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Compositionality in Game Theory: an Operadic View

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1 Introduction

An important problem, frequently glossed over in most applications of Game Theory is how to address the fact that any agent participates in different interactions, each of which interpreted as a game. The outcomes in these interactions impact on the payoffs of the agent, and thus create incentives for behaving differently as when participating in a single game with other agents. The result can be seen as a larger game, obtained by composing the particular games.

The goal of this contribution is to analyze the mathematical conditions for the compositionality of games. In this initial version we will consider only games in strategic form and how to compose them to obtain larger strategic form games. In future versions we will tackle extensive form games, instead of their strategic versions. Furthermore, we will analyze *resource allocation* games. They will contribute to the extension of our results to general interactive systems, making them applicable as general models of social and economic systems.

We will show how to formalize the cases in which players participate in several games, composing their behaviors in each of the individual games. A basic category of games allows the definition of a symmetric monoidal structure which captures the characterization of new games up from smaller ones (starting from the individual players). A hypergraph category based on this structure includes a fundamental component, namely a lax monoidal functor, defining an algebra of *equilibria* that parallels the compositional features of the monoid of games.[1]

The presentation in this paper is self-contained, although we are aware that most of the mathematical apparatus applied in this investigation is not familiar

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[1] A closely related research, based on a different categorical treatment can be found in [2].

for most economists. We hope this will serve as a primer in the subject and encourage further research on this and related topics. In turn, the elements of Game Theory used in this article are quite basic, but we have chosen to adopt the formal language used in [4].

2 Mathematical Preliminaries

There are various mathematical notions that are relevant for our treatment. They provide a framework for the connection ("wiring up") generic entities. The first one is the concept of a *cospan*.

Consider a central objects G with input and output ports, X and Y respectively. G is called the *apex* of the cospan, while X and Y are its *feet*. The cospan can be written as: $X \stackrel{f}{\to} G \stackrel{g}{\leftarrow} Y$. Suppose now that another cospan shares Y with the previous one: $Y \stackrel{h}{\to} G' \stackrel{l}{\leftarrow} Z$. These cospans can be composed by connecting the two apexes to yield a new one, with input X and output Z.

In this way, a cospan $X \stackrel{f}{\to} G \stackrel{g}{\leftarrow} Y$ can be seen as a morphism between X and Y. In order to be such, the appex and the feet must be objects in a category $\mathcal{C}$. Furthermore, we need (as the identity morphism) to define a cospan $A \stackrel{\mathrm{id}_A}{\to} A \stackrel{\mathrm{id}_A}{\leftarrow} A$ for each object A. This is ensured by the very fact that $\mathcal{C}$ is a category.

This definition of a cospan as a morphism allows us to define a category $\mathbf{Cospan}_\mathcal{C}$ in which the objects are the same as in $\mathcal{C}$, while the morphisms are cospans among objects in $\mathcal{C}$. We need to ensure that they can be composed. This obtains if pushouts can be defined in the basis category $\mathcal{C}$.

Since we also want compositions in parallel, besides a composition we need to be able to define a *monoidal operation* $+$ between cospans. For this is enough to have both an *initial object* and *pushouts* between any pair of objects in $\mathcal{C}$.

If these conditions are fulfilled, $\mathbf{Cospan}_\mathcal{C}$ is a *monoidal symmetric* category, i.e. it has the following components:

- An initial object $\emptyset$, such that for each object A in $\mathcal{C}$ there exists a unique morphism $\emptyset \stackrel{!}{\to} A$.
- The *coproduct* $\sqcup$ in $\mathcal{C}$, provides the definition of the functor (monoidal product) $+ : \mathbf{Cospan}_\mathcal{C} \times \mathbf{Cospan}_\mathcal{C} \to \mathbf{Cospan}_\mathcal{C}$.
- For each pair of objects A and B, $A + B$ is isomorphic to $B + A$.

This is ensured if pushouts and an initial object exist in $\mathcal{C}$. This is the case if it is a category with *finite colimits.* One important case arises when $\mathcal{C}$ is

FinSet, whose objects are finite sets. We can restrict our attention further to *typed finite sets* ($\mathbf{TFS}_\Lambda$). We define a set Λ of types and each object in $\mathbf{TFS}_\Lambda$ is (X, τ_X) such that $\tau_X : X \to \Lambda$. Each X can be interpreted a set of *ports* and τ_X indicates the type of each port. It is known that $\mathbf{TFS}_\Lambda$ has also finite colimits [3].

We will use $\mathbf{W}_\Lambda$ to denote $\mathbf{Cospan}_{\mathbf{TFS}_\Lambda}$. Its objects are called *interfaces*. The morphisms (cospans) $X \overset{f_1}{\to} N \overset{f_2}{\leftarrow} Y$ are called *wiring diagrams* and the apex N is the family of *connections*. We can thus connect three objects X_1, X_2 and X_3 to obtain a new object Y as a cospan $X_1 + X_2 + X_3 \overset{g_1}{\to} C \overset{g_2}{\leftarrow} Y$.

A *Hypergraph category* ([1]) is a monoidal symmetric category in which the wiring diagrams constitute *networks* (that is, cables can be joined and can bifurcate).[2] It can be described as (Λ, H) where Λ is a class of types and $H : \mathbf{W}_\Lambda \to \mathbf{Set}$ is a *lax* functor such that, given an operation $\otimes$ in $\mathbf{Set}$, $H(X) \otimes H(Y) \to H(X + Y)$ and, given the unit of object I, $I \to H(\emptyset)$.

The importance of the existence of a lax functor on $\mathbf{W}_\Lambda$ is that in $\mathbf{Set}$ we can represent *behaviors* associated to the *structures* represented as wiring diagrams. This is particularly true in the case of $\mathbf{W}_\Lambda$.

3 A Categorical Representation of Games

Let us consider a category $\mathcal{G}$ of *games*. Each object G in the category corresponds to a game $G = \langle (I_G, S_G, \mathbf{O}_G, \rho_G), \pi_G \rangle$, where

- $(I_G, S_G, \mathbf{O}_G, \rho_G)$ is a game form:
  - I_G is the class of players.
  - $S_G = \prod_{i \in I_G} S_i^G$ is the *strategy set* of the game, where $S_i^G \subseteq S_i$ is the set of strategies that player i can deploy in game G, for each $i \in I_G$.[3]
  - $\mathbf{O}_G$ is the class of *outcomes* of the game and $\rho_G : S_G \to \mathbf{O}_G$ is a one-to-one function that associates each profile of strategies in the game with one of its outcomes.
- $\pi_G = \prod_{i \in I} \pi_i^G$ is a *profile of payoff functions*, where $\pi_i^G : \mathbf{O}_G \to \mathbb{R}^+$ is the payoff function of player i in game G, for each $i \in I_G$.

A game is defined in terms of the interactions of *players*. Each player can be seen as been described in terms of the strategies she can play and the payoffs she can receive from the results of her action (jointly with those of the other players).

[2] Technically, each object is equipped with a *special commutative Frobenius monoid*. In the instances to be considered here, this condition is automatically satisfied.

[3] S_i is the set of all the strategies that player i can play in the games in which she participates.

We can define a category $\mathcal{G}$, where the objects are games. Given two games

$$G = \langle (I_G, S_G, \mathbf{O}_G, \rho_G), \pi_G \rangle \quad \text{and} \quad G' = \langle (I_{G'}, S_{G'}, \mathbf{O}_{G'}, \rho_{G'}), \pi_{G'} \rangle,$$

a morphism of games

$$G \to G'$$

is such that:

- $I_G \subseteq I_{G'}$.
- $S_i^G \subseteq S_i^{G'}$ for each $i \in I_G$.
- There exist two functions:
  - an inclusion $p_{\mathbf{O}_G}^{\mathbf{O}_{G'}} : SO_{G'} \hookrightarrow \mathbf{O}_G$ for $SO_{G'} \subseteq \mathbf{O}_{G'}$
  - a projection $p_{S_G}^{S_{G'}} : S_{G'} \to S_G$, i.e. $p_{S_G}^{S_{G'}}(s_1^{G'}, \ldots, s_i^{G'}, \ldots, s_{|I_{G'}|}) \in \prod_{i \in I_G} S_i^{G'} = S_G$.

  These functions verify the following condition:

  - For every $s' \in S_{G'}$, $s = p_{S_G}^{S_{G'}}(s') \in S_G$ is such that $\rho_G(s) = p_{\mathbf{O}_G}^{\mathbf{O}_{G'}}(\rho_{G'}(s'))$.

Thus, if a morphism $G \to G'$ exists, G can be conceived as a *subgame form* of G'.

To complete the characterization of $\mathcal{G}$ notice that it is immediate that we can define *pushouts* and an *initial object* in this category:

- **Pushouts**: Consider three objects G, G' and G'' and morphisms $G \xrightarrow{f} G'$ and $G \xrightarrow{g} G''$. Then, take the coproduct of G' and G'', denoted $G' + G''$, obtained as the direct sums of the strategies sets and the outcomes of both games. By identifying the subgame forms of G' and G'' corresponding to G we obtain the *pushout* of

$$G' \quad \xleftarrow{f} \quad G \quad \xrightarrow{g} \quad G''$$

- **Initial object**: Consider the *empty game* $G^{\emptyset}$, where $I_{G^{\emptyset}} = \emptyset$ and consequently $S_{G^{\emptyset}} = \emptyset$ and $\mathbf{O}_G = \emptyset$ (thus $\pi_{G^{\emptyset}}$ must be the empty function). It is immediate to see that $G^{\emptyset} \to G$ for every G in $\mathcal{G}$.

Then we have

Proposition 1 *$\mathcal{G}$ is a category with* colimits.

Since $\mathcal{G}$ is a category with colimits we can define *cospans* in it. Consider again three objects G, G' and G'' and two morphisms $G \xrightarrow{f} G'' \xleftarrow{g} G'$. This is called a cospan from G to G'. The interpretation of such a cospan is that G and G' are subgame forms of the same game (G'').

4 Games as Boxes

We can conceive each game G in $\mathcal{G}$ as a *box*, $G = (\text{in}^G, \text{out}^G)$, where in^G and out^G are, respectively *input* and *output* ports. in^G has type $\mathbf{O}_G$, i.e. the input is an outcome of G. In turn, the out^G port has type S_G, being each output a profile in G.

Notice that each player i can be conceived as a game $(\text{in}^i, \text{out}^i)$, where in^i has type $\cup_{G:i\in I_G}\mathbf{O}_G$ and out^i has type S_i.

Up to this point, our definition of morphisms in $\mathcal{G}$ does not involve the payoffs. They can be incorporated by redefining the games as *modal boxes*, in which an additional component are the *internal states* of the game. More precisely, given any G and the class of its internal states, Σ_G, we can identify G as a triple $\langle \text{in}^G, \text{out}^G, \Sigma_G\rangle$, associated to two correspondences:

- **payoff**: $\phi^1_G : \bar{\text{in}}^G \times \Sigma_G \to \mathbb{R}^{+\mathbf{O}_G}$, such that for the vector $o \in \bar{\text{in}}^G$ (the vector of all possible inputs of G, each entry being an outcome of the game) and state σ, $\phi^1_G(o,\sigma) = (\pi^i_G(o))_{o\in\mathbf{O}_G}$. That is, it yields the vector of payoffs corresponding to all the outcomes of G.
- **choice**: $\phi^2_G : \Sigma_G \to \bar{\text{out}}^G$, such that for any state σ, $\phi^2_G(\sigma) = s \in \bar{\text{out}}^G$ (the class of all possible strategy profiles in S_G) is a profile of strategies that may be chosen at that state.

Particularly relevant for our analysis is the definition of the internal states of each player i, Σ_i. Consider a game G such that $i \in I_G$, and a sequence of morphisms in $\mathcal{G}$

$$G^0_i \to G^1_i \to \quad \ldots \quad \to G^{n-1}_i \to G^n_i$$

where G^0_i is a game in which i is the only player and $G = G^n_i$. We identify the state of player i when playing G as a sequence $\sigma^i_G = \langle\sigma^i_0, \ldots, \sigma^i_{n-1}\rangle$, where $\sigma^i_k \in \Sigma_{G^k_i}$, for $k = 0\ldots, n-1$. Then, a distinguished object $\sigma^i_* \in \Sigma_i$ is defined, such that σ^i_G is one of its initial segments.[4]

Therefore, for each game G, σ^i_* can be instantiated yielding the corresponding state, and therefore the payoffs and the choices of player i in the game. The state σ_G of the entire game just obtains as the profile of states of its players.

A simple example is $\sigma^i_{G^n}$ yielding as payoff for i the product of the payoffs she gets in the subgames of G^n. This case will be elaborated a bit more in Example 1, below.

[4]Thus, σ^i_* has a *forest* structure.

5 An Operadic View of Games

We can define the category of cospans in $\mathcal{G}$, denoted $\text{cospan}_{\mathcal{G}}$ which has a symmetric monoidal structure. Its objects are the same as those of $\mathcal{G}$ and a morphism $G \xrightarrow{h} G^{'}$ is a cospan from G to G', indicating that there exists a game of which G and G' are subgame forms. Thus, morphisms in $\text{cospan}_{\mathcal{G}}$ are actually isomorphisms.

Given two morphisms in $\text{cospan}_{\mathcal{G}}$, $G \xrightarrow{f} G'$ and $G' \xrightarrow{g} G''$ there exists a morphism $G \xrightarrow{g \circ f} G''$ that obtains as a composition of the corresponding cospans.

The monoidal structure of $\text{cospan}_{\mathcal{G}}$ is given by:

- The unit is $G^{\emptyset}$, the initial object in $\mathcal{G}$.
- The monoidal product of G and G', is the coproduct $G + G'$.

We now present a diagram language for open games. We start by considering the symmetric monoidal category $\mathbf{W}_{\mathcal{G}}$. By definition, we have that:

$$\mathbf{W}_{\mathcal{G}} \;=\; \text{cospan}_{\mathcal{G}}$$

Each object, i.e. a game G, is seen as a $\langle \text{in}^G, \text{out}^G, \Sigma_G \rangle$-labeled *interface*, satisfying ϕ^1_G and ϕ^2_G. On the other hand, morphisms $G \to C \leftarrow G'$, are called $\langle \text{in}, \text{out}, \Sigma \rangle$-labeled *wiring diagrams*. The interpretation is that C is the overarching game that connects the subgames (not just the game forms) G and G'.

We write $\psi : G_1, G_2, \ldots, G_n \to \bar{G}$ to denote the wiring diagram $\phi : G_1 + G_2 + \ldots + G_n \to \bar{G}$. We can, in turn see this as

$$G_1 + G_2 + \ldots + G_n \xrightarrow{f} C \xleftarrow{\bar{f}} \bar{G}$$

which indicates that, being f and $\bar{f}$ isomorphisms,

Proposition 2 *$\bar{G}$ is the minimal game that includes the direct sum of $G_1, \ldots, G_n$ as a subgame.*

6 Hypergraph Categories and Equilibria

We define a *hypergraph category* $\langle \mathcal{G}, \text{Eq} \rangle$ with $\text{Eq} : \mathbf{W}_{\mathcal{G}} \to \prod_i S_i$, such that, for every object G in $\mathbf{W}_{\mathcal{G}}$, $\text{Eq}(G)$ is a class of vectors in $\prod_{i \in I} S_i^G$, the strategy set of game G. We assume that $\text{Eq}(G)$ is a class of *equilibria* of G, for some notion of equilibrium (as for instance, dominant strategies equilibrium, admissible strategies, or Nash equilibrium).

Example 1 *Consider two games, G between players 1 and 2:*[5]

[5]This a *Battle of the Sexes* game, where $S_1 = S_2 = \{\text{Bx}, \text{Bll}\}$.

		Player 2	
		Bx	Bll
Player 1	Bx	1, 1	0, 0
	Bll	0, 0	1, 2

and G' between players 2 and 3:[6]

		Player 3	
		C	D
Player 2	C	2, 2	0, 3
	D	3, 0	1, 1

In red we have highlighted $Eq(G) = \{(Bx,Bx),(Bll, Bll)\}$ *and* $Eq(G') = \{(D,D)\}$*, where Eq corresponds to Nash equilibrium.*[7]

Let us represent now $G + G'$*. We start by building its corresponding game form. We obtain two tables, where the first one corresponds to player 3 choosing C:*

		Player 2			
		Bx/C	Bx/D	Bll/C	Bll/D
Player 1	Bx	$o_{1,1}$	$o_{1,2}$	$o_{1,3}$	$o_{1,4}$
	Bll	$o_{2,1}$	$o_{2,2}$	$o_{2,3}$	$o_{2,4}$

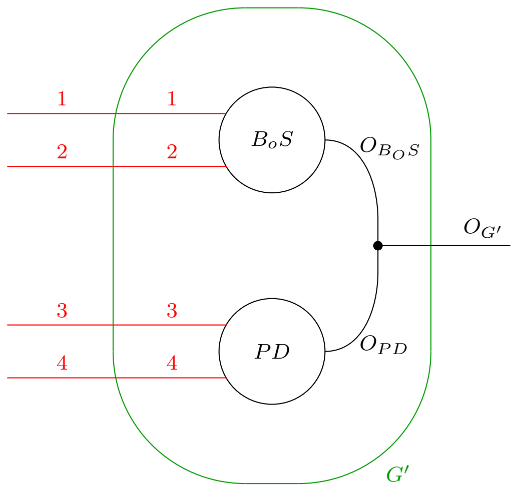
and another corresponding to player 3 choosing D:

		Player 2			
		Bx/C	Bx/D	Bll/C	Bll/D
Player 1	Bx	$o'_{1,1}$	$o'_{1,2}$	$o'_{1,3}$	$o'_{1,4}$
	Bll	$o'_{2,1}$	$o'_{2,2}$	$o'_{2,3}$	$o'_{2,4}$

Here, for instance, $\langle Bx, Bx/C, C\rangle$*, yielding the outcome* $o_{1,1}$*, indicates that 1 and 2 go to Box and 2 and 3 Cooperate. On the other hand,* $\langle Bx, Bx/C, D\rangle$*, with result* $o'_{1,1}$ *indicates that, again 1 and 2 go to Box, but while 2 keeps Cooperating, 3 Defects. The other entries can be interpreted likewise.*

[6] A *Prisoner's Dilemma*, where $S_2 = S_3 = \{C, D\}$.

[7] Notice that here player 2, participates in two games.



Suppose that the internal states of the players, σ_*^1, σ_*^2 and σ_*^3 are such that instantiated on $G + G'$ yield the following payoffs and choices:

If 3 chooses C:

		Player 2			
		Bx/C	Bx/D	Bll/C	Bll/D
Player 1	Bx	$2,1 \times 2,2$	$2,1 \times 3,0$	$0,0 \times 2,2$	$0,0 \times 3,0$
	Bll	$0,0 \times 2,2$	$0,0 \times 3,0$	$1,2 \times 2,2$	$1,2 \times 3,0$

while if 3 chooses D:

		Player 2			
		Bx/C	Bx/D	Bll/C	Bll/D
Player 1	Bx	$2,1 \times 0,3$	$2,1 \times 1,1$	$0,0 \times 0,3$	$0,0 \times 1,1$
	Bll	$0,0 \times 0,3$	$0,0 \times 1,1$	$1,2 \times 0,3$	$1,2 \times 1,1$

In words, players 1 and 3 keep the payoffs they get in the subgames, while 2 takes the product of the payoffs in G and G'. In red, we have highlighted the equilibria of $G + G'$, under this specification.

Let us define an operation $\hat{\cup}$ such that given two equilibria $s \in \text{Eq}(G)$ and $s' \in \text{Eq}(G')$, yields a new profile $s/s' \in \text{Eq}(G)\hat{\cup}\text{Eq}(G')$ verifying that for each player $i \in I_G \cap I_{G'}$, a new strategy obtains combining s_i and s'_i, while in on all other cases the individual strategies are the same as in G and G'. Furthermore,

$\pi_i^{G\hat{\cup}G'}(s/s') = \pi_i^G(s) \times \pi_i^{G'}(s^{'})$ for $i \in I_G \cap I_{G'}$.[8]

In our example, since $\mathrm{Eq}(G+G') = \{(\mathrm{Bx, Bx/D, D}), (\mathrm{Bll, Bll/D, D})\}$, we have that

$$\mathrm{Eq}(G)\hat{\cup}\mathrm{Eq}(G') = \mathrm{Eq}(G+G').$$

This example illustrates the following claim:

Proposition 3 *For any pair of games G and G', $Eq(G)\hat{\cup}Eq(G') = Eq(G+G')$.*

Proof: *Trivial. If $I_G \cap I_{G'} = \emptyset$, $G+G' = G \cup G'$ with $G \cap G' = \emptyset$. Thus, each equilibrium of $G+G'$ is just the disjoint combination of equilibria in G and G'.*

If, on the other hand, $I_G \cap I_{G'} \neq \emptyset$, given $i \in I_G \cap I_{G'}$, her strategy set in $G+G'$ is $S_i^G \times S_i^{G'}$, where S_i^G and $S_i^{G'}$ are her strategy sets in G and G', respectively. Now suppose that s_i^G and $s_i^{G'}$ are equilibrium strategies of i in the individual games but that $(s_i^G, s_i^{G'})$ does not belong to an equilibrium in $G+G'$. Then, there exist an alternative combined strategy $(\hat{s}_i^G, \hat{s}_i^{G'})$ such that on the new profile π_i yields a higher payoff, but since this equilibrium can be decomposed in two profiles, one in G and the other in G', the payoff of i is the product of the payoffs over those two profiles. But then either $\hat{s}_i^G$ yields a higher payoff than s_i^G or $\hat{s}_i^{G'}$ yields a higher payoff than $s_i^{G'}$ (recall that they are all positive real numbers). Thus, either s_i^G or $s_i^{G'}$ is not an equilibrium in the corresponding game. Absurd. $\square$

If we denote $+$ the monoidal operation in $\mathbf{W}_{\mathcal{G}}$, if we take $\otimes = \hat{\cup}$ as monoidal operation in $\prod_i S_i$, Proposition 3 indicates that there exist a trivial *natural isomorphism*

$$\mathrm{Eq}(G) \otimes \mathrm{Eq}(G^{'}) \ \rightarrow \ \mathrm{Eq}(G+G^{'})$$

Furthermore, taking the unit in $\prod_i S_i$ to be the empty set, we have also that $\emptyset = \mathrm{Eq}(G^{\emptyset})$, where $G^{\emptyset}$ is the initial object in $\mathcal{G}$ and thus in $\mathbf{W}_{\mathcal{G}}$.

We have that

Proposition 4 *Eq is a lax monoidal functor.*

Thus, the corresponding algebra allows to associate the composition of games with the equilibria of the components.

[8] An alternative yielding also Proposition 3 obtains if, instead, we take $\pi_i^{G\hat{\cup}G'}(s/s') = \pi_i^G(s) + \pi_i^{G'}(s^{'})$ for $i \in I_G \cap I_{G'}$.

7 An Alternative Definition of the Lax Functor

Proposition 4 depends critically on the possibility of defining $\otimes$ in terms of a function $\mathbf{f}$, defined as follows. Given a player $i \in I_G \cap I_{G'}$, a combined strategy s_i/s'_i is such that for $s = (s_i, s_{-i}) \in \mathrm{Eq}(G)$ and $s' = (s'_i, s'_{-i}) \in \mathrm{Eq}(G')$, satisfying $\pi_i(s/s') = \mathbf{f}(\pi_i^G(s), \pi_i^{G'}(s'))$ and with $s/s' \in \mathrm{Eq}(G+G')$. As we saw above if $\mathbf{f}$ is the arithmetic product or sum, Eq will be indeed a lax monoidal functor.

But this restricts the compositionality of games to just trivial cases. We are interested in more general and non-obvious cases. In order to do that consider an alternative characterization of the hypergraph category $\langle \mathcal{G}, \mathrm{Eq} \rangle$:

$$\mathrm{Eq} : \mathrm{W}_{\mathcal{G}} \rightarrow \prod_i S_i \times \cup_{G \in \mathtt{Obj}(\mathcal{G})} \Sigma_G$$

Furthermore, we need another definition of $\otimes$:

$$\otimes : (\prod_i S_i \times \cup_{G \in \mathtt{Obj}(\mathcal{G})} \Sigma_G) \times (\prod_i S_i \times \cup_{G \in \mathtt{Obj}(\mathcal{G})} \Sigma_G) \ \rightarrow \ \prod_i S_i \times \bigcup_{G \in \mathtt{Obj}(\mathcal{G})} \Sigma_G$$

such that given two games G and G' with $s \in \prod_{i \in I_G} S_i$ and σ_G, and $s' \in \prod_{i \in I_{G'}} S_i$ and $\sigma_{G'}$ we have:

$$(s, \sigma_G) \otimes (s', \sigma_{G'}) = (\bar{s}, \sigma_{G+G'}) \in \prod_{i \in I_{G+G'}} S_i \times \Sigma_{G+G'}$$

where $\bar{s} \in S_{G+G'}$ is a Nash equilibrium if and only if s and s' are Nash equilibria of G and G' respectively.

$\otimes$ is well-defined. To see this, just recall that, by definition $G + G'$ obtains in terms of the game forms of G and G' (the strategy sets and the outcomes), allowing different possible internal states and thus payoffs. The view of games as boxes presented in Section 4 indicates that there exist sequences of internal states of games, in parallel to sequences of morphisms between games, allowing to define $\sigma_{G+G'}$, and thus payoffs that make $\bar{s}$ a Nash equilibrium if s and s' are also equilibria.

We can see that $\prod_i S_i \times \bigcup_{G \in \mathtt{Obj}(\mathcal{G})} \Sigma_G$ with $\otimes$, defined as above can be interpreted as a monoidal category, with morphisms defined in terms of those of $\mathcal{G}$, with $(\emptyset, \emptyset)$ as its initial object. allows to define Eq in such a way that by definition:

Proposition 5 *Eq is a lax functor satisfying* $Eq(G + G') = Eq(G) \otimes Eq(G')$.

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