *and* $Y \sim U(0, 1)$*. We have* $w^* \in (0, 1)$ *for* $\tau \in \left(0, \frac{1}{2}\right)$ *and* $w^* = 0$ *for* $\tau \in \left(\frac{1}{2}, 1\right)$*. See Figure 4. Its left panel plots the optimal allocation* $w^*(\tau)$*, while its right panel plots the* τ*-quantiles of* X*,* Y *and the optimal portfolio* $S_{w^*(\tau)}$*.*

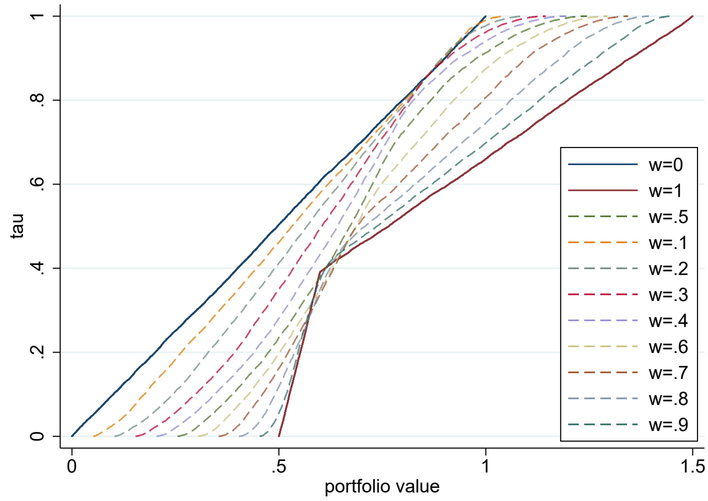


Figure 3: CDF of $S_w = wX + (1-w)Y$ indexed by $w \in [0, 1]$ when (X, Y) are defined in the Example 3.7.

Example 3.9. *Consider* $X \sim U(0.25, 1.25)$ *and* $Y \sim U(0, 1)$. *We have* $w^* \in (0, 1)$ *for* $\tau \in (0, 0.25)$ *and* $w^* = 1$ *for* $\tau > 0.25$. *See Figure 5. As in the previous case, the left panel plots* $w^*(\tau)$, *while its right panel plots the* τ*-quantiles of* X, Y *and the optimal portfolio* $S_{w^*(\tau)}$.

3.3.3 No diversification for large quantiles

The examples above suggest that for high τ, the optimal choice is $w^* \in \{0, 1\}$. This is indeed correct, as the following result establishes. It shows that the optimal portfolio choice for values of τ close to 1 is not interior.

Theorem 3. *Assume that* $\underline{x} = \underline{y} < \overline{y} = \overline{x} < \infty$. *Fix* $\overline{\tau} \in (0, 1)$. *Let* M *and* m *be such that* $m \leqslant f(x, y) \leqslant M$ *for all* $(x, y) \in [x_{\overline{\tau}}, \overline{x}] \times [\underline{y}, \overline{y}] \cup [\underline{x}, \overline{x}] \times [x_{\overline{\tau}}, \overline{y}]$.[13] *If* $x_{\overline{\tau}} - \underline{x} \geqslant \frac{M(\overline{x}-\underline{x})}{2m}$, *then* $w^*(\tau) \in \{0, 1\}$ *for all* $\tau \in [\overline{\tau}, 1)$.

Note that in this scenario we may have one or two solutions for τ sufficiently close to 1. In other words, there is no portfolio diversification in the upper tail of the distribution. These findings are related to Ibragimov and Walden (2007). These authors find that for truncated versions of heavy-tailed distributions with unbounded support diversification may increase value at risk as long as the random variables are concentrated on a sufficiently large interval. The results in this section generalize Ibragimov and Walden (2007) by considering a wider class of distribution functions characterized by assumption 1. In particular, we derive the conditions

[13] The inferior limit m may be taken over a limited region, not over the whole support. That is, we can accommodate cases in which the support is infinite so that $f(x, y) \to 0$ when $y \to \infty$.

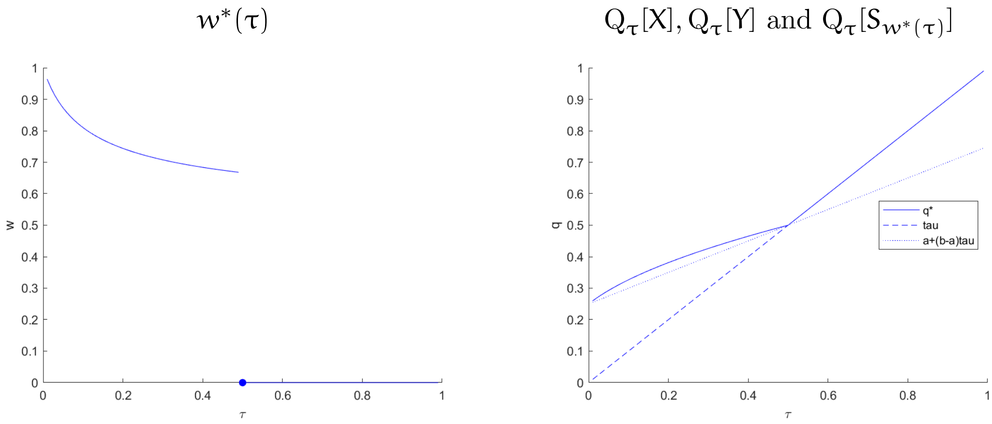


Figure 4: Illustration of Example 3.8—Optimal w^* for $X \sim U(0.25, 0.75)$ and $Y \sim U(0, 1)$.

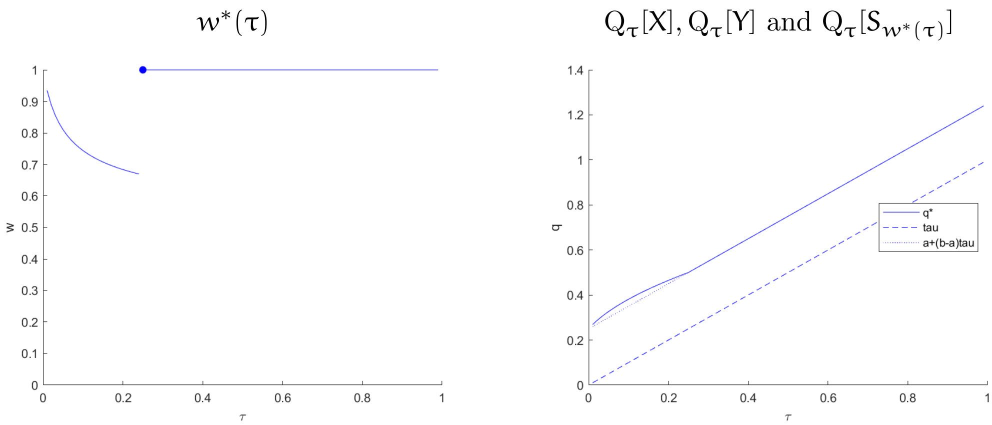


Figure 5: Illustration of Example 3.9—Optimal w^* for $X \sim U(0.25, 1.25)$ and $Y \sim U(0, 1)$.

under which diversification in the tails may or may not be an optimal outcome for individuals endowed with QP.

In the following section we proceed to characterize the existence of interior solutions to the QP optimal portfolio choice problem.

3.3.4 Characterization of the interior solution in QP optimal portfolio choice

The preceding section has presented conditions on the support of the random variables X and Y that lead to either full diversification or null diversification. We have also extended the results to the tails of the distributions by showing that for lower bound supports far apart diversification is not an optimal outcome. In this section, we build upon these results and provide a characterization of the interior solutions to the optimal portfolio choice problem under QP theory. The section also establishes properties of the portfolio selection problem in a QP framework.

The proof of the results below will depend on some new definitions. First, let us denote $\Pr(S_w \leqslant q)$ by $h(w,q)$, that is,

$$h(w,q) \equiv \int_{\mathcal{I}_X} \int_{\mathcal{I}_Y \cap \left(-\infty, \frac{q-wx}{1-w}\right]} f(x,y)\,dy\,dx, \tag{8}$$

if $w \in (0,1)$, $h(0,q) \equiv F_Y(q)$ and $h(1,q) \equiv F_X(q)$. Let us focus on the case $w \in (0,1)$. From Assumption 1, we know that as long as $\left\{\left(x, \frac{q-wx}{1-w}\right) : x \in \mathbb{R}\right\} \cap \mathcal{I}_X \times \mathcal{I}_Y$ contains more than one point, the equation $h(w,q) = \tau \in (0,1)$ implicitly defines the quantile $q = q_{w,\tau}$ of S_w as a function of $w \in (0,1)$ and $\tau \in (0,1)$, that is,

$$h(w, q_{w,\tau}) = \tau \implies q(w,\tau) \equiv Q_\tau[S_w].$$

In what follows, we will omit τ from $q(w,\tau)$ such that $q(w,\tau) \equiv q(w)$.

Let us define $\mathcal{I}_Z = \mathcal{I}_Z^{w,q} = \{x \in \mathbb{R} : \left(x, \frac{q-wx}{1-w}\right) \in \mathcal{I}_X \times \mathcal{I}_Y\}$ and assume that w and q are such that $\mathcal{I}_Z = \mathcal{I}_Z^{w,q}$ is a proper interval. Now, let us define a random variable $Z = Z_{w,q}$ that is characterized by the following density function:

$$f_Z(x) \equiv \frac{f\left(x, \frac{q-wx}{1-w}\right)}{\int_{\mathcal{I}_Z} f\left(t, \frac{q-tw}{1-w}\right) dt}, \tag{9}$$

for $x \in \mathcal{I}_Z$. This density function is strictly positive on $\mathcal{I}_Z$ and zero otherwise. In order to understand what Z corresponds to, consider Figure 6. This figure plots the region $\left\{\left(x, \frac{q-wx}{1-w}\right) : x \in \mathbb{R}\right\} \cap [0,1] \times [0,1]$ in a XY plane for the case of two standard uniform distributions. The left plot in Figure 6 describes the problem for small quantiles ($\tau \leqslant \frac{1}{2}$) and $w = \frac{1}{2}$, and the right plot for large quantiles ($\tau > \frac{1}{2}$) and $w = \frac{1}{2}$. More generally, the random variable Z can be interpreted as the projection of $Y = \frac{q-wX}{1-w}$ onto X, and has support drawn in red and density given by the values in the probability distribution function along the blue line.[14]

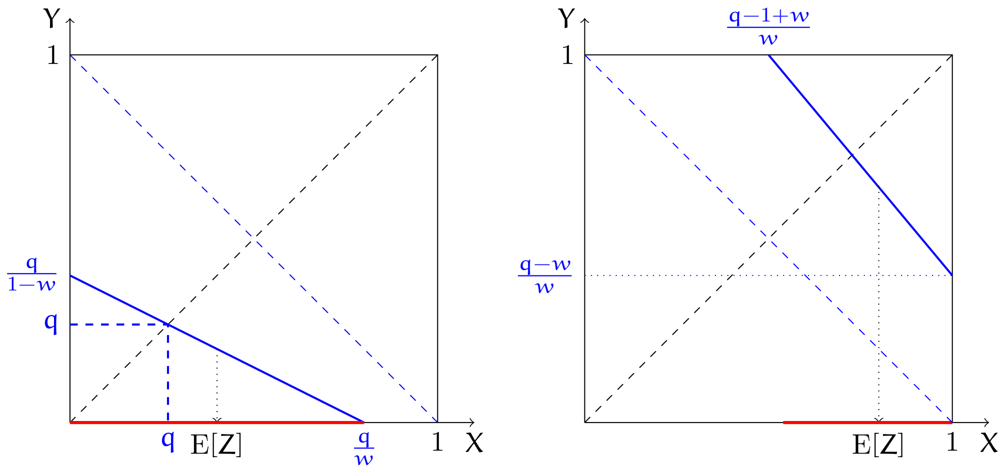
The definition of the random variable Z through the characterization of its density function (9), allow us to state our first result in this section.

Proposition 3.10. *Let X and Y be random variables satisfying assumption 1, and let $Z = Z_{w,q}$ be a random variable characterized by the density function (9). Then, the function $q(w)$ is differentiable at $w \in (0,1)$ and*

$$q'(w) = \frac{1}{1-w}\left(\mathrm{E}[Z] - q(w)\right), \text{ for } \tau \in (0,1), \tag{10}$$

provided that one of the following cases hold: (1) $\mathcal{I}_X = \mathcal{I}_Y = \mathbb{R}$; (2) $\mathcal{I}_X = \mathcal{I}_Y = \mathbb{R}_+$; (3)

[14] The two illustrations (a) and (b) correspond, respectively, to $\tau \leqslant \Pr\left(\left[\frac{X+Y}{2} \leqslant q\right]\right)$ and $\tau > \Pr\left(\left[\frac{X+Y}{2} \leqslant q\right]\right)$. In the first case (a), the line $wX + (1-w)Y = q$ is below the dashed blue line joining $(1,0)$ to $(0,1)$. In case (b) $\tau > \Pr\left(\left[\frac{X+Y}{2} \leqslant q\right]\right)$, the line $wX + (1-w)Y = q$ is above the dashed blue line.



(a) $\tau \leqslant \Pr\left(\left[\frac{X+Y}{2} \leqslant q\right]\right)$ (b) $\tau > \Pr\left(\left[\frac{X+Y}{2} \leqslant q\right]\right)$

Figure 6: An illustration of Z for the case $\mathcal{I}_X = \mathcal{I}_Y = [0, 1]$.

$\mathcal{I}_X = \mathcal{I}_Y = [0, c]$, *with* $c > 0$ *a constant defining a compact support.*

This result allows one to derive the optimality condition characterizing the optimal portfolio choice.

Corollary 3.11. *Let* $w^* \in (0, 1)$ *be an interior solution to the maximization problem* (7) *for a given* $\tau \in (0, 1)$. *Under the conditions of Proposition 3.10, the optimal* w^* *satisfies the condition* $E[Z_{w^*,q}] = q(w^*)$, *with* $\frac{\partial}{\partial w} E[Z_{w^*,q}] < 0$.

To understand the optimality condition in Corollary 3.11, we revisit Example 3.3 by considering again the particular case of two standard uniform random variables, $X, Y \sim U(0, 1)$.

Example 3.12. *Let* X *and* Y *be* iid $U(0, 1)$ *random variables. In this case, if* $\tau \leqslant \frac{1}{2}$, *we have* $E[Z] = \frac{q}{2w}$. *Then,* $q(w)$ *is increasing if*

$$E[Z] > q \iff \frac{q}{2w} > q \iff w < \frac{1}{2}.$$

The maximum is achieved at $w^* = \frac{1}{2}$ *given that* $E[Z] = q \iff \frac{q}{2w^*} = q \iff w^* = \frac{1}{2}$. *To show that* w^* *is a maximum we also note that the function* $q(w)$ *is strictly decreasing for* $w > \frac{1}{2}$ *and* $\tau \leqslant \frac{1}{2}$. *This case is illustrated in Figure 6(a). On the other hand, if* $\tau > \frac{1}{2}$, *Figure 6(b) depicts* Z. *In this case, we can see that*

$$E[Z] = \frac{1}{2}\left(\frac{q-1+w}{w} + 1\right) = 1 - \frac{1-q}{2w} > q \iff \frac{1-q}{2w} < 1 - q \iff w > \frac{1}{2},$$

which implies that the function $q(w)$ *is increasing for* $w > \frac{1}{2}$ *and achieves the maximum at* $w^* = 1$*, for all* $\tau > \frac{1}{2}$.

In this example there is a separation in the optimal portfolio allocation between risk-averse individuals and risk lovers. The optimal portfolio allocation of risk-averse individuals, characterized by $\tau \in (0, \frac{1}{2}]$, is full diversification given by $w^* = \frac{1}{2}$. Risk lovers, characterized by $\tau \in (\frac{1}{2}, 1)$, maximize the objective function (7) at $w^* = 1$. This important result separating the optimal portfolio allocation into two regions broadly representing the risk preferences of risk-averse and risk-loving individuals can be generalized to other density functions under the following assumptions.

Assumption 2. *The joint density function* $f(x, y)$ *of the bivariate random variable* (X, Y) *is* C^1 *and satisfies that, for all fixed* $\mu \in \mathbb{R}$, $f(\mu + \varepsilon, \mu - \frac{w}{1-w}\varepsilon)$ *is unimodal on* $\varepsilon \in \mathbb{R}$*, with mode at* $\varepsilon = 0$.

Assumption 3. *The joint density function* $f(x, y)$ *of the bivariate random variable* (X, Y) *is such that there exists some* $w^* \in (0, 1)$ *satisfying the condition*

$$f\left(\mu + \varepsilon, \mu - \frac{w^*}{1 - w^*}\varepsilon\right) = f\left(\mu - \varepsilon, \mu + \frac{w^*}{1 - w^*}\varepsilon\right) \qquad (11)$$

for all $\mu \in \mathbb{R}$ *and* $\varepsilon > 0$.

Assumption 2 evaluated at the optimal w^* guarantees the unimodality of the density function f_Z defined in (9). Similarly, Assumption 3 guarantees its symmetry. Note that condition (11) does not necessarily imply the symmetry of $f(x, y)$ in the sense $f(x, y) = f(y, x)$ unless $w^* = \frac{1}{2}$.

The following result constitutes one of the main results of this study, namely, we show the existence of two regimes in the optimal portfolio allocation for individuals characterized by QP and maximizing the objective function (7).

Proposition 3.13. *Let* $f(x, y)$ *be the joint density function of the pair* (X, Y)*. Under assumptions 1 to 3, the solution to the maximization problem* (7)*, for all* $\tau \in (0, \tau_0]$*, is* $w^* \in (0, 1)$ *that satisfies expression* (11)*. For* $\tau > \tau_0$*, the solution is* $w^* \in \{0, 1\}$*. More specifically,* $w^* = 0$ *for* $\tau > \tau_0$ *if* $w^* \geqslant \frac{1}{2}$ *in the interval* $(0, \tau_0]$*, and* $w^* = 1$ *for* $\tau > \tau_0$ *if* $w^* \leqslant \frac{1}{2}$ *in the interval* $\tau \in (0, \tau_0]$*. The threshold* τ_0 *is defined by the condition* $\tau \leqslant P(Z \leqslant q)$ *for* $\tau \in (0, \tau_0]$*, and* $P(Z \leqslant q) < \tau$ *for* $\tau \in (\tau_0, 1)$.

This result accommodates the $\mathtt{iid}$ case as a particular example.

Corollary 3.14. *Let* X *and* Y *be two* $\mathtt{iid}$ *random variables with unimodal density function* $f_X(\cdot)$ *that satisfies assumption 1. The solution to the maximization problem* (7) *is* $w^* = \frac{1}{2}$ *for* $\tau \in (0, \tau_0]$*, and* $w^* \in \{0, 1\}$*, indistinctively, for* $\tau > \tau_0$.

Proposition 3.13 and Corollary 3.14 offer very interesting insights about portfolio allocation and the importance of diversification. Under some general assumptions that accommodate, among other examples, very popular families of density functions such as the Normal distribution and the Student-t distribution, diversification is optimal for individuals with preferences characterized by risk aversion (low τ-quantiles). Individuals characterized by these preferences choose the same portfolio, regardless their specific τ, that is given by an interior solution. For iid random variables or for symmetric bivariate random variables the optimal allocation is full diversification, interpreted as an equal contribution of each asset to the portfolio. In contrast, individuals with preferences characterized by high values of τ, do not diversify at all. The optimal investment strategy of these agents is full investment in the asset with more upside potential, that in these cases corresponds to the asset with higher downside risk too. In this case the specific value of τ characterizing the individual's quantile preference is not relevant for determining the optimal portfolio allocation as long as this value is greater than the cut-off point τ_0. For symmetric distributions, $\tau_0 = 0.5$.

3.4 Numerical results

In this section we study optimal asset allocation under QP for portfolios given by the mixture of two continuous random variables X and Y with payoff realizations defined as $S_w = wX + (1-w)Y$. We also compare the results with the optimal asset allocation obtained under EU. Here, for simplicity and brevity, we concentrate on the simple case of two Gaussian independent and identically distributed random variables. Nevertheless, Appendix C presents a more comprehensive examination across different scenarios of the problem allowing for dependence and different configurations of the Gaussian and Chi-squared distributions.

In order to compute the optimal portfolio $w^*(\tau) : (0, 1) \to [0, 1]$ we are required to compute the cumulative distribution function F_{S_w} and quantile function $Q_\tau[S_w]$, for which in many cases we do not have an analytical expression to work with. We approximate the optimal portfolio allocation w^* and quantile function through simulation. The following example aims to illustrate the practical selection and diversification of portfolios, as well as the theoretical findings of the previous sections.

3.4.1 Design

In this numerical study we seek to compute the optimal portfolio weights $w^*(\tau)$ for the QP model in equation (5) for $\tau \in (0, 1)$. To simulate a portfolio, we draw $n = 10,000$ realizations of the random variables (X, Y), which are chosen from different distributions, say Gaussian and Chi-squared. We investigate several alternative scenarios using independence and dependence of the variables X and Y. In addition, we vary the mean and variance of the distribution functions.

For each case below, we report two figures. The figures on the left panel plot the cumulative distribution functions generated by the portfolio value S_w, that is, we plot F_{S_w} for $w \in \{0.1, \ldots, 0.9\}$. These figures intuitively characterize the solutions because for any τ we can draw a horizontal line and find the optimal QP solution by searching for the largest CDF. Second, figures on the right panel plot the graph of the portfolio selection $\{w^*(\tau), \tau\}$ for different values of τ.

3.4.2 Computation of portfolio

To implement the optimization, we use a simple grid search. Let $T_j = \{0 < \tau_1 < \ldots < \tau_j < 1\}$ be a grid of values for the quantile index τ with j values, and let $\Lambda_l = \{0 \leqslant w_1 < \ldots < w_l \leqslant 1\}$ be a grid of values for w with l values. For each fixed $\tau \in T_j$ we solve the following portfolio problem numerically

$$\widehat{w}_n^*(\tau) = \arg \max_{\{w \in \Lambda_m\}} \widehat{Q}_\tau[S_w], \tag{12}$$

where $\widehat{Q}_\tau$ is the sample counterpart of the quantile function Q_τ.

We consider T_j with grid spacing of 0.01 ($\#T_j = 99$, with $\tau_1 = 0.01$ and $\tau_{99} = 0.99$) and Λ_l with grid spacing of 0.01 ($\#\Lambda_j = 101$, with $w_1 = 0$ and $\tau_{101} = 1$).

Consider a simulation exercise given by a very simple case where we consider a weighted combination of two independent standard Gaussian random variables, $X, Y \sim \text{iid } N(0, 1)$. It is well known that $S_w = wX + (1 - w)Y$ also follows a Gaussian distribution with $S_w \sim N(0, w^2 + (1 - w)^2)$. In this scenario, under risk aversion and for very general forms of the utility function, e.g. mean-variance, CRRA and CARA utility functions embedded in the optimal portfolio decision, risk-averse EU individuals choose the investment portfolio that minimizes the variance of the random variable S_w. In this setting this is given by $w^* = 0.5$.

The results for the QP model presented on the right panel of Figure 7 confirm these findings for low quantiles of the distribution but show more heterogeneity in the optimal portfolio decision for the upper quantiles. The findings of this simulation exercise provide empirical evidence showing that the optimal QP choice depends on the quantile $\tau \in (0, 1)$. In particular, for this specification of the random variables X and Y the optimal weight w^* is a function of τ such that $\widehat{w}_n^*(\tau) = 0.5$ for $\tau \leqslant 0.5$, and either $\widehat{w}_n^*(\tau) = 0$ or $\widehat{w}_n^*(\tau) = 1$ for $\tau > 0.5$. This result is a consequence of the fact that the distributions of X and Y are identical and confirm the predictions in Corollary 3.14 about a lack of diversification in the upper tail of the distribution of portfolio returns. The left panel of the figure plots all combinations of distributions $\widehat{F}_{S_w}$ as a function of $w \in [0, 1]$. These distributions reveal a unique single-crossing point at $\tau_0 = 0.5$ for all w.

This simple exercise sheds light on portfolio selection under QP. As mentioned above, the QP do not rely on any functional form of the utility function. In addition, QP models have the advantage of allowing for heterogeneity through the quantiles because it offers a family of preferences indexed by τ, with the risk attitude under the QP being captured by the quantile.

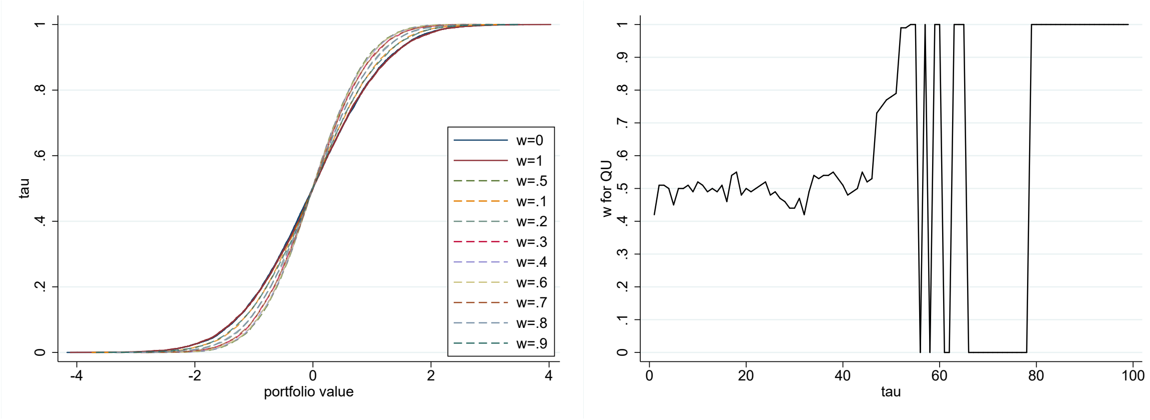


Figure 7: $X, Y \sim N(0, 1)$. Left panel plots the CDF of S_w. Right panel plots QP portfolio selection.

The results in Figure 7 show that individuals with QP encompass both risk-averse and risk-loving behavior. Those economic agents concerned with downside risk (low quantiles) diversify by investing equally in each asset, whereas individuals with preferences driven by the upper quantiles of the distribution of portfolio returns do not diversify at all. The critical point that determines whether an individual diversifies or not is $\tau_0 = 0.5$.[15]

This neat separation between risk-aversion and risk-loving behaviors characterized by the quantile τ_0 is also observed, more generally, for distribution functions satisfying the conditions of Proposition 3.13. The result in the proposition accommodates departures from the iid case, otherwise, if the conditions of the proposition are not satisfied then the optimal asset allocation under QP is $\tau-$dependent and shifts smoothly from a risk-averse optimal allocation to a risk-loving optimal allocation. Appendix C illustrates this case and collects several additional results across different scenarios, including the analysis of portfolios with a risk-free asset. The results provide strong support to the theoretical findings in this paper.

4 Empirical application to a portfolio of stocks, bonds and cash

This section provides an empirical application illustrating the methods developed in the paper. We investigate the optimal portfolio choice of an investor with QP preferences that can allocate wealth among three assets: a risk-free asset (one-month Treasury bill rate), a bond index (G0Q0 Bond Index), and a stock index (S&P 500). We consider monthly data collected from Bloomberg on the S&P 500 and G0Q0 Bond Index for the period January 1980 to December 2016. The G0Q0 Bond Index is a Bank of America and Merrill Lynch U.S. Treasury Index that tracks the performance of U.S. dollar denominated sovereign debt publicly issued by the U.S. government in its domestic market. The nominal yield on the U.S. one-month risk-free

[15]The simulation algorithm reflects this issue by selecting either $w = 0$ or $w = 1$ for $\tau_0 = 0.5$.

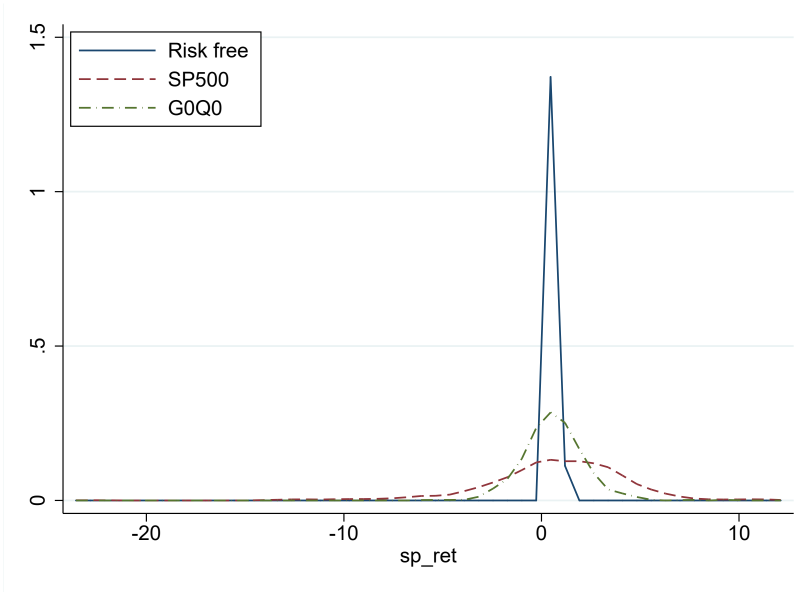


Figure 8: Nonparametric kernel estimates of the unconditional densities of monthly log-returns on the U.S. one-month Treasury bill, the G0Q0 bond index and the S&P 500 index. Monthly data are collected from Bloomberg on the S&P 500 and G0Q0 Bond Index for the period January 1980 to December 2016. The risk-free rate is obtained from Kenneth French website.

rate is obtained from Kenneth French website.

Prior to computing the portfolio weights, we report nonparametric kernel estimates of the unconditional density functions of the returns on the three assets. The results are given in Figure 8. Visual inspection of the densities shows that the three density functions are unimodal and exhibit similar mean returns but very different standard deviations. A formal statistical analysis rejects, however, the null hypothesis of equality of means for all pairwise combinations of the U.S. Treasury bill, the G0Q0 index and the S&P 500 index, and the null hypothesis of symmetry of the three density functions. More specifically, the mean return and standard deviation for the U.S. one-month Treasury bill are 0.363 and 0.296, respectively; the mean return and standard deviation for the G0Q0 bond index are 0.608 and 1.584, respectively, and 0.680 and 3.635 for the S&P 500 index. These summary statistics suggest that the equity index has the highest expected return and variance, and is followed by the bond index with regards to expected return and risk. In contrast, the U.S. Treasury bill has the lowest mean and variance.

We compute the quantile preferences (QP) portfolio optimal weights by solving the maximization problem (5) numerically, as in equation (12). We divide the analysis in two. First, we study the optimal portfolio allocation between the U.S. one-month Treasury bill and the G0Q0 index, and between the U.S. one-month Treasury bill and the S&P 500 index. Figure 9 clearly confirms the predictions of previous sections. The optimal portfolio allocation of a QP investor

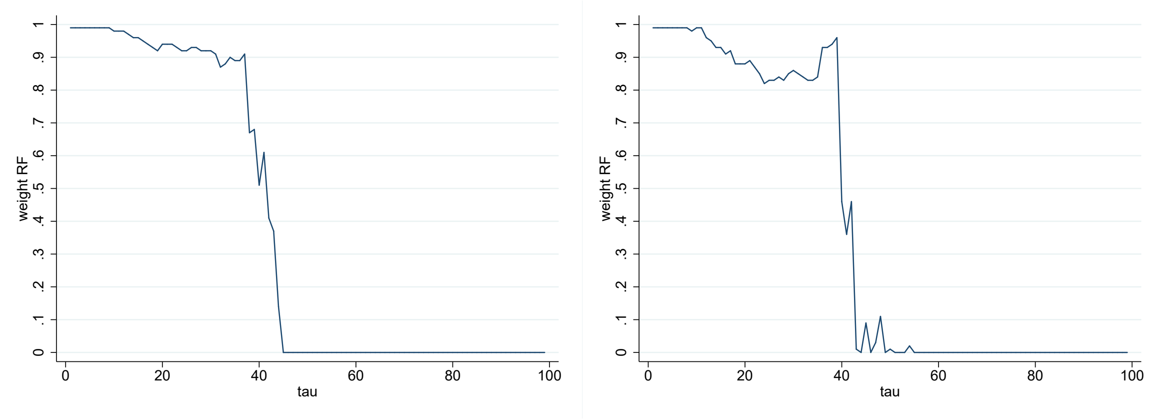


Figure 9: Left panel reports the optimal portfolio allocation between the U.S. risk-free asset and G0Q0 bond index. Right panel reports the optimal portfolio allocation between the U.S. risk-free asset and the S&P 500 index. Monthly data are collected from Bloomberg on the S&P 500 and G0Q0 Bond Index for the period January 1980 to December 2016. The risk-free rate is obtained from Kenneth French website.

can be divided into two regions indexed by $\tau \in (0, 1)$. For low values of τ, individuals are risk averse and choose to minimize risk by allocating all the wealth on the risk-free asset. However, for high values of τ, individuals become risk lovers and choose the riskiest strategy that brings the highest upside potential. Interestingly, the results in Figure 9 highlight the role of the U.S. one-month Treasury bill as a risk-free investment and are consistent with the insights discussed in Section 3.2 on the mutual fund separation theorem for the QP case. For low quantiles, the optimal choice is to fully invest on the risk-free asset, whereas for high quantiles, the optimal choice is to invest fully on the risky alternative.

Second, we study the allocation problem where the investment universe comprises the three assets. In this case we report the weights of all the three assets. The results are given in Figure 10 and provide similar insights about the optimal portfolio allocation exercise. We observe a separation between the risk-free and riskiest asset for low and high quantiles. In particular, for very low quantiles the optimal portfolio allocation is given by only investing in the risk-free asset. In contrast, for values of τ beyond the mode, the optimal asset allocation is given by fully investing on the S&P 500 index. As τ increases, we find that the optimal portfolio allocation is given by a combination of the three assets, with a large share of investment on the risk-free asset and a small share of investment distributed equally between the G0Q0 bond index and the equity index. The allocation to the S&P 500 index with respect to the other two assets increases as the tolerance of the individual towards risk grows.

We conclude the section by comparing these results with the optimal portfolio allocation of a EU investor with preferences characterized by two different types of utility function: mean-

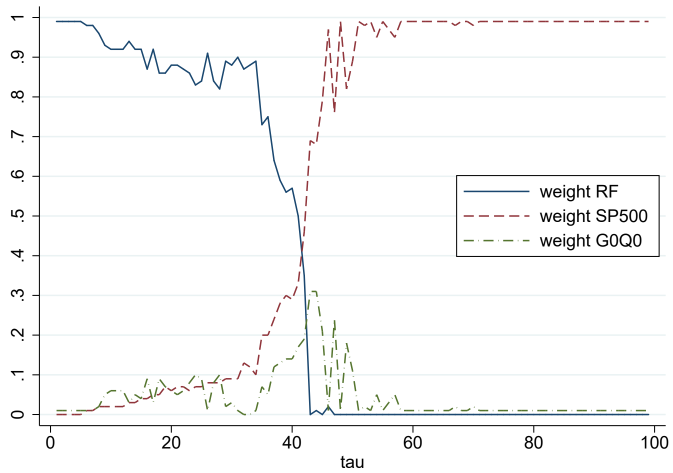


Figure 10: The optimal portfolio allocation between the U.S. risk-free asset, G0Q0 bond index and the S&P 500 index. Monthly data are collected from Bloomberg on the S&P 500 and G0Q0 Bond Index for the period January 1980 to December 2016. The risk-free rate is obtained from Kenneth French website.

variance and power utility functions. The mean-variance utility function is defined as

$$U(\mu, C) = w'\mu - \frac{\alpha}{2} w' C w, \tag{13}$$

with μ the vector of mean returns and C the covariance matrix of returns; the vector w denotes the optimal portfolio weights. This is performed for different degrees of risk aversion α. The left panel of Figure 11 presents the optimal portfolio allocations for values of α between 0 and 0.5. In this range, the optimal portfolio allocation leads to a diversified portfolio that contains non-zero combinations of the three assets. For values of α close to zero, corresponding to risk neutrality, the optimal portfolio allocation is mainly driven by investment in the S&P 500 index. However, as the tolerance to risk decreases, investment in the bond index and the US Treasury bill gains importance. Investment in the risk-free asset is monotonically increasing on α and dominates the portfolio for values greater than 0.3. Comparison of both sets of results suggests interesting similarities and differences across types of individuals. Thus, there is a mapping between the optimal portfolio choice of the QP investor with risk preferences characterized by τ in the interval $(0.40, 0.60)$ and the optimal portfolio choice of the mean-variance investor with values of α in the range 0.05 to 0.45. The QP optimal allocation for values of τ close to zero is also similar to the mean-variance allocation for values of α greater than 0.5, signalling risk aversion. The main differences are, however, for behaviors related to risk-loving attitudes. Thus, for values of τ greater than 0.5 we find that the allocation of the QP individual is concentrated on the equity index. This result is only found for mean-variance

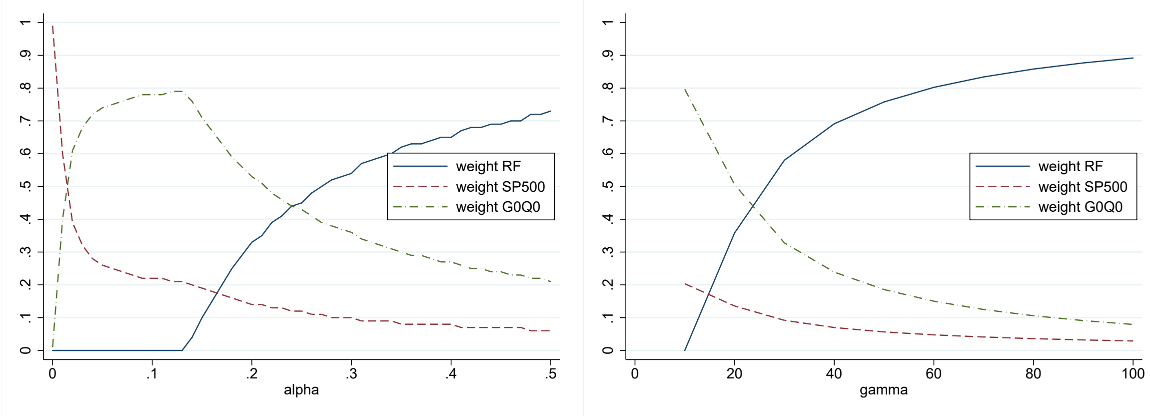


Figure 11: Left panel reports the optimal portfolio allocation for a mean-variance investor with risk preferences modeled by $\alpha \in [0, 0.5]$. Right panel reports the optimal portfolio allocation for an investor with a CRRA utility function with risk aversion coefficient given by $\gamma \in [10, 100]$. Monthly data are collected from Bloomberg on the S&P 500 and G0Q0 Bond Index for the period January 1980 to December 2016. The risk-free rate is obtained from Kenneth French website.

investors for values of α very close to zero that reflect no risk penalty in the utility function (13).

For completeness, the right panel of Figure 11 reports the optimal asset allocation problem for a EU individual with a power utility function characterized by different degrees of relative risk aversion. Interestingly, the results are very similar to those obtained for the mean-variance case. In this example, risk-loving attitudes for EU individuals take place for values of γ that converge to zero.

5 Conclusion

This paper studies the optimal asset allocation problem for individuals with quantile preferences (QP). The proposed QP model has several attractive features: (i) the portfolio choice is independent of the utility function and related to the risk attitude τ; (ii) the ability to capture heterogeneity by varying the quantiles; (iii) robustness; and (iv) it has a solid axiomatic foundation.

We divide the portfolio allocation problem in two scenarios: with and without a risk-free asset. In the former case, we find that an investor with quantiles preferences with risk aversion characterized by a small τ reacts to the presence of a risk-free asset by fully investing on it. Otherwise, for higher values of τ the optimal strategy is to fully invest on the risky portfolio. This result is in stark contrast with the standard mutual fund separation theorem that shows that the optimal combination between the risk-free asset and a risky portfolio is convex and determined by the investor's risk aversion profile. Under quantile preferences, we observe an all-or-nothing behavior, instead. For the case of two risky assets, we derive theoretically conditions on the support of the random variables under which the optimal portfolio decision has an interior solution. This result provides the setup under which diversification strategies are optimal for investors endowed with quantile preferences. These insights are in clear contrast to the EU paradigm that claims that diversification is always an optimal strategy under very general forms of risk aversion.

The paper has also explored the optimality of diversification strategies in the tails of the distribution of portfolio returns. In particular, we have derived conditions under which diversification in the tails is outperformed by fully investing in one risky asset. Additionally, we have characterized the optimal portfolio allocation under quantile preferences when an interior solution exists. Under unimodality and a symmetry condition on the bivariate distribution of the random lotteries, the individual's optimal portfolio decision under quantile preferences is characterized by two regions: for quantiles below the median full diversification is optimal, for quantiles above the median diversification is dominated by fully investing on the asset with highest upside potential. This strategy characterizes the optimal portfolio decision of risk-loving individuals.

Appendix

A Proofs

This appendix collects the proofs for the results in the main text. We will use two properties of quantiles that are easy to verify. First, quantiles are non-decreasing, that is, $\tau \leqslant \bar{\tau}$ implies $Q_\tau[X] \leqslant Q_{\bar{\tau}}[X]$. Second, for any random variable X with CDF F_X,

$$F_X(t) \geqslant \tau \iff Q_\tau[X] \leqslant t. \tag{14}$$

For reader's convenience, we provide below a detailed proof of Proposition 2.2, but the result can also be found in Manski (1988, Proposition 3, p. 95).

Proof of Proposition 2.2:

Proof. Let Y be a $\bar{\tau}$-quantile-preserving spread of X and let $q = Q_{\bar{\tau}}[Y] = Q_{\bar{\tau}}[X]$, so that

$$t < q \implies F_Y(t) \geqslant F_X(t); \text{ and} \tag{15}$$

$$t > q \implies F_Y(t) \leqslant F_X(t). \tag{16}$$

To show (i), assume for a contradiction that $\tau \leqslant \bar{\tau}$ and $t \equiv Q_\tau[X] < Q_\tau[Y]$. Since $Q_\tau[Y] > t$, by (14), $F_Y(t) < \tau$. Since $Q_\tau[X] = t$ implies $F_X(t) \geqslant \tau$ again by (14), we conclude that

$$F_Y(t) < \tau \leqslant F_X(t). \tag{17}$$

Since quantiles are non-decreasing, we have $t < Q_\tau[Y] \leqslant Q_{\bar{\tau}}[Y] = q$. But then (17) contradicts (15). The contradiction establishes (i).

Similarly to show (ii) with a contradiction, assume that $\tau \geqslant \bar{\tau}$ and $t \equiv Q_\tau[Y] < Q_\tau[X]$. Since $Q_\tau[X] > t$, by (14), $F_X(t) < \tau$. Since $Q_\tau[Y] = t$ implies $F_Y(t) \geqslant \tau$ again by (14), we conclude that

$$F_X(t) < \tau \leqslant F_Y(t). \tag{18}$$

Since quantiles are non-decreasing, we have $t > Q_\tau[Y] \geqslant Q_{\bar{\tau}}[Y] = q$. But then (18) contradicts (16). The contradiction establishes (ii). □

Proof of Proposition 2.4:

Proof. (1) $\Rightarrow$ (4) : Let $\tau \geqslant \tau'$ and Y be a $\bar{\tau}$-quantile-preserving spread of X and $Y \succ_{\tau'} X \Leftrightarrow Q_{\tau'}[Y] > Q_{\tau'}[X]$. By Proposition 2.2(i), $\tau' > \bar{\tau}$, which implies $\tau > \bar{\tau}$. By Proposition 2.2(ii), $Q_\tau[X] \leqslant Q_\tau[Y] \Rightarrow Y \succeq_\tau X$.

$(4) \Rightarrow (3)$: Assume that Y be a $\bar{\tau}$-quantile-preserving spread of X and $X \succ_\tau Y$. For a contradiction, assume that $\neg (X \succeq_{\tau'} Y) \Leftrightarrow Y \succ_{\tau'} X$. By (4), this implies that $Y \succeq_\tau X$, which contradicts $X \succ_\tau Y$.

$(3) \Rightarrow (1)$: For a contradiction, assume that $\tau < \tau'$ and let Y be a $\bar{\tau}$-quantile-preserving spread of X satisfying

$$Q_{\hat{\tau}}[X] = Q_{\hat{\tau}}[Y] \Rightarrow \hat{\tau} = \bar{\tau}, \tag{19}$$

for some fixed $\bar{\tau} \in (\tau, \tau')$, that is, the quantile functions of Y and X only coincide at $\bar{\tau}$. Since $\tau < \bar{\tau}$, Proposition 2.2(i) implies that $Q_\tau[X] \geqslant Q_\tau[Y]$, which must be $Q_\tau[X] > Q_\tau[Y]$ because of (19). By (3), we must have $Q_{\tau'}[X] \geqslant Q_{\tau'}[Y]$. Since $\tau' > \bar{\tau}$, by Proposition 2.2(ii), $Q_{\tau'}[X] \leqslant Q_{\tau'}[Y]$. Therefore, $Q_{\tau'}[X] = Q_{\tau'}[Y]$, which contradicts (19) since $\tau' > \bar{\tau}$.

$(1) \Rightarrow (2)$: Let $\tau \geqslant \tau'$. Since quantiles are monotonic, $Q_\tau[X] \geqslant Q_{\tau'}[X]$. Therefore,

$$\begin{aligned} q \succeq_\tau X \iff q \geqslant Q_\tau[X] \implies q \geqslant Q_{\tau'}[X] \iff q \succeq_{\tau'} X; \text{ and} \\ q \succ_\tau X \iff q > Q_\tau[X] \implies q > Q_{\tau'}[X] \iff q \succ_{\tau'} X. \end{aligned}$$

$(2) \Rightarrow (1)$: Assume that $\succeq_{\tau'}$ is more uncertainty averse than $\succeq_\tau$ and, for a contradiction, that $\tau < \tau'$. By monotonicity, $Q_\tau[X] \leqslant Q_{\tau'}[X]$ for any X. Let X be such that $Q_\tau[X] < Q_{\tau'}[X]$ and $q \equiv Q_\tau[X] \Rightarrow q \succeq_\tau X$. Since $\succeq_{\tau'}$ is more uncertainty averse than $\succeq_\tau$, this implies $q \succeq_{\tau'} X \Leftrightarrow q = Q_\tau[X] \geqslant Q_{\tau'}[X]$, which contradicts $Q_\tau[X] < Q_{\tau'}[X]$. □

Proof of Lemma 3.1:

Proof. The objective function of the primal problem in (4) is

$$\max_{w \in [0,1]^n} Q_\tau \left[u(S_w)\right].$$

Noticing that $u(\cdot)$ is continuous and increasing, and that the quantile is invariant with respect to monotone transformations, then the above maximization argument is given by

$$\begin{aligned} \arg\max_w Q_\tau \left[u(S_w)\right] &= \arg\max_w u\left(Q_\tau \left[S_w\right]\right) \\ &= \arg\max_w Q_\tau \left[S_w\right]. \end{aligned}$$

□

Proof of Lemma 3.2:

Proof. It is sufficient to show that $w \mapsto Q_\tau\left[wX + (1-w)Y\right] = Q_\tau[S_w]$ is continuous. But this follows from Assumption 1, which implies that the CDF of $S_w = wX + (1-w)Y$ is strictly

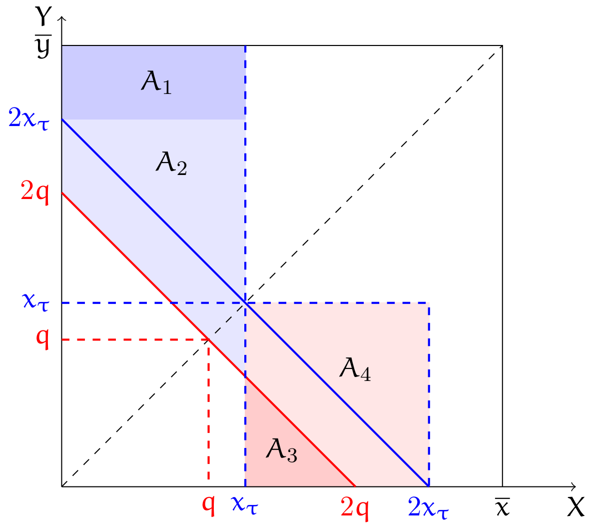


Figure 12: Illustration of the proof.

increasing, thus making its quantile continuous. □

Proof of Theorem 1:

Proof. (1) By appealing to translations if necessary, we may assume without loss of generality that $\underline{x} = \underline{y} = 0$, that is, if this is not the case, we may consider the random variables $X' = X - \underline{x}$ and $Y' = Y - \underline{y}$. We may also assume without loss of generality that $\overline{x}, \overline{y} < \infty$, for if $\overline{x} = \infty$ or $\overline{y} = \infty$, we may truncate the distribution at large $\hat{x}, \hat{y}$ so that $\Pr(\{(x, y) : x > \hat{x} \text{ or } y > \hat{y}\}) < \epsilon$ for some small $\epsilon > 0$. Thus, without loss of generality, we may assume that there exists $m, M \in \mathbb{R}_+$ such that $0 < m < f(x, y) < M < \infty$ for all $(x, y) \in [0, \overline{x}] \times [0, \overline{y}]$.

We will prove that we can choose τ small enough so that $w = 1$ is dominated by $w = \frac{1}{2}$. The argument that $w = 0$ is also dominated by $w = \frac{1}{2}$ for sufficiently small τ is analogous. For this, we will assume that $w = 1$ is better than $w = \frac{1}{2}$ and find a contradiction for τ small enough. Let x_τ denote the τ-quantile of X and assume that $q = q\left(\frac{1}{2}, \tau\right) \leqslant x_\tau$. See Figure 12 where $q < x_\tau$. From the definition, the probability under the red line $\frac{X+Y}{2} \leqslant 2$ is τ, which is the same of the probability of the vertical band from 0 to x_τ. Therefore, the blue area in the graph (both light and dark blue together) have to be the same as the dark red that corresponds to the small triangle from x_τ to $2q$ in the horizontal axis and from 0 to $2q - x_\tau$. To better follow the argument, let A_1 correspond to the probability in the dark blue area and A_2, the probability of the light blue area. Let A_3 correspond to the probability of the dark red area and A_4, the probability of the light red area. Therefore, as observed above, by definition,

$$A_1 + A_2 = A_3. \tag{20}$$

We will show, however, that we can choose τ sufficiently small so that

$$A_1 > A_3 + A_4. \tag{21}$$

Observe that (21) implies

$$A_1 + A_2 > A_1 > A_3 + A_4 > A_3,$$

which contradicts (20). In order to obtain (21), observe that

$$\begin{aligned} A_1 &= \int_0^{x_\tau} \left[\int_{2x_\tau}^{\overline{y}} f(x,y) dy \right] dx \\ A_3 + A_4 &= \int_{x_\tau}^{2x_\tau} \left[\int_0^{x_\tau} f(x,y) dy \right] dx. \end{aligned}$$

Since $m < f(x,y) < M$ for $(x,y) \in [0,\overline{x}]^2$, we have

$$\begin{aligned} A_1 &> \int_0^{x_\tau} \left[\int_{2x_\tau}^{\overline{y}} m dy \right] dx = m x_\tau (\overline{y} - 2x_\tau) \\ A_3 + A_4 &< \int_{x_\tau}^{2x_\tau} \left[\int_0^{x_\tau} M dy \right] dx = M x_\tau^2. \end{aligned}$$

Therefore, we obtain (21) if

$$\begin{aligned} & m x_\tau (\overline{y} - 2x_\tau) > M x_\tau^2 \\ \iff & x_\tau [m\overline{y} - (2m + M)x_\tau] > 0 \\ \iff & x_\tau < \frac{\overline{y}}{\left(\frac{M}{m} + 2\right)}. \end{aligned}$$

Since $\overline{y}$, M and m are given constants, the above inequality is satisfied if x_τ is small enough, that is, if τ is sufficiently close to 0.

$\square$

Proof of Theorem 2:

Proof. Note that by translating the distribution (along the 45° line) if necessary, we may assume without loss of generality that $\underline{y} = 0$.

Let us fix any $\tau \in (0, \overline{\tau}]$ and let x_τ, y_τ denote the τ-quantiles of X and Y, respectively. Of course, $x_\tau > \underline{x}$. Figure 13 below will be useful to illustrate the reasoning in this proof.

Let $\bar{M}$ be the average $f_X(x)$ density on the region $x \in [\underline{x}, x_\tau]$, that is, $\bar{M} \equiv \frac{\tau}{x_\tau - \underline{x}}$. Therefore, $m \leqslant \bar{M} \leqslant M$ and $m \leqslant f(x,y) \leqslant M$ for all $(x,y) \in [\underline{x}, x_{\overline{\tau}}] \times [\underline{y}, \overline{y}]$. We want to show that $q(w,\tau) < x_\tau$ for all $w \in [0,1)$. To show this, it is enough to show that for any $q \geqslant x_\tau$ and

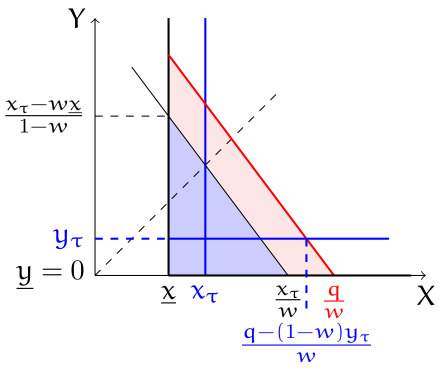


Figure 13: Illustration of the proof.

$w \in [0, 1)$, the probability under the line $wx + (1-w)y \leqslant q$ is larger than τ, that is,

$$h(w, q) = \int_{\underline{x}}^{\frac{x_\tau}{w}} \left[\int_0^{\frac{x_\tau - wx}{1-w}} f(x, y) dy \right] dx > \tau.$$

For this, it is sufficient to show that the area in blue in Figure 13 is larger than τ already for $q = x_\tau$, that is,

$$q \geqslant x_\tau \implies h(w, q) \geqslant h(w, x_\tau) = \int_{\underline{x}}^{\frac{x_\tau}{w}} \left[\int_0^{\frac{x_\tau - wx}{1-w}} f(x, y) dy \right] dx > \tau. \tag{22}$$

Since $f(x, y) > m$ for all $(x, y) \in [\underline{x}, x_{\overline{\tau}}] \times [\underline{y}, \overline{y}] \cup [\underline{x}, \overline{x}] \times [\underline{y}, x_{\overline{\tau}}]$, we have

$$h(w, x_\tau) > \frac{m}{2} \left(\frac{x_\tau}{w} - \underline{x} \right) \left(\frac{x_\tau - w\underline{x}}{1-w} \right) = \frac{m(x_\tau - w\underline{x})^2}{2w(1-w)}.$$

Thus, to establish (22), it is sufficient to show that

$$\begin{aligned} \frac{m(x_\tau - w\underline{x})^2}{2w(1-w)} > \tau &\iff m(x_\tau - w\underline{x})^2 > 2\tau w(1-w) \\ &\iff m\left[(x_\tau)^2 - 2w\underline{x}x_\tau + w^2\underline{x}^2\right] - 2\tau w + 2w^2\tau > 0 \\ &\iff w^2\left(2\tau + m\underline{x}^2\right) - 2w\left(m\underline{x}x_\tau + \tau\right) + mx_\tau^2 > 0. \end{aligned}$$

Let us define the quadratic polynomial:

$$p(w) \equiv w^2\left(2\tau + m\underline{x}^2\right) - 2w\left(m\underline{x}x_\tau + \tau\right) + mx_\tau^2. \tag{23}$$

Notice that $p(0) = mx_\tau^2 > 0$ and for $w = 1$:

$$p(1) = \left(2\tau + m\underline{x}^2\right) - 2\left(m\underline{x}x_\tau + \tau\right) + mx_\tau^2 = m\left(\underline{x}^2 - 2\underline{x}x_\tau + x_\tau^2\right) = m(x_\tau - \underline{x})^2 > 0.$$

Let w_V denote the vertex of the quadratic $p(w)$, which is given by

$$w_V = \frac{mx_\tau\underline{x} + \tau}{2\tau + m\underline{x}^2}.$$

Note that w_V is always positive.

We can conclude that the quadratic form $p(w)$ in (23) is positive if $p(w_V) > 0$. To verify this, we can substitute w_V in the quadratic form to obtain:

$$\begin{aligned} p(w_V) &= \left[\frac{mx_\tau\underline{x} + \tau}{2\tau + m\underline{x}^2}\right]^2 \left(2\tau + m\underline{x}^2\right) - 2\left[\frac{mx_\tau\underline{x} + \tau}{2\tau + m\underline{x}^2}\right]\left(m\underline{x}x_\tau + \tau\right) + mx_\tau^2 \\ &= \frac{-\left(mx_\tau\underline{x} + \tau\right)^2}{2\tau + m\underline{x}^2} + mx_\tau^2 = \frac{mx_\tau^2(2\tau + m\underline{x}^2) - \left(mx_\tau\underline{x} + \tau\right)^2}{2\tau + m\underline{x}^2}, \end{aligned}$$

which is positive as long as

$$\begin{aligned} &2\tau mx_\tau^2 + m^2x_\tau^2\underline{x}^2 > \left(mx_\tau\underline{x} + \tau\right)^2 = m^2x_\tau^2\underline{x}^2 + 2\tau mx_\tau\underline{x} + \tau^2 \\ &\iff 2\tau mx_\tau\left(x_\tau - \underline{x}\right) > \tau^2 \\ &\iff 2mx_\tau\left(x_\tau - \underline{x}\right) > \tau. \end{aligned}$$

Since $\tau = \bar{M}(x_\tau - \underline{x})$, the above condition is equivalent to

$$2mx_\tau > \bar{M} \iff x_\tau > \frac{\bar{M}}{2m}.$$

Since $\bar{M} \leqslant M$ and $x_\tau > \underline{x} \geqslant \frac{M}{2m}$, we have $x_\tau > \frac{\bar{M}}{2m}$, thus proving the result. □

Remark A.1. *One can also conclude that* $p(w) > 0$ *for all* $w \in [0, 1]$ *if* $w_V > 1$. *Unfortunately, the condition for this is stronger than the one given in Theorem 2. Indeed,* $w_V > 1$ *if*

$$w_v = \frac{mx_\tau\underline{x} + \tau}{2\tau + m\underline{x}^2} \geqslant 1,$$

which is equivalent to

$$\begin{aligned} &mx_\tau\underline{x} + \tau \geqslant 2\tau + m\underline{x}^2 \\ &\iff m\underline{x}(x_\tau - \underline{x}) \geqslant \tau. \end{aligned}$$

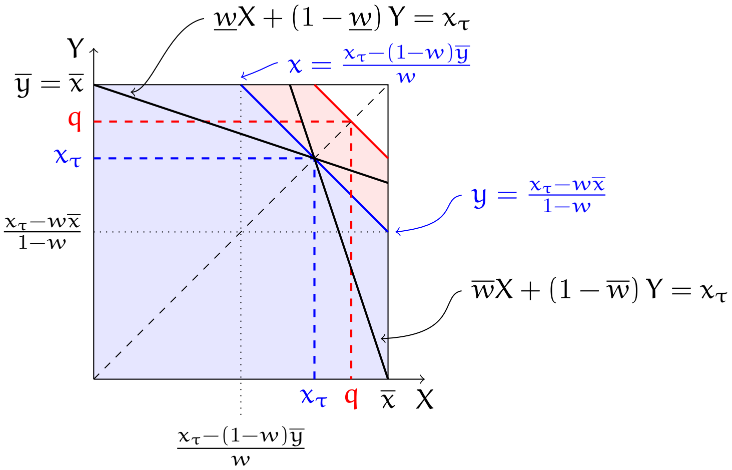


Figure 14: Illustration for the proof of Theorem 3: the case of τ close to 1.

Since $\tau = \bar{M}(x_\tau - \underline{x})$*, the above condition simplifies to*

$$m\underline{x} \geqslant \bar{M} \iff \underline{x} \geqslant \frac{\bar{M}}{m}.$$

Since $\bar{M} \leqslant M$*, if we have* $\underline{x} \geqslant \frac{M}{m}$ *then* $\underline{x} \geqslant \frac{\bar{M}}{m}$*. However, this assumption* $\underline{x} \geqslant \frac{M}{m}$ *is stronger than the condition in Theorem 2.*

Proof of Theorem 3:

Proof. Fix $\tau \geqslant \overline{\tau}$ and assume that $\underline{x} = \underline{y}$ and $\overline{x} = \overline{y}$. We want to show that $w^*(\tau) = 1$ for sufficiently high τ, which is equivalent to show that $q(w, \tau) < x_\tau$ for all $w \in (0, 1)$ and $\tau \geqslant \overline{\tau}$. To show this, it is enough to show that for any $q > x_\tau$ the probability under the line $wx + (1 - w)y \leqslant q$ is larger than τ, that is,

$$q > x_\tau \implies h(w, q) > h(w, x_\tau) = \int_{A(w,x_\tau)} f(x, y)dydx \geqslant \tau,$$

where $A(w, q) \equiv \{(x, y) \in [\underline{x}, \overline{x}] \times [\underline{x}, \overline{x}] : wx + (1 - w)y \leqslant q\}$. This is equivalent to show that

$$1 - h(w, x_\tau) = \int_{A^c(w,x_\tau)} f(x, y)dydx \leqslant 1 - \tau.$$

For estimating this integral, we have to consider three intervals for w: $(0, \underline{w})$, $[\underline{w}, \overline{w}]$ and $(\overline{w}, 1)$, where $\underline{w}$ and $\overline{w}$ correspond to the lines that pass by $(\underline{x}, \overline{x})$ and $(\overline{x}, \underline{x})$ respectively, that is,

$$\underline{w}\,\underline{x} + (1-\underline{w})\,\overline{x} = x_\tau \iff \underline{w} = \frac{\overline{x} - x_\tau}{\overline{x} - \underline{x}};$$
$$\overline{w}\,\overline{x} + (1-\overline{w})\,\underline{x} = x_\tau \iff \overline{w} = \frac{x_\tau - \underline{x}}{\overline{x} - \underline{x}}.$$

Note that $\underline{w} = 1 - \overline{w} < \overline{w}$ for large τ.

Figure 14 shows the set $A(w, x_\tau)$ in blue, for $w \in [\underline{w}, \overline{w}]$. Thus, if $w \in [\underline{w}, \overline{w}]$,

$$1 - h(w, x_\tau) = \int_{\frac{x_\tau - (1-w)\underline{x}}{w}}^{\overline{x}} \left[\int_{\frac{x_\tau - wx}{1-w}}^{\overline{x}} f(x, y)dy \right] dx.$$

For $w \in (0, \underline{w})$,

$$1 - h(w, x_\tau) = \int_{A^c(w, x_\tau)} f(x, y)dydx = \int_{\underline{x}}^{\overline{x}} \left[\int_{\frac{x_\tau - wx}{1-w}}^{\overline{x}} f(x, y)dy \right] dx,$$

and, for $w \in (\overline{w}, 1)$,

$$1 - h(w, x_\tau) = \int_{A^c(w, x_\tau)} f(x, y)dydx = \int_{\underline{x}}^{\overline{x}} \left[\int_{\frac{x_\tau - (1-w)y}{w}}^{\overline{x}} f(x, y)dx \right] dy.$$

We will consider each of the above cases.

Case 1: $w \in [\underline{w}, \overline{w}]$**.**

Since $m \leqslant f(x, y) \leqslant M$ for all $(x, y) \in [x_{\overline{\tau}}, \overline{x}] \times [\underline{x}, \overline{x}]$, we have

$$1 - h(w, x_\tau) \leqslant \frac{M}{2} \left[\overline{x} - \frac{x_\tau - (1-w)\overline{x}}{w} \right] \left(\overline{x} - \frac{x_\tau - w\overline{x}}{1-w} \right).$$

Using $\overline{x} = \overline{x}$, the left hand side becomes $\frac{M(\overline{x} - x_\tau)^2}{2w(1-w)}$. Thus, it is enough to prove that

$$\frac{M\,(\overline{x} - x_\tau)^2}{2w(1-w)} \leqslant 1 - \tau \iff w(1-w) \geqslant \frac{M\,(\overline{x} - x_\tau)^2}{2(1-\tau)}.$$

Since $w \in [\underline{w}, \overline{w}]$, we have

$$w(1-w) \geqslant \min\{\underline{w}(1-\underline{w}), \overline{w}(1-\overline{w})\} = \underline{w}\overline{w} = \frac{x_\tau - \underline{x}}{\overline{x} - \underline{x}} \cdot \frac{\overline{x} - x_\tau}{\overline{x} - \underline{x}}.$$

Thus, it is sufficient to show that:

$$\frac{(x_\tau - \underline{x})(\overline{x} - x_\tau)}{(\overline{x} - \underline{x})^2} \geqslant \frac{M\,(\overline{x} - x_\tau)^2}{2(1-\tau)}$$
$$\iff 2(1-\tau)(x_\tau - \underline{x}) \geqslant M(\overline{x} - \underline{x})^2\,(\overline{x} - x_\tau)\,. \tag{24}$$

We now define $\bar{m} = \frac{(\overline{x} - x_\tau)(\overline{x} - \underline{x})}{1-\tau}$ so that $1 - \tau = \bar{m}(\overline{x} - x_\tau)(\overline{x} - \underline{x}) = \bar{m}(\overline{x} - x_\tau)(\overline{x} - \underline{x})$. Since

$$1 - \tau = \int_{x_\tau}^{\overline{x}} \left[\int_{\underline{x}}^{\overline{x}} f(x, y)\,dy\right] dx \geqslant m(\overline{x} - x_\tau)(\overline{x} - \underline{x}),$$

we have $\bar{m} \geqslant m \implies \frac{M}{\bar{m}} \leqslant \frac{M}{m}$. Therefore, the assumption gives

$$x_\tau - \underline{x} \geqslant x_{\overline{\tau}} - \underline{x} \geqslant \frac{M(\overline{x} - \underline{x})}{2m} \geqslant \frac{M(\overline{x} - \underline{x})}{2\bar{m}},$$

as we wanted to show.

Case 2: $w \in (0, \underline{w})$**.**

Since $m \leqslant f(x, y) \leqslant M$ for all $(x, y) \in [x_{\overline{\tau}}, \overline{x}] \times [\underline{x}, \overline{x}]$, we have

$$1 - h(w, x_\tau) \leqslant \frac{M}{2}\,(\overline{x} - \underline{x})\left(\overline{x} - \frac{x_\tau - w\overline{x}}{1 - w}\right) = \frac{M\,(\overline{x} - \underline{x})\,(\overline{x} - x_\tau)}{2(1 - w)}.$$

Thus, it is enough to prove

$$\frac{M\,(\overline{x} - \underline{x})\,(\overline{x} - x_\tau)}{2(1 - w)} \leqslant 1 - \tau \iff 1 - w \geqslant \frac{M\,(\overline{x} - \underline{x})\,(\overline{x} - x_\tau)}{2(1 - \tau)}.$$

Since $w \in (0, \underline{w})$, we have $1 - w > 1 - \underline{w} = \frac{x_\tau - \underline{x}}{\overline{x} - \underline{x}}$. Thus, it is sufficient to show that:

$$2(1-\tau)(x_\tau - \underline{x}) \geqslant M(\overline{x} - \underline{x})^2\,(\overline{x} - x_\tau)\,.$$

This is exactly condition (24) above, which we have shown to be implied by the assumption.

Case 3: $w \in (\overline{w}, 1)$**.**

Since $m \leqslant f(x, y) \leqslant M$ for all $(x, y) \in [x_{\overline{\tau}}, \overline{x}] \times [\underline{x}, \overline{x}]$, we have

$$1 - h(w, x_\tau) \leqslant \frac{M}{2}\,(\overline{x} - \underline{x})\left[\overline{x} - \frac{x_\tau - (1 - w)\overline{x}}{w}\right] = \frac{M\,(\overline{x} - \underline{x})\,(\overline{x} - x_\tau)}{2w}.$$

Thus, it is enough to prove

$$\frac{M\,(\overline{x} - \underline{x})\,(\overline{x} - x_\tau)}{2w} \leqslant 1 - \tau \iff 2(1-\tau)w \geqslant M\,(\overline{x} - \underline{x})\,(\overline{x} - x_\tau)\,.$$

Since $w \in (\overline{w}, 1)$, $w > \overline{w} = \frac{x_\tau - \underline{x}}{\overline{x} - \underline{x}}$, it is sufficient to show again condition (24) above, which we have shown to be implied by the assumption of Theorem 3. □

Proof of Proposition 3.10:

Proof. Assume first that $q(w)$ is differentiable at $w \in (0,1)$. Taking the total derivative with respect to w on the equation $h(w,q) = \tau$, we obtain

$$\partial_w h(w,q) + \partial_q h(w,q) \cdot q'(w) = 0 \implies q'(w) = -\frac{\partial_w h(w,q)}{\partial_q h(w,q)}. \tag{25}$$

From (8), it is clear that h is differentiable and that $\partial_q h(w,q) > 0$. Therefore, by the Implicit Function Theorem, q is differentiable and given by (25). Now, we will calculate $\partial_w h(w,q)$ and $\partial_q h(w,q)$. Define the function $y(w,q) \equiv \frac{q - wx}{1-w}$. Then,

$$\begin{aligned}
\partial_q y(w,q) &= \frac{1}{1-w}, \text{ and} \\
\partial_w y(w,q) &= \frac{(-x)(1-w) - (q - wx)(-1)}{(1-w)^2} = \frac{q - x}{(1-w)^2}.
\end{aligned}$$

Support on $\mathbb{R}$

Here we consider first the case in which $\mathcal{I}_X = \mathcal{I}_Y = \mathbb{R}$. The other two cases are similar and considered below.[16] It is clear from (8) and Assumption 1 that h is C^1 and:

$$\partial_w h(w,q) = \int_{-\infty}^{\infty} f\left(x, \frac{q - wx}{1-w}\right) \left[\frac{q - x}{(1-w)^2}\right] dx \tag{26}$$

and

$$\partial_q h(w,q) = \int_{-\infty}^{\infty} f\left(x, \frac{q - wx}{1-w}\right) \frac{1}{1-w} dx. \tag{27}$$

From this,

$$\begin{aligned}
\partial_w h(w,q) &= \frac{q}{(1-w)^2} \int_{-\infty}^{\infty} f\left(x, \frac{q - wx}{1-w}\right) dx - \frac{1}{(1-w)^2} \int_{-\infty}^{\infty} x f\left(x, \frac{q - wx}{1-w}\right) dx \\
&= \frac{q}{1-w} \partial_q h(w,q) - \frac{1}{(1-w)} \mathrm{E}[Z] \cdot \partial_q h(w,q) \\
&= \frac{1}{1-w} \cdot (q - \mathrm{E}[Z]) \cdot \partial_q h(w,q).
\end{aligned}$$

Therefore,

$$q'(w) = -\frac{\partial_w h(w,q)}{\partial_q h(w,q)} = \frac{1}{1-w} (\mathrm{E}[Z] - q),$$

as we wanted to show. □

[16] When $\mathcal{I}_X$ or $\mathcal{I}_Y$ are not $\mathbb{R}$, then we have to consider limits that make the derivatives more complex.

Support on $[0, \infty)$

Now, we consider the case $\mathcal{I}_X = \mathcal{I}_Y = \mathbb{R}_+ = [0, \infty)$. In this case, we have

$$h(w, q) = \int_0^{\frac{q}{w}} \int_0^{\frac{q-wx}{1-w}} f(x, y) dy dx. \tag{28}$$

Let us define:

$$g(w, q, x) \equiv \int_0^{\frac{q-wx}{1-w}} f(x, y) dy,$$

for $x < \frac{q}{w}$ and $g\left(w, q, \frac{q}{w}\right) = 0$ so that g is continuous and differentiable. Also,

$$h(w, q) = \int_0^{\frac{q}{w}} g(w, q, x) dx.$$

Then,

$$\begin{aligned}
\partial_w h(w, q) &= g\left(w, q, \frac{q}{w}\right)\left(\frac{-q}{w^2}\right) + \int_0^{\frac{q}{w}} \partial_w g(w, q, x) dx \\
&= \frac{1}{(1-w)^2} \int_0^{\frac{q}{w}} (q - x) f\left(x, \frac{q - wx}{1-w}\right) dx,
\end{aligned}$$

and

$$\begin{aligned}
\partial_q h(w, q) &= g\left(w, q, \frac{q}{w}\right)\left(\frac{1}{w}\right) + \int_0^{\frac{q}{w}} \partial_q g(w, q, x) dx \\
&= \frac{1}{1-w} \int_0^{\frac{q}{x}} f\left(x, \frac{q - wx}{1-w}\right) dx.
\end{aligned}$$

From this, we observe that the same expressions remain valid:

$$\begin{aligned}
\partial_w h(w, q) &= \frac{q}{(1-w)^2} \int_0^{\frac{q}{w}} f\left(x, \frac{q - wx}{1-w}\right) dx - \frac{1}{(1-w)^2} \int_0^{\frac{q}{w}} x f\left(x, \frac{q - wx}{1-w}\right) dx \\
&= \frac{1}{1-w} \cdot (q - \mathrm{E}[Z]) \cdot \partial_q h(w, q)
\end{aligned}$$

and

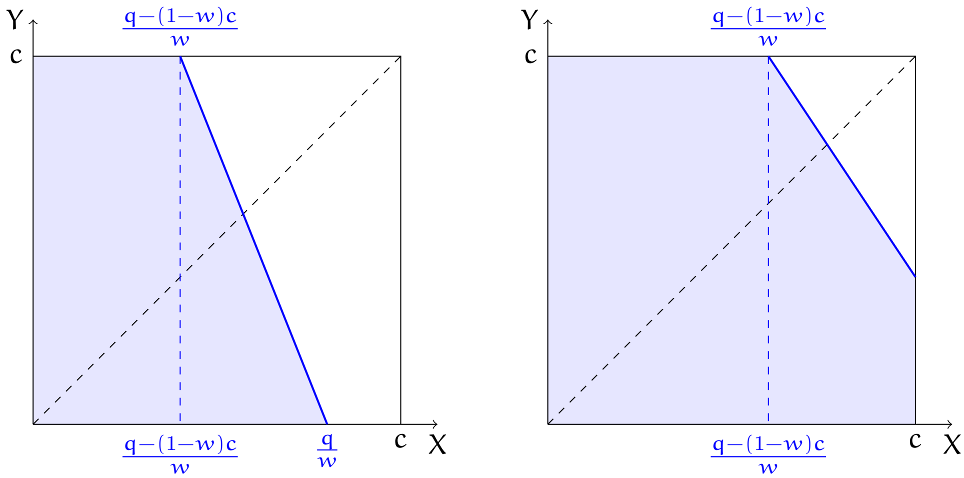
$$q'(w) = -\frac{\partial_w h(w, q)}{\partial_q h(w, q)} = \frac{1}{1-w} (\mathrm{E}[Z] - q), \tag{29}$$

as before.

Support on $[0, c]$

We now consider the case $\mathcal{I}_X = \mathcal{I}_Y = [0, c]$. There are two subcases to consider, as show in Figure 15.

Case (a)



(a) $\tau \leqslant \Pr\left(\left[\frac{X+Y}{2} \leqslant q\right]\right)$ (b) $\tau > \Pr\left(\left[\frac{X+Y}{2} \leqslant q\right]\right)$

Figure 15: An illustration of Z for the case $\mathcal{I}_X = \mathcal{I}_Y = [0, c]$. The blue area has probability τ.

In this case,

$$h(w, q) = \int_0^{\frac{q-(1-w)c}{w}} \int_0^c f(x, y) dy dx + \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} \int_0^{\frac{q-wx}{1-w}} f(x, y) dy dx.$$

Let

$$g(\alpha, \beta, x) \equiv \int_\alpha^\beta f(x, y) dy.$$

Therefore,

$$h(w, q) = \int_0^{\frac{q-(1-w)c}{w}} g(0, c, x) dx + \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} g(0, \frac{q-wx}{1-w}, x) dx.$$

Thus,

$$\begin{aligned} \partial_w h(w, q) = {} & g\left(0, c, \frac{q-(1-w)c}{w}\right) \partial_w \left[\frac{q-(1-w)c}{1-w}\right] \\ & + \int_0^{\frac{q-(1-w)c}{w}} \partial_w \left[g(0, c, x)\right] dx \\ & + g\left(0, 0, \frac{q}{w}\right)\left(\frac{-q}{w^2}\right) - g\left(0, c, \frac{q-(1-w)c}{w}\right) \partial_w \left[\frac{q-(1-w)c}{1-w}\right] \\ & + \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} \partial_w \left[g(0, \frac{q-wx}{1-w}, x)\right] dx. \end{aligned}$$

Since $g(0,0,\cdot)=0$ and $\partial_w g(0,c,x)=0$, and $g\left(0,c,\frac{q-(1-w)c}{w}\right)\partial_w\left[\frac{q-(1-w)c}{1-w}\right]$ appear with + and - signs, we have

$$\begin{aligned}
\partial_w h(w,q) &= \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} \partial_w\left[g(0,\frac{q-wx}{1-w},x)\right]dx \\
&= \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} f\left(x,\frac{q-wx}{1-w}\right)\partial_w\left[\frac{q-wx}{1-w}\right]dx \\
&= \frac{q-x}{(1-w)^2}\int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} f\left(x,\frac{q-wx}{1-w}\right)dx.
\end{aligned}$$

We have:

$$\begin{aligned}
\partial_q h(w,q) &= g\left(0,c,\frac{q-(1-w)c}{w}\right)\partial_q\left[\frac{q-(1-w)c}{1-w}\right] \\
&+ \int_0^{\frac{q-(1-w)c}{w}} \partial_q\left[g(0,c,x)\right]dx \\
&+ g\left(0,0,\frac{q}{w}\right)\left(\frac{1}{w}\right) - g\left(0,c,\frac{q-(1-w)c}{w}\right)\partial_q\left[\frac{q-(1-w)c}{1-w}\right] \\
&+ \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} \partial_q\left[g(0,\frac{q-wx}{1-w},x)\right]dx.
\end{aligned}$$

Anagously to the previous case,

$$\begin{aligned}
\partial_q h(w,q) &= \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} \partial_q\left[g(0,\frac{q-wx}{1-w},x)\right]dx \\
&= \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} f\left(x,\frac{q-wx}{1-w}\right)\partial_q\left[\frac{q-wx}{1-w}\right]dx \\
&= \frac{1}{(1-w)^2}\int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} (q-x)f\left(x,\frac{q-wx}{1-w}\right)dx.
\end{aligned}$$

Note that in this case,

$$f_Z(z) = \frac{f\left(z,\frac{q-wz}{1-w}\right)}{\int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} f\left(x,\frac{q-wx}{1-w}\right)dx},$$

so that

$$\mathrm{E}[Z] = \frac{\int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} xf\left(x,\frac{q-wx}{1-w}\right)dx}{\int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} f\left(x,\frac{q-wx}{1-w}\right)dx}.$$

From this, we observe that the same expressions remain valid:

$$\begin{aligned}
\partial_w h(w,q) &= \frac{q}{(1-w)^2}\int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} f\left(x, \frac{q-wx}{1-w}\right) dx - \frac{1}{(1-w)^2}\int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} xf\left(x, \frac{q-wx}{1-w}\right) dx \\
&= \frac{q}{1-w}\partial_q h(w,q) - \frac{1}{(1-w)}\mathrm{E}[Z]\cdot\partial_q h(w,q) \\
&= \frac{1}{1-w}\cdot(q-\mathrm{E}[Z])\,\partial_q h(w,q),
\end{aligned}$$

and

$$q'(w) = -\frac{\partial_w h(w,q)}{\partial_q h(w,q)} = \frac{1}{1-w}\left(\mathrm{E}[Z]-q\right),$$

as before.

Case (b)

In this case,

$$h(w,q) = \int_0^{\frac{q-(1-w)c}{w}}\int_0^c f(x,y)\,dy\,dx + \int_{\frac{q-(1-w)c}{w}}^{c}\int_0^{\frac{q-wx}{1-w}} f(x,y)\,dy\,dx,$$

and

$$h(w,q) = \int_0^{\frac{q-(1-w)c}{w}} g(0,c,x)\,dx + \int_{\frac{q-(1-w)c}{w}}^{c} g(0, \frac{q-wx}{1-w}, x)\,dx.$$

Thus,

$$\begin{aligned}
\partial_w h(w,q) = {} & g\left(0, c, \frac{q-(1-w)c}{w}\right)\partial_w\left[\frac{q-(1-w)c}{1-w}\right] \\
&+ \int_0^{\frac{q-(1-w)c}{w}} \partial_w\left[g(0,c,x)\right] dx \\
&- g\left(0, c, \frac{q-(1-w)c}{w}\right)\partial_w\left[\frac{q-(1-w)c}{1-w}\right] \\
&+ \int_{\frac{q-(1-w)c}{w}}^{c} \partial_w\left[g(0, \frac{q-wx}{1-w}, x)\right] dx.
\end{aligned}$$

Since $\partial_w g(0,c,x) = 0$, and $g\left(0, c, \frac{q-(1-w)c}{w}\right) \partial_w \left[\frac{q-(1-w)c}{1-w}\right]$ appear with + and - signs,

$$\begin{aligned}
\partial_w h(w,q) &= \int_{\frac{q-(1-w)c}{w}}^{c} \partial_w \left[g(0, \frac{q-wx}{1-w}, x)\right] dx \\
&= \int_{\frac{q-(1-w)c}{w}}^{c} f\left(x, \frac{q-wx}{1-w}\right) \partial_w \left[\frac{q-wx}{1-w}\right] dx \\
&= \frac{q-x}{(1-w)^2} \int_{\frac{q-(1-w)c}{w}}^{c} f\left(x, \frac{q-wx}{1-w}\right) dx.
\end{aligned}$$

Also,

$$\begin{aligned}
\partial_q h(w,q) = g&\left(0, c, \frac{q-(1-w)c}{w}\right) \partial_q \left[\frac{q-(1-w)c}{1-w}\right] \\
&+ \int_0^{\frac{q-(1-w)c}{w}} \partial_q \left[g(0,c,x)\right] dx \\
&- g\left(0, c, \frac{q-(1-w)c}{w}\right) \partial_q \left[\frac{q-(1-w)c}{1-w}\right] \\
&+ \int_{\frac{q-(1-w)c}{w}}^{c} \partial_q \left[g(0, \frac{q-wx}{1-w}, x)\right] dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\partial_q h(w,q) &= \int_{\frac{q-(1-w)c}{w}}^{c} \partial_q \left[g(0, \frac{q-wx}{1-w}, x)\right] dx \\
&= \int_{\frac{q-(1-w)c}{w}}^{c} f\left(x, \frac{q-wx}{1-w}\right) \partial_q \left[\frac{q-wx}{1-w}\right] dx \\
&= \frac{1}{(1-w)^2} \int_{\frac{q-(1-w)c}{w}}^{c} (q-x) f\left(x, \frac{q-wx}{1-w}\right) dx.
\end{aligned}$$

Note that in this case,

$$f_Z(z) = \frac{f\left(z, \frac{q-wz}{1-w}\right)}{\int_{\frac{q-(1-w)c}{w}}^{c} f\left(x, \frac{q-wx}{1-w}\right) dx},$$

so that

$$E[Z] = \frac{\int_{\frac{q-(1-w)c}{w}}^{c} x f\left(x, \frac{q-wx}{1-w}\right) dx}{\int_{\frac{q-(1-w)c}{w}}^{c} f\left(x, \frac{q-wx}{1-w}\right) dx}.$$

From this, we observe that the same expressions remain valid:

$$\begin{aligned}\partial_w h(w,q) &= \frac{q}{(1-w)^2}\int_{\frac{q-(1-w)c}{w}}^{c} f\left(x, \frac{q-wx}{1-w}\right) dx - \frac{1}{(1-w)^2}\int_{\frac{q-(1-w)c}{w}}^{c} xf\left(x, \frac{q-wx}{1-w}\right) dx \\ &= \frac{q}{1-w}\partial_q h(w,q) - \frac{1}{(1-w)}E[Z]\cdot\partial_q h(w,q) \\ &= \frac{1}{1-w}\cdot(q-E[Z])\,\partial_q h(w,q)\end{aligned}$$

and

$$q'(w) = -\frac{\partial_w h(w,q)}{\partial_q h(w,q)} = \frac{1}{1-w}\left(E[Z]-q\right),$$

as before.

□

Proof of Corollary 3.11:

Proof. The proof of this result is immediate from imposing $q'(w^*) = 0$ in Proposition 3.10, and noting that $q''(w) = \frac{\partial}{\partial w}E[Z_{w,q}]$, with $q''(\cdot)$ the second derivative with respect to w. To show this, we take the first derivative of $q'(w)$ in (10). Then,

$$q''(w) = \frac{\left(\frac{\partial}{\partial w}E[Z] - q'(w)\right)(1-w) + E[Z] - q(w)}{(1-w)^2}. \tag{30}$$

Then, noting that $E[Z] = q(w^*)$ and $q'(w^*) = 0$, we have

$$q''(w) = \frac{1}{(1-w)^2}\frac{\partial}{\partial w}E[Z]. \tag{31}$$

□

Proof of Proposition 3.13:

Proof. The proof of this result is shown in three different stages. First, we note from Corollary 3.11 that the first order condition characterizing an extremum of the optimization problem (7) is $E[Z] = q(w^*)$, with $w^* \in (0,1)$. Now, given that the density function $f_Z(z)$ is evaluated over the line $y = \frac{q(w^*)-w^*x}{1-w^*}$, it is not difficult to see that the projection of this line on the y-axis is $y^* = E[Z]$ for $x^* = E[Z]$. This result implies that for different quantile values $q(w^*) \equiv q(w^*,\tau)$ indexed by $\tau \in (0,1)$, the mean value of the random variable Z characterized by the density function $f_Z(z)$ in (9) is in the 45% degree line such that $(x^*, y^*) = (E[Z], E[Z])$.

Second, we prove that condition (11) evaluated at w^* guarantees the symmetry around $E[Z]$ of the density function f_Z for $x \in \mathcal{I}_Z = \mathcal{I}_Z^{w^*,q} = \{x \in \mathbb{R} : \left(x, \frac{q-w^*x}{1-w^*}\right) \in \mathcal{I}_X \times \mathcal{I}_Y\}$. More formally, condition (11) implies $f_Z(E[Z]+\varepsilon) = f_Z(E[Z]-\varepsilon)$ for all $\varepsilon > 0$. This is so by noting that $f_Z(E[Z]+\varepsilon) = \frac{f\left(E[Z]+\varepsilon, E[Z]-\frac{w^*}{1-w^*}\varepsilon\right)}{\int_{\mathcal{I}_Z} f\left(t, \frac{E[Z]-tw^*}{1-w^*}\right)dt}$ and $f_Z(E[Z]-\varepsilon) = \frac{f\left(E[Z]-\varepsilon, E[Z]+\frac{w^*}{1-w^*}\varepsilon\right)}{\int_{\mathcal{I}_Z} f\left(t, \frac{E[Z]-tw^*}{1-w^*}\right)dt}$.

Now, we show that under assumption 2, the density function f_Z is unimodal. To show this, we note that $f_Z(z) = \frac{f\left(x, \frac{E[Z]-w^*x}{1-w^*}\right)}{\int_{\mathcal{I}_Z} f\left(t, \frac{E[Z]-tw^*}{1-w^*}\right)dt}$, such that under the change of variable $x = E[Z]+\varepsilon$, we obtain $f_Z(z) = \frac{f\left(E[Z]+\varepsilon, E[Z]-\frac{w^*}{1-w^*}\varepsilon\right)}{\int_{\mathcal{I}_Z} f\left(t, \frac{E[Z]-tw^*}{1-w^*}\right)dt}$. Assumption 2 evaluated at $\mu = E[Z]$ implies that the numerator of this expression increases with ε up to $\varepsilon = 0$ and then decreases. This condition is sufficient to show that $f_Z(z)$ is unimodal with mode at $E[Z]$.

These findings (unimodality and symmetry of $f_Z(\cdot)$) apply to every $\tau \in (0,1)$ and imply that the point $(x^*, y^*) = (E[Z], E[Z])$ divides the line $\frac{q(w^*)-w^*x}{1-w^*}$ in two equal segments for all quantile values $q(w^*) \equiv q(w^*, \tau)$ indexed by $\tau \in (0,1)$. This property also implies that $E[Z]$ is the median of the distribution of Z with support the projection of the line on the x-axis. In this scenario no other combination $\widetilde{w}$, with $\widetilde{w} \neq w^*$, and such that $q(\widetilde{w}, \tau)$ and $q(w^*, \tau)$ defines two different lines for the same $\tau \in (0,1)$, yields a line $y = \frac{q(\widetilde{w})-\widetilde{w}x}{1-\widetilde{w}}$ that intersects $y = x$ at $(x^*, y^*) = (E[Z], E[Z])$. Let $(\widetilde{x}, \widetilde{x})$ denote such intersection, and let $\tau_0 \in (0,1)$ be defined by the condition $\tau \leqslant P(Z \leqslant q)$ for $\tau \leqslant \tau_0$, and $P(Z \leqslant q) < \tau$ for $\tau \in (\tau_0, 1]$. Then, the condition $\tau \leqslant P(Z \leqslant q)$, for all $\tau \in (0, \tau_0]$, implies that the projection of the crossing point $(\widetilde{x}, \widetilde{x})$ on the x-axis, given by $\widetilde{x}$, is smaller than the corresponding projection for w^*, that is $x^* = E[Z]$. Then, for all $\widetilde{w} \in (0,1)$ with $\widetilde{w} \neq w^*$, it follows that $\widetilde{x} < E[Z]$, that is equivalent to the condition $q(\widetilde{w}) < q(w^*)$ since $q(\widetilde{w}) = \widetilde{w}x + (1-\widetilde{w})y$. Then, for $x = y = \widetilde{x}$, we have $q(\widetilde{w}) = \widetilde{x}$ and for $x = y = E[Z]$, we have $q(w^*) = E[Z]$. Hence, the quantity w^* maximizes the quantile function for all $\tau \in (0, \tau_0]$.

It remains to see that the condition $q(w^*) = E[Z]$ characterizes a minimum of the optimization problem (7) for $\tau > \tau_0$. In this scenario, the solution to the optimization problem (7) is also in the 45^o degree line, however, for $\tau \in (\tau_0, 1]$, it follows that $P(Z \leqslant q) < \tau$ (see Figure 6(b)). Then, for any $\widetilde{w} = w^* \pm \varepsilon$, with $\varepsilon > 0$, the projection of the crossing point $(\widetilde{x}, \widetilde{x})$ on the x-axis is larger than the projection of the crossing point $(E[Z], E[Z])$ associated to w^*. Thus, for any $\widetilde{w} = w^* \pm \varepsilon$ with $\varepsilon > 0$, it follows that $\widetilde{x} > E[Z]$ and, using the above arguments, $q(\widetilde{w}) > q(w^*)$ for all $\widetilde{w} \neq w^*$, with $w^*, \widetilde{w} \in (0,1)$.

The continuity of $q(w)$ with respect to w implies that the solution to (7) is a corner solution. For $w^* = \frac{1}{2}$ in the region $\tau \in (0, \tau_0]$, the solution for $\tau > \tau_0$ is indistinctively zero or one due to the symmetry of condition (11). Otherwise, if $w^* \neq \frac{1}{2}$ in the region $\tau \in (0, \tau_0]$, the solution to the maximization problem for $\tau > \tau_0$ is given by the random variable with largest upside potential. More formally, it will be one if $F_X(z) < F_Y(z)$ uniformly over z, for z sufficiently large. In contrast, the solution will be zero for $\tau > \tau_0$ if $F_X(z) > F_Y(z)$ uniformly over z, for z sufficiently large. This property is determined by whether $w^* < \frac{1}{2}$ in condition (11) or not.

□

Proof of Corollary 3.14:

Proof. To prove the result in the corollary, it is sufficient to show that the iid assumption implies assumptions 2 and 3 with $w^* = \frac{1}{2}$. Then, applying Proposition 3.13 the result follows.

To prove assumption 2, we note that under the iid condition, for any $\mu \in \mathbb{R}$, it holds that $f(\mu+\varepsilon, \mu-\varepsilon) = f_X(\mu+\varepsilon) f_X(\mu-\varepsilon)$. Now, taking the first derivative with respect to ε, we obtain

$$\frac{\partial f(\mu+\varepsilon,\mu-\varepsilon)}{\partial \varepsilon} = \frac{\partial f_X(\mu+\varepsilon)}{\partial \varepsilon} f_X(\mu-\varepsilon) - \frac{\partial f_X(\mu-\varepsilon)}{\partial \varepsilon} f_X(\mu+\varepsilon). \tag{32}$$

Let m_X denote the mode of the random variable X. For $\mu = m_X$, the unimodality of $f_X(\cdot)$ implies that $\frac{\partial f_X(\mu+\varepsilon)}{\partial \varepsilon} > 0$ for $\varepsilon < 0$, $\frac{\partial f_X(\mu+\varepsilon)}{\partial \varepsilon} = 0$ at $\varepsilon = 0$ and $\frac{\partial f_X(\mu+\varepsilon)}{\partial \varepsilon} < 0$ for $\varepsilon > 0$. Under these conditions, expression (32) yields $\frac{\partial f(\mu+\varepsilon,\mu-\varepsilon)}{\partial \varepsilon} > 0$ for $\varepsilon < 0$; $\frac{\partial f(\mu+\varepsilon,\mu-\varepsilon)}{\partial \varepsilon} = 0$ at $\varepsilon = 0$ and $\frac{\partial f(\mu+\varepsilon,\mu-\varepsilon)}{\partial \varepsilon} < 0$ for $\varepsilon > 0$, as stated in assumption 2.

The proof is a bit more complex for $\mu \neq m_X$. In this case, for $\mu < m_X$ there exists an interval $|\varepsilon| < m_X - \mu$ that needs to be carefully evaluated. More specifically, for values of ε inside the interval, we note that expression (32) is positive if $\varepsilon < 0$ and negative if $\varepsilon > 0$. This follows from noting that for $\varepsilon < 0$ and $\mu - m_X < \varepsilon$, the following condition is satisfied:

$$\frac{\partial f_X(\mu+\varepsilon)/\partial \varepsilon}{\partial f_X(\mu-\varepsilon)/\partial \varepsilon} > \frac{f_X(\mu+\varepsilon)}{f_X(\mu-\varepsilon)}. \tag{33}$$

To show this condition it is sufficient to show that $\frac{\partial f_X(\mu+\varepsilon)}{\partial \varepsilon} > \frac{\partial f_X(\mu-\varepsilon)}{\partial \varepsilon}$ given that $f_X(\mu+\varepsilon) < f_X(\mu-\varepsilon)$ for $\varepsilon < 0$. This condition is, however, fulfilled for $x \in (-\infty, m_X)$ if the density function f_X is unimodal. Similarly, for $\varepsilon > 0$ and $\varepsilon < m_X - \mu$, the following condition is satisfied:

$$\frac{\partial f_X(\mu+\varepsilon)/\partial \varepsilon}{\partial f_X(\mu-\varepsilon)/\partial \varepsilon} < \frac{f_X(\mu+\varepsilon)}{f_X(\mu-\varepsilon)}. \tag{34}$$

This condition can be shown using the above arguments and noting that $\varepsilon > 0$.
For values of ε outside the interval $(0, m_x - \mu)$ the above proof follows straightforwardly.

For $\mu > m_X$, there exists an interval $|\varepsilon| < \mu - m_X$ that needs to be carefully evaluated. Nevertheless, the proof in this case follows as in the previous case. Also, for values of ε outside the interval the above proof follows similarly.

To verify Assumption 3, it is sufficient to show that for $w^* = \frac{1}{2}$ and $\varepsilon > 0$, equation (11) becomes

$$f(\mu+\varepsilon, \mu-\varepsilon) = f(\mu-\varepsilon, \mu+\varepsilon), \tag{35}$$

for all $\mu \in \mathbb{R}$, with $f(\cdot,\cdot)$ the joint density function of the random variables X and Y. This condition is, however, satisfied by construction under the iid assumption. This is so because

$$f(\mu+\varepsilon, \mu-\varepsilon) = f_X(\mu+\varepsilon) f_X(\mu-\varepsilon) = f(\mu-\varepsilon, \mu+\varepsilon), \tag{36}$$

for all $\mu \in \mathbb{R}$.

$\square$

B Mixture of Two Uniform Random Variables

This section describes the portfolio allocation exercise for different mixtures of two uniform random variables.

Without loss of generality, let $X \sim U(a, b)$ with $b > a$, and $Y \sim U(0, 1)$, two independent uniform random variables. We wish to derive the optimal portfolio allocation for $S_w(\tau) = w(\tau)X + (1 - w(\tau))Y$, for $\tau \in (0, 1)$. We first calculate the quantile function of the random variable $S_w(\tau)$, and second, we state the formal proposition with the optimal combination defined by $w^*(\tau)$ and the corresponding proof.

We have for each uniform

$$f_{wX}(z) = \begin{cases} \frac{1}{w(b-a)}, & \text{if } z \in [wa, wb] \\ 0, & \text{otherwise,} \end{cases}$$

$$f_{(1-w)Y}(z) = \begin{cases} \frac{1}{(1-w)}, & \text{if } z \in [0, 1-w] \\ 0, & \text{otherwise,} \end{cases}$$

The density function for the sum is given by

$$f_S(z) = \int f_{wX}(z - \xi) f_{(1-w)Y}(\xi) d\xi.$$

Now we need to examine the limits for integration. The integrand above will be zero unless $z - \xi \in [wa, wb]$ and $\xi \in [0, 1 - w]$. This leads to the following restrictions that need to be satisfied

$$\begin{cases} z - wb \leqslant \xi \leqslant z - wa \\ 0 \leqslant \xi \leqslant 1 - w. \end{cases}$$

Therefore, we can define the integration limits as following:

$$\text{inferior limit} : \max\{z - wb, 0\}$$
$$\text{superor limit} : \min\{1 - w, z - wa\}.$$

Note that the limits of integration depend on w. Now we consider the four possible cases.

Case I.

$$\text{inferior limit} : \max\{z - wb, 0\} = z - wb \iff z \geqslant wb$$
$$\text{superor limit} : \min\{1 - w, z - wa\} = 1 - w \iff z \geqslant wa + (1 - w).$$

In this case we have that

$$\begin{aligned} f_S(z) &= \frac{1}{w(b-a)}\frac{1}{(1-w)}\int_{z-wb}^{(1-w)} 1dx \\ &= \frac{1}{w(b-a)}\frac{1}{(1-w)}((1-w)+wb-z). \end{aligned}$$

Case II.

$$\begin{aligned} &\text{inferior limit} : \max\{z-wb, 0\} = z-wb \iff z > wb \\ &\text{superor limit} : \min\{(1-w), z-wa\} = z-wa \iff z < wa+(1-w). \end{aligned}$$

In this case we have that

$$\begin{aligned} f_S(z) &= \frac{1}{w(b-a)}\frac{1}{(1-w)}\int_{z-wb}^{z-wa} 1dx \\ &= \frac{1}{w(b-a)}\frac{1}{(1-w)}w(b-a). \end{aligned}$$

Case III.

$$\begin{aligned} &\text{inferior limit} : \max\{z-wb, 0\} = 0 \iff z < wb \\ &\text{superor limit} : \min\{(1-w), z-wa\} = z-wa \iff z < wa+(1-w). \end{aligned}$$

In this case we have that

$$\begin{aligned} f_S(z) &= \frac{1}{w(b-a)}\frac{1}{(1-w)}\int_{0}^{z-wa} 1dx \\ &= \frac{1}{w(b-a)}\frac{1}{(1-w)}(z-wa). \end{aligned}$$

Case IV.

$$\begin{aligned} &\text{inferior limit} : \max\{z-wb, 0\} = 0 \iff z < wb \\ &\text{superor limit} : \min\{(1-w), z-wa\} = (1-w) \iff z > wa+(1-w). \end{aligned}$$

In this case we have that

$$\begin{aligned} f_S(z) &= \frac{1}{w(b-a)}\frac{1}{(1-w)}\int_{0}^{(1-w)} 1dx \\ &= \frac{1}{w(b-a)}\frac{1}{(1-w)}(1-w). \end{aligned}$$

There are two different scenarios given by $0 < b-a \leqslant \frac{1-w}{w}$ and $b-a > \frac{1-w}{w}$ that lead to two different orderings in the limits of integration. Thus, for $b - a > \frac{1-w}{w}$, we obtain

$$f_S(z) = \begin{cases} 0, & z < wa \\ \frac{z-wa}{w(b-a)(1-w)}, & wa \leqslant z < wa + (1-w) \\ \frac{1}{w(b-a)} & wa + (1-w) \leqslant z < wb \\ \frac{(1-w)+wb-z}{w(b-a)(1-w)} & wb \leqslant z < wb + (1-w) \\ 0 & z \geqslant wb + (1-w). \end{cases}$$

The cumulative distribution function can be written as

$$F_S(z) = \begin{cases} 0, & z < wa \\ \frac{(z-wa)^2}{2w(b-a)(1-w)}, & wa \leqslant z < wa + (1-w) \\ \frac{1-w}{2w(b-a)} + \frac{z-wa-(1-w)}{w(b-a)}, & wa + (1-w) \leqslant z < wb \\ 1 - \frac{1}{2(b-a)}\left(b - \frac{z-(1-w)}{w}\right)\left(1 - \frac{z-wb}{1-w}\right), & wb \leqslant z < wb + (1-w) \\ 1, & z \geqslant wb + (1-w). \end{cases}$$

Operating with the expression in the $wb \leqslant z < wb + (1-w)$ bracket, we obtain $\tau = 1 - \frac{1}{2(b-a)}\left(b - \frac{z-(1-w)}{w}\right)\left(1 - \frac{z-wb}{1-w}\right)$. Then, $2(1-\tau)(b-a) = \left(b - \frac{z-(1-w)}{w}\right)\left(1 - \frac{z-wb}{1-w}\right)$. We define $x = \frac{z-(1-w)}{w}$ such that the previous expression reads as $2(1-\tau)(b-a) = (b-x)\frac{w}{1-w}(b-x)$ and $2(1-\tau)(b-a)\frac{1-w}{w} = (b-x)^2$. Then, $x = b \pm \sqrt{2(1-\tau)(b-a)\frac{1-w}{w}}$. Furthermore, the constraint $wb \leqslant z < wb + (1-w)$ implies that $x < b$. Then, $x = b - \sqrt{2(1-\tau)(b-a)\frac{1-w}{w}}$. Now, changing the variable x for z we obtain $z = wb + (1-w) - \sqrt{2(1-\tau)(b-a)w(1-w)}$. Then, the quantile function can be written as

$$q(\tau; w) = \begin{cases} wa + \sqrt{2\tau w(1-w)(b-a)}, & 0 \leqslant \tau < \tau_1 \\ wa + \tau w(b-a) + \frac{1-w}{2}, & \tau_1 \leqslant \tau < \tau_2 \\ wb + (1-w) - \sqrt{2(1-\tau)w(1-w)(b-a)} & \tau \geqslant \tau_2, \end{cases} \tag{37}$$

with $\tau_1 = F_S(wa+(1-w))$ and $\tau_2 = F_S(wb)$. Note that in the previous definition we explicitly include w.

Similarly, for $0 < b - a \leqslant \frac{1-w}{w}$, the density function is

$$f_S(z) = \begin{cases} 0, & z < wa \\ \frac{z-wa}{w(b-a)(1-w)}, & wa \leqslant z < wb \\ \frac{1}{(1-w)}, & wb \leqslant z < wa + (1-w) \\ \frac{(1-w)+wb-z}{w(b-a)(1-w)}, & wa + (1-w) \leqslant z < wb + (1-w) \\ 0, & z \geqslant wb + (1-w). \end{cases}$$

The corresponding cumulative distribution function is

$$F_S(z) = \begin{cases} 0, & z < wa \\ \frac{(z-wa)^2}{2w(b-a)(1-w)}, & wa \leqslant z < wb \\ \frac{w}{1-w}\frac{b-a}{2} + \frac{z-wb}{1-w}, & wb \leqslant z < wa + (1-w) \\ 1 - \frac{1}{2(b-a)}\left(b - \frac{z-(1-w)}{w}\right)\left(1 - \frac{z-wb}{1-w}\right), & wa + (1-w) \leqslant z < wb + (1-w) \\ 1, & z \geqslant wb + (1-w). \end{cases}$$

Using the same expressions as before, we obtain the following quantile function:

$$q(\tau; w) = \begin{cases} wa + \sqrt{2\tau w(1-w)(b-a)}, & 0 \leqslant \tau < \tilde{\tau}_1 \\ wb + (1-w)\tau - w\frac{b-a}{2}, & \tilde{\tau}_2 \leqslant \tau < \tilde{\tau}_2 \\ wb + (1-w) - \sqrt{2(1-\tau)w(1-w)(b-a)} & \tau \geqslant \tilde{\tau}_2, \end{cases} \tag{38}$$

with $\tilde{\tau}_1 = F_S(wb)$ and $\tilde{\tau}_2 = F_S(wa + (1-w))$.

Proposition B.1. *Suppose we have two independent uniform random variables* $X \sim U(a, b)$ *with* $b > a$ *and* $Y \sim U(0, 1)$, *and let* $S_{w(\tau)} = w(\tau)X + (1 - w(\tau))Y$ *with* $w(\tau)$ *a function* $w : [0, 1] \to [0, 1]$. *Then, the optimal combination for portfolio QP maximization* $w^*(\tau)$ *is the following.*

If $a \geqslant 1$ *then* $w^*(\tau) = 1$, $\forall\tau$.

If $b < 0$ *then* $w^*(\tau) = 0$, $\forall\tau$.

Otherwise, define

$$w^*(\tau) = \begin{cases} \tilde{w}^*(\tau), & \max\{q(\tau; \tilde{w}^*(\tau)), \tau, \tau(b-a)\} = q(\tau, \tilde{w}^*(\tau)) \\ 0, & \max\{q(\tau; \tilde{w}^*(\tau)), \tau, \tau(b-a)\} = \tau \\ 1, & \max\{q(\tau; \tilde{w}^*(\tau)), \tau, \tau(b-a)\} = \tau(b-a), \end{cases}$$

where $\tilde{w}^*(\tau)$ *is defined in the following way. For* $b - a \geqslant \frac{1-\tilde{w}^*(\tau)}{\tilde{w}^*(\tau)}$, *with* $\tau \in [0, 1]$, $\tilde{w}^*(\tau)$ *is*

$$\tilde{w}^*(\tau) = \begin{cases} \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{1+\frac{a^2}{2\tau(b-a)}}}, & 0 \leqslant \tau < \tau_1 \\ \frac{1}{1+b-a}, & \tau_1 \leqslant \tau < \min(\max(\frac{\frac{1}{2}-a}{b-a}, \tau_1), \tau_2) \\ 1, & \tau \geqslant \min(\max(\frac{\frac{1}{2}-a}{b-a}, \tau_1), \tau_2), \end{cases}$$

with $\tau_1 = F_S(\tilde{w}^*(\tau)a + (1 - \tilde{w}^*(\tau)))$ *and* $\tau_2 = F_S(\tilde{w}^*(\tau)b)$.

For $0 < b - a \leqslant \frac{1-\tilde{w}^*(\tau)}{\tilde{w}^*(\tau)}$*, with* $\tau \in [0,1]$*,* $\tilde{w}^*(\tau)$ *is*

$$\tilde{w}^*(\tau) = \begin{cases} \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{1}{1+\frac{a^2}{2\tau(b-a)}}}, & 0 \leqslant \tau < \tilde{\tau}_1 \\ \frac{1}{1+b-a}, & \tilde{\tau}_1 \leqslant \tau < \min(\max(\frac{a+b}{2}, \tilde{\tau}_1), \tilde{\tau}_2) \\ 0, & \tau \geqslant \min(\max(\frac{a+b}{2}, \tilde{\tau}_1), \tilde{\tau}_2), \end{cases}$$

with $\tilde{\tau}_1 = F_S(\tilde{w}^*(\tau)b)$ *and* $\tilde{\tau}_2 = F_S(\tilde{w}^*(\tau)a + (1 - \tilde{w}^*(\tau)))$*.*

Proof. Using the quantile processes derived above, we can obtain the optimal $\tilde{w}^*(\tau)$ for different regions inside $\tau \in [0,1]$. To do this, we maximize the quantile functions $q(\tau)$ in (37) and (38) with respect to w.

For $b-a \geqslant \frac{1-w}{w}$, the quantile function is (37) and the first regime corresponds to $0 < \tau < \tau_1$, with $\tau_1 = F_S(wa + 1 - w)$. Then, the optimal w satisfies that

$$a + \tau(1-2w)\sqrt{\frac{b-a}{2\tau w(1-w)}} = 0.$$

This equation is equivalent to $a = \frac{2w-1}{\sqrt{2w(1-w)}}\sqrt{\tau(b-a)}$. Taking squares in both sides, we obtain $2a^2 w(1-w) = \tau(1 - 4w + 4w^2)(b-a)$. After some further algebra, we obtain

$$(4\tau(b-a) + 2a^2)w^2 - (4\tau(b-a) + 2a^2)w + \tau(b-a) = 0,$$

that is equivalent to $w^2 - w + y = 0$, with $y = \frac{\tau(b-a)}{4\tau(b-a)+2a^2}$. Then, the solution to this problem for $0 < \tau < \tau_1$ is $w^*(\tau) = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{1}{1+\frac{a^2}{2\tau(b-a)}}}$. Furthermore, the condition $b - a > \frac{1-w}{w}$ implies that $w > \frac{1}{1+b-a}$. Then, for each τ in $[0, \tau_1)$, we have $w^*(\tau) = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{1+\frac{a^2}{2\tau(b-a)}}}$.

For the second term, we note that for a given τ with $\tau_1 \leqslant \tau < \tau_2$, it follows that $q'(w) = a + \tau(b-a) - \frac{1}{2}$. This function is strictly increasing for $\frac{\frac{1}{2}-a}{b-a} < \tau$ such that $w^*(\tau) = 1$ in this region; and $q'(w)$ is strictly decreasing for $\frac{\frac{1}{2}-a}{b-a} > \tau$ such that $w^*(\tau) = \frac{1}{1+b-a}$ given the constraint $\tilde{w}^*(\tau) \geqslant \frac{1}{1+b-a}$ entailed by the condition $b - a \geqslant \frac{1-\tilde{w}^*(\tau)}{\tilde{w}^*(\tau)}$.

For the third term given by $\tau \geqslant \tau_2$, the relevant quantile function is $q(\tau; w) = wb + (1-w) - \sqrt{2(1-\tau)w(1-w)(b-a)}$ and the first derivative with respect to w is $q'(w) = b - 1 - \sqrt{\frac{(1-\tau)(b-a)}{2w(1-w)}}(1-2w)$. This function is strictly positive implying that $w^*(\tau) = 1$ in this case. Then,

$$\tilde{w}^*(\tau) = \begin{cases} \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{1+\frac{a^2}{2\tau(b-a)}}}, & 0 \leqslant \tau < \tau_1 \\ \frac{1}{1+b-a}, & \tau_1 \leqslant \tau < \min(\max(\frac{\frac{1}{2}-a}{b-a}, \tau_1), \tau_2) \\ 1, & \min(\max(\frac{\frac{1}{2}-a}{b-a}, \tau_1), \tau_2) \leqslant \tau < \tau_2 \\ 1, & \tau \geqslant \tau_2, \end{cases}$$

with $\tau_1 = F_S(\tilde{w}^*(\tau)a + (1 - \tilde{w}^*(\tau)))$ and $\tau_2 = F_S(\tilde{w}^*(\tau)b)$.

Similarly, for $b - a \leqslant \frac{1-w}{w}$, the quantile function is (38) and the first regime corresponds to $0 < \tau < \tilde{\tau}_1$ with $\tilde{\tau}_1 = F_S(wb)$. The first order conditions of $q(\tau; w)$ obtained from (38) yield the same condition as before: $w^*(\tau) = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{1}{1+\frac{a^2}{2\tau(b-a)}}}$. The condition $w \leqslant \frac{1}{1+b-a}$ for $0 \leqslant \tau < \tilde{\tau}_1$ implies that the optimal weight function in this regime is $w^*(\tau) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{1}{1+\frac{a^2}{2\tau(b-a)}}}$.

For the second term, we note that for a given τ with $\tilde{\tau}_1 \leqslant \tau < \tilde{\tau}_2$, it follows that $q'(w) = \frac{a+b}{2} - \tau$. This function is strictly increasing for $\frac{a+b}{2} > \tau$ such that $w^*(\tau) = \frac{1}{1+b-a}$ given the constraint $w^*(\tau) \leqslant \frac{1}{1+b-a}$; $q'(w)$ is strictly decreasing for $\frac{a+b}{2} < \tau$ such that $w^*(\tau) = 0$.

For the third term given by $\tau \geqslant \tilde{\tau}_2$, the relevant quantile function is $q(\tau; w) = wb + (1 - w) - \sqrt{2(1-\tau)w(1-w)(b-a)}$ and the first derivative with respect to w is $q'(w) = b - 1 - \sqrt{\frac{(1-\tau)(b-a)}{2w(1-w)}}(1 - 2w)$. This function is strictly negative implying that $w^*(\tau) = 0$ in this case. Then,

$$\tilde{w}^*(\tau) = \begin{cases} \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{1}{1+\frac{a^2}{2\tau(b-a)}}}, & 0 \leqslant \tau < \tilde{\tau}_1 \\ \frac{1}{1+b-a}, & \tilde{\tau}_1 \leqslant \tau < \min(\max(\frac{a+b}{2}, \tilde{\tau}_1), \tilde{\tau}_2) \\ 0, & \min(\max(\frac{a+b}{2}, \tilde{\tau}_1), \tilde{\tau}_2) \leqslant \tau < \tilde{\tau}_2 \\ 0, & \tau \geqslant \tilde{\tau}_2, \end{cases}$$

with $\tilde{\tau}_1 = F_S(\tilde{w}^*(\tau)b)$ and $\tilde{\tau}_2 = F_S(\tilde{w}^*(\tau)a + (1 - \tilde{w}^*(\tau)))$.

□

C Numerical simulation study

This Appendix presents additional results on the optimal asset allocation under quantile preferences (QP) for a portfolio $S_w = wX + (1 - w)Y$ given by the mixture of two continuous random variables X and Y, which are chosen from different distribution functions. For illustrative purposes, we consider pairs of Gaussian and Chi-squared random variables, under mutual independence and also with dependence. These results are also compared with the optimal asset allocation obtained under EU. Numerical computation of the portfolios are as described in Section 3.4.

C.1 Independent and identically distributed random variables

Two Chi-Squared independent and identically distributed random variables

To illustrate the optimal portfolio allocation problem for asymmetric distributions, we investigate the case of two independent and identically distributed standard Chi-squared distributions, X and Y. Under risk aversion, the iid assumption is sufficient to guarantee that the optimal portfolio choice of a EU individual will be a fully diversified portfolio given by $w^* = 0.5$. For QP, the same result holds true for values of τ identified with risk aversion. The left panel of Figure 16 shows that the family of distribution functions F_{S_w} satisfies the single-crossing condition, see Figure 1 illustrating this condition. The right panel of Figure 16 shows the optimal portfolio allocation is divided into two regions: a first region given by $\widehat{w}_n^*(\tau) = 0.5$, for $\tau \leqslant \tau_0 \approx 0.80$, and a second region given by $\widehat{w}_n^*(\tau) = \{0, 1\}$, for $\tau > \tau_0$. The shift in the cut-off point τ_0 compared to $\tau_0 = 0.5$ is due to the asymmetry of the Chi-squared distribution.

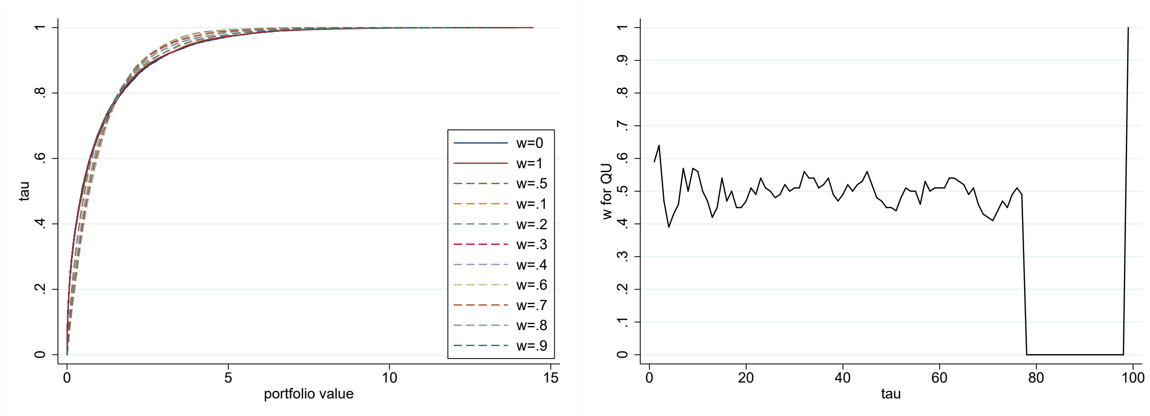


Figure 16: $X, Y \sim \chi_1^2$. Left box plots the CDF of S_w. Right box plots QP portfolio selection.

C.2 Independently distributed random variables

Two Gaussian independent random variables with different variances

We extend the Gaussian case discussed above and consider the case of independent random variables X and Y with different variances. Let $X \sim N(0, 1)$ and $Y \sim N(0, 2)$, independent of each other. In this case Y is a mean-preserving spread of the random variable X. This observation implies that X second order stochastically dominates Y and, using Fishburn (1977), X is preferred to Y for EU individuals endowed with an increasing and concave utility function. We can extend this result to derive the optimal portfolio allocation of a EU risk averse individual. In this case, the combination S_{w^*} with $w^* = 2/3$ has zero mean and minimizes the variance of the family of random variables S_w. This combination second order stochastically dominates any other convex combination including X and Y. Then, using Fishburn (1977), S_{w^*} is the optimal strategy for EU individuals endowed with an increasing and concave utility function.

For individuals endowed with QP, we note that condition (11) is satisfied for $w^* = 2/3$. Then, Proposition 3.13 implies that the optimal combination of a QP individual is $w^* = \frac{2}{3}$ for $\tau \leqslant \tau_0$, with $\tau_0 = \frac{1}{2}$. This result is illustrated in the numerical exercise on the right panel of Figure 17. Diversification takes place for $\tau \leqslant \frac{1}{2}$ and no diversfication is the optimal result for $\tau > \frac{1}{2}$.

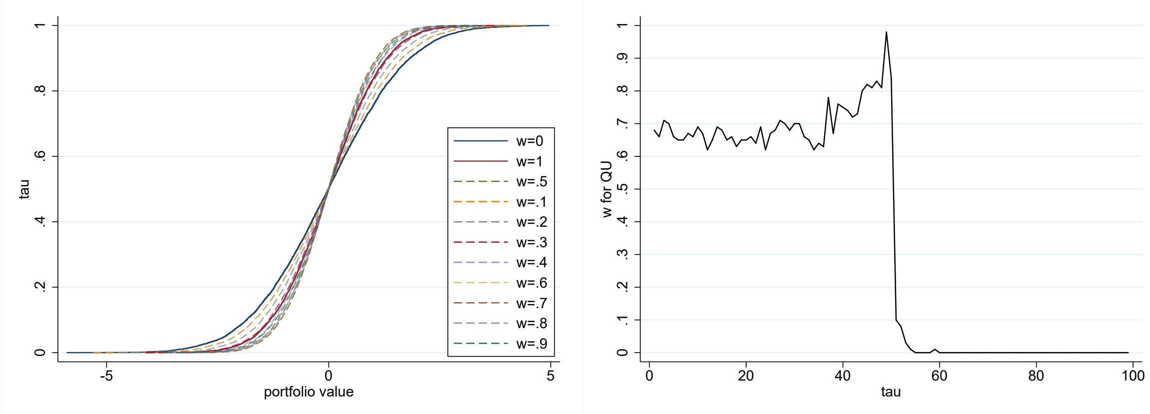


Figure 17: $X \sim N(0, 1)$ and $Y \sim N(0, 2)$. Left panel plots the CDF of S_w. Right panel plots QP portfolio selection.

Two Gaussian independent random variables with different means

Consider now the case of two independent Gaussian random variables with different means. Let $X \sim N(1, 1)$ and $Y \sim N(1.5, 1)$, independent of each other. In this case the random variable Y first order stochastically dominates the random variable X. In this scenario the optimal portfolio decision of an EU individual has to be calculated for each utility function separately because it depends on how the individual faces the trade-off between risk and return.

The results for the QP case are provided in Figure 18. The results show the presence of an interior solution for $\tau \leqslant \tau_0$, with τ_0 around 0.30, and a corner solution $\widehat{w}_n^*(\tau) = 0$ for $\tau > 0.30$. The solution in the right tail of the distribution is rationalized by the first order stochastic dominance of Y over any other convex combination of X and Y for τ sufficiently large.

C.3 Dependent random variables

We study now the case of two dependent random variables. We differentiate between symmetric and asymmetric random variables.

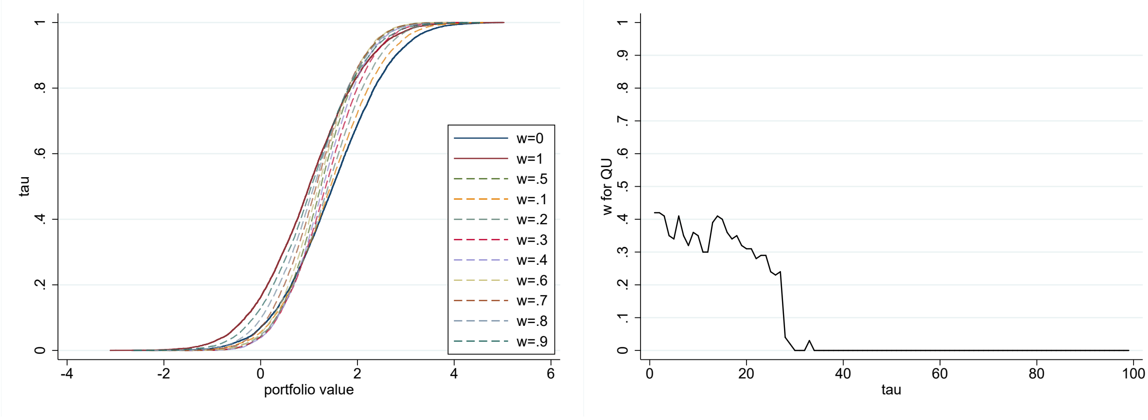


Figure 18: $X \sim N(1,1)$ and $Y \sim N(1.5,1)$. Left box plots the CDF of S_w. Right box plots QP portfolio selection.

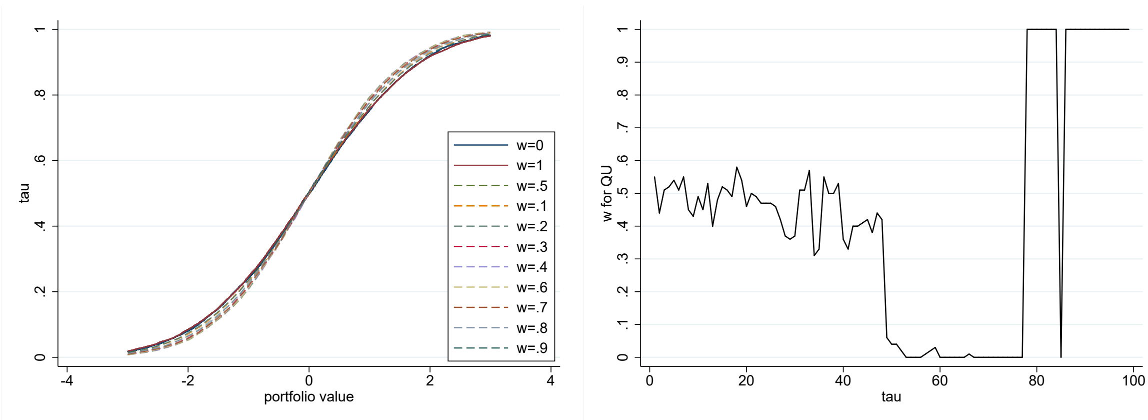


Figure 19: $(X, Y) \sim N(0, \Sigma)$ with $\Sigma = [1\ 0.5; 0.5\ 1]$. Left box plots CDF of S_w. Right box plots QP portfolio selection.

Two Gaussian dependent random variables with same mean

In this exercise, we consider a bivariate Normal random variable (X, Y) with covariance matrix $\Sigma = [1\ \rho; \rho\ 1]$, where $\rho = 0.5$ is the correlation parameter. This scenario is rationalized by Proposition 3.13. More specifically, condition (11) is satisfied for $\widehat{w}_n^*(\tau) = 0.5$, such that an interior solution is obtained for $\tau \leqslant \tau_0 = 0.5$. This result is observed in Figure 19. For values of τ greater than 0.5 the optimal solution is, indistinctively, $\widehat{w}_n^*(\tau) = \{0, 1\}$.

Chi-squared dependent random variables

We complete the case of dependent assets with asymmetric distribution functions. Suppose that $X \sim \chi_1^2 + 1$ and $Y \sim \chi_2^2 + X - 2$ two dependent assets with expected value equal to two in both cases. For QP individuals, the optimal portfolio allocation has an interior solution for

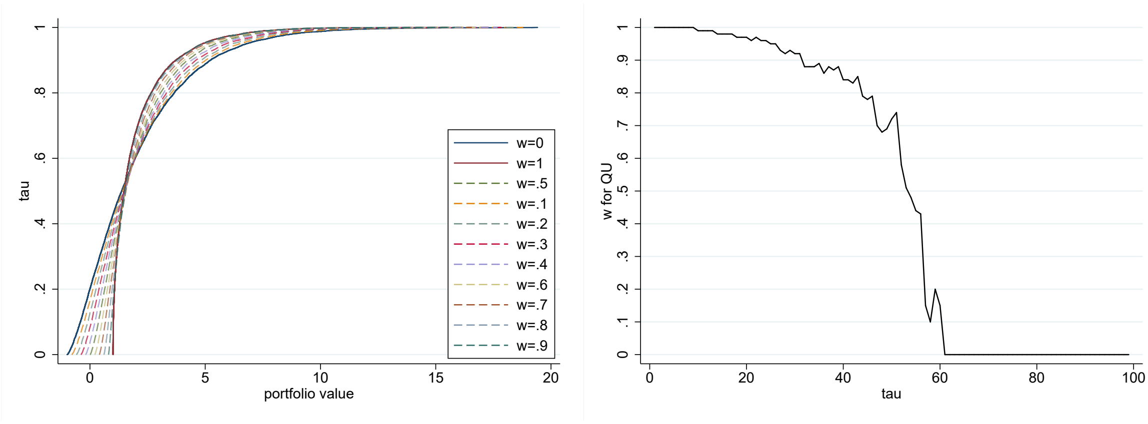


Figure 20: $X \sim \chi_1^2 + 1$ and $Y \sim \chi_2^2 + X - 2$. Left box plots the CDF of S_w. Right box plots QP portfolio selection.

$\tau \leqslant \tau_0$, with $\tau_0 = 0.60$. The presence of an interior solution for small values of τ is rationalized by Theorem 1. However, in contrast to previous examples, assumptions 3 is not satisfied implying that the optimal portfolio allocation varies over τ and there is no separation between risk aversion and risk loving behavior. Figure 20 reports the optimal portfolio allocation in this case.

C.4 Optimal portfolio allocation when there is a risk-free asset

Building upon previous insights of the QP theory in Section 3.2, we can extend the mutual fund separation theorem to the case of one risk free asset with returns $\bar{r}$ and two risky assets R_1, R_2 with distribution functions F_{R_1}, F_{R_2}, respectively, and such that F_{R_1} crosses F_{R_2} from below at point x_{12}. This assumption implies that R_2 is riskier than R_1. The portfolio return is defined by the convex combination $R_p = w_0\bar{r} + w_1 R_1 + w_2 R_2$, with $w_0 + w_1 + w_2 = 1$. The investor's optimization problem is

$$\arg \max_{\{w_0, w_1, w_2\}} Q_\tau[w_0\bar{r} + w_1 R_1 + w_2 R_2]. \tag{39}$$

In this case the individual's optimal portfolio choice is given by the risk-free rate when $Q_\tau[R_p(w)] < \bar{r}$, for any combination of weights $w = \{w_0, w_1, w_2\}$ that is different from $w^* = \{w_0^* = 1, w_1^* = w_2^* = 0\}$. For higher quantiles, the solution to the maximization problem (39) is, in principle, quantile-specific. We illustrate this scenario by simulating the returns on three assets with returns $\bar{r}$, $R_1 \sim N(\mu_1, 1)$ and $R_2 \sim N(\mu_2, 1)$. The solution for $0 \leqslant \tau \leqslant 1$ is obtained by simulating $n = 10,000$ realizations of the random variables.

The left panel of Figure 21 reports the case $\bar{r} = 0.5$, $\mu_1 = \mu_2 = 0$. The right panel considers the case $\bar{r} = 0.25$, $\mu_1 = \mu_2 = 0$. As discussed above, for $Q_\tau[R_p(w)] < \bar{r}$, the optimal allocation

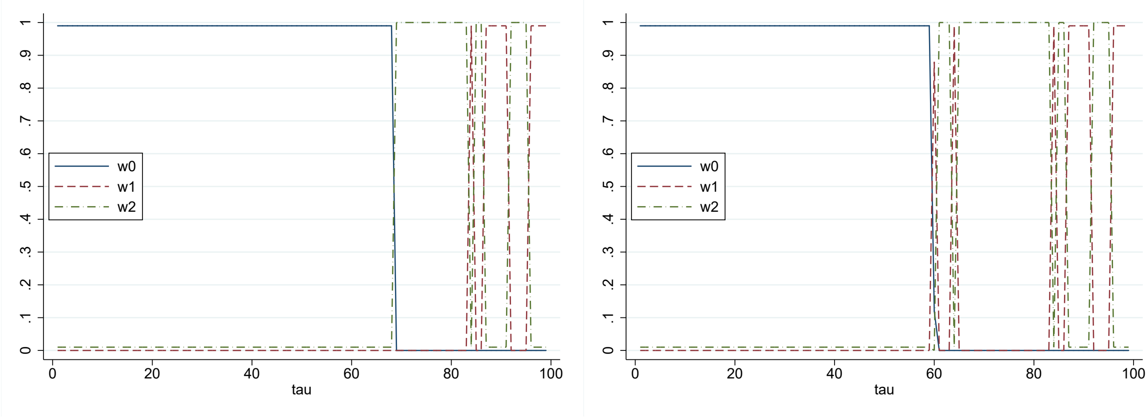


Figure 21: $R_p = w_0\bar{r} + w_1R_1 + w_2R_2$, with $\bar{r}$ constant, $R_1 \sim N(\mu_1, 1)$ and $R_2 \sim N(\mu_2, 1)$. Left box plots $\bar{r} = 0.5$, $\mu_1 = \mu_2 = 0$. Right box plots $\bar{r} = 0.25$, $\mu_1 = \mu_2 = 0$.

to the portfolio is given by investing on the risk-free asset. For the parameterization $\bar{r} = 0.5$, $\mu_1 = \mu_2 = 0$ - left panel - the value of τ that yields the condition $Q_\tau[R_p(w^*)] = \bar{r}$ is $\tau_0 \approx 0.7$. For values of $\tau > 0.7$, the optimal allocation to the risk-free asset is zero, and the QP individual is indifferent between assets 1 and 2. The right panel characterized by a smaller risk-free return ($\bar{r} = 0.25$) presents a similar outcome. In this scenario the value of τ that satisfies the condition $Q_\tau[R_p(w^*)] = \bar{r}$ is $\tau_0 \approx 0.6$. For higher quantiles, the optimal portfolio allocation is the same as for the left panel and entails a zero allocation to the risk-free rate.[17]

[17]The case of two risky assets with different means is available from the authors upon request.

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