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MARRIAGE MARKET WITH INDIFFERENCES: A LINEAR PROGRAMMING APPROACH*

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Resumen

We study stable ans strongly stable matchings in the marriage market with indifference in their preferences. We characterize the stable matchings as integer extreme points of a convex polytope. We give an alternative proof for the integrity of the strongly stable matching polytope. Also, we compute men-optimal (women-optimal)stable and strongly stable matchings using linear programming. When preferences are strict we find the men-optimal (women-optimal) stable matching.
tenxtbfKeywords: Matching markets, The marriage market with indifferences, Optimal Stable matchings, Linear programming

1. Introduction

The marriage market describes a matching problem in which agents are divided into two disjoint subsets: the set of men and the set of women. The objective of this market is to assign a woman to a man, allowing the possibility for men and/or women to stay single. In this paper, we allow agents to be indifferent among agents on the other side of the market.

Many results for the matching market when preferences are strict cannot be extended when agents have preferences with indifferences.[1]

In matching markets, stability is a desirable property to be satisfied by any matching. Unlike the marriage market with strict preferences, in which there is a unique stability notion, when indifferences are allowed there are several notions. A matching is stable if each agent is matched to an acceptable partner, and there is no man-woman pair such that they are unmatched to each other and strictly prefer each other to their current partners.[2] Irving [10] formulates two other possible definitions of stability for the marriage market with indifferences. A matching is strongly stable if each agent is matched to an acceptable partner, and there is no man-woman pair such that they are unmatched to each other and one of them strictly prefers the other one to their current partner, the other weakly prefers the other one to their current partner. A matching is super stable if each agent is matched to an acceptable partner, and there is no

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[1]See Roth and Sotomayor [17] for a more detailed explanation.

[2]Irving [10] refers to stable matchings as weakly stable matchings.

man-woman pair such that they are unmatched to each other and weakly prefer each other to their current partners.

Gale and Shapley [8] show that at least one stable matching for the marriage market always exists, even when agents may have indifferences in their preferences. Usually, the procedure to compute a stable matching is breaking ties and then applying Gale and Shapley's Deferred Acceptance Algorithm. How these indifferences may be ordered has both strategic and welfare consequences. (See Erdil and Ergin [6] and Abdulkadiroğlu *et al.* [1]). On the other hand, strongly stable matchings and super stable matchings may not exist. Irving [10] presents an algorithm that computes a strongly stable matching when it exists. The same algorithm can also compute a super stable matching. [15] shows that the set of strongly stable matchings forms a distributive lattice. Ghosal *el al.* [9] present a polynomial-time algorithm the generation of all strongly stable matchings. They also prove that the set of strongly stable matching forms a distributive lattice (an alternative proof).

Many instances of matching problems are studied using a linear programming approach. Rothblum [19] introduces a list of linear inequalities which generate a convex polytope.[3] He characterizes the stable matchings of the marriage market with strict preferences as extreme points of this convex polytope. Roth *et al.* [18] present a linear program and use linear programming theory to give alternative proofs to already well-know results in the marriage market with strict preferences.

Kwanashie and Manlove [14] study the hospital resident market with indifferences. They present an integer linear program for calculating a maximum stable matching. That is, a stable matching with a maximum number of pairs assigned. Finding this stable matching is known to be NP-hard. (See Irving *et al.* [15]). Kwanashie and Manlove [14] introduce a list of linear inequalities that generate a convex polytope. They show that the integer extreme points are the stable matchings.

The case in which all hospitals have quota equal to one is called a marriage market. In this market when preferences are strict, the convex polytope of Kwanashie and Manlove [14] may have non-integer vertices. The constraints of Kwanashie and Manlove's linear program, do not coincide with the linear inequalities of our linear program. Even in the marriage market when preferences are strict, the linear inequalities of Kwanashie and Manlove [14] do not coincide with the linear inequalities of Rothblum [19]. Anyway, the integer solutions of both convex polytopes coincide with the stable matchings. In this paper, we generalize the linear inequalities of Rothblum [19].

Here, we present a linear inequality system that characterizes the stable matchings for the marriage market with indifferences. The convex polytope of stable matchings may have strictly fractional extreme points (see Example 1).

Kunysz [13] study strongly stable matchings in the marriage market with indifferences. He analyses this market as an undirected bipartite graph. For this market, he considering a weight function that does not depends on the agents' preferences. He finds a strongly stable matching that maximizes this weight function. To this end, he presents a linear inequality system that characterizes the strongly stable matchings as extreme points of a convex polytope. In this paper, we give an alternative proof for the integrality of this convex polytope. Unlike Kunsy's proof, which uses graph theory techniques, our proof uses matching techniques. The structure of our proof is inspired by the one presented in Roth *et al.* [18], (for the marriage market with strict preferences).

For the marriage market with indifferences, Spieker [20] proves that the set of super stable matchings is the intersection of the set of stable matchings each of which is for a possible tie-breaking. That is, a super stable matching is a stable matching in each marriage market with

[3]Vande Vate [21] characterizes stable matchings as extreme points of a linear inequality system in a market when all agents are mutually acceptable, and the set of men and women has the same number of agents.

strict preferences obtained by breaking ties in some strict order. To obtain a system of linear inequalities that characterizes super stable matchings, it is only necessary to list the linear inequalities for each marriage market with strict preferences (using the characterization result presented in Rothblum [19] for each market with strict preferences).

A stable matching is men-optimal at a preference profile if it is not dominated by another stable matching (Pareto dominated) according to men's opinions. In the marriage market with strict preferences, a unique men-optimal stable matching always exists. However, when indifferences in preferences are allowed, the optimal stable matching may not be unique. That is, some times there are more than one stable matchings that are not Pareto dominated by another stable matching.

A social planner may need to compute an optimal stable matching for one side of the market, for instance, a men-optimal stable matching. One can be tempted to break ties and use the Deferred Acceptance Algorithm (Gale and Shapley [8]) to compute the men-optimal stable matching at the strict preference profile associated. Despite this, the men-optimal stable matching at this strict preference profile may not be a men-optimal stable matching at the original preference profile (with indifferences) (See Example 2). Erdil and Ergin [7] establish an algorithm that computes optimal stable matchings in the college admission problem with indifferences. To this end, they break ties and apply Pareto improvement cycles and Pareto improvement chains.

In this paper, we present an integer linear program that computes one of the men-optimal (women-optimal) stable matchings in the marriage market with indifferences without using any tie-breaking. For the same market, using the linear inequality system presented in Kunysz [13], we present a linear program (not integer) that computes a men-optimal (women-optimal) strongly stable matching. In both cases, we define a new objective function in the linear program that is correlated with men's (women's) preferences. To each pair of agents, we associate a weight that depends on the preferences of the agents.

Other authors study stable matchings using linear programming in matching markets with strict preferences. Baiou and Balinski [2] compute the optimal stable matching in a many-to-one matching market with strict preferences. Given a pair of agents (a, u), they define a weight w_{au} to be a cost or profit associated with the assignment of agent a to agent u. To solve this problem, once they find the set of stable matchings, they compute the stable matching μ such that maximizes $\sum_{(i,j)\in\mu} w_{ij}$. Despite this, for computing the optimal stable matching, they need to compute all stable matchings ex-ante. The main difference with our approach is that we compute directly the optimal stable matching (without computing all stable matchings).

On the other hand, Kiraly and Pap [11] introduce weights that do not depend on the preferences of agents and study the stable marriage polytope with strict preferences. Chen, Ding, Hu, and Zang [5] study the problem of finding the maximum-weight stable matching in a more general strict market, which is known to be NP-hard. They use linear programming, polyhedral approaches and graph theory to study this problem. They present a polynomial-time algorithm for the maximum-weight stable matching problem under certain conditions.

The paper is organized as follows. In Section 2, we introduce the market, preliminary notations and definitions. In Section 3, we characterize the set of stable matchings as integer extreme points of a convex polytope. For the strongly stable matchings polytope, we present an alternative proof that this polytope is integral. In Section 4, we present two linear programs that compute optimal stable matchings and optimal strongly stable matchings. Finally, an Appendix with two examples.

2. The Marriage Market

In the marriage market with indifferences, there are two finite sets of agents, $M = \{m_1, ..., m_n\}$ of men and the set $W = \{w_1, ..., w_p\}$ of women. Each agent $i \in M \cup W$ has a complete and transitive preference order for the agents on the other side of the market and the prospect of being alone. A preference profile $R = (R_i)_{i \in M \cup W}$ is a vector of weak orders. We denote by P_i and I_i the antisymmetric and symmetric parts of the binary relation R_i, respectively. Then, P_i is an antisymmetric, transitive and irreflexive (strict preference relation), and I_i is reflexive, symmetric and transitive (indifference preference relation).

For instance, the preferences R_m for the man m, where $w_1 P_m w_2$, $w_2 P_m w_3$, $w_2 P_m w_4$, $w_3 I_m w_4$ (and by transitivity $w_1 P_m w_3$ and $w_1 P_m w_4$), will be denoted by

$$R_m : w_1, w_2, [w_3, w_4] .$$

A preference profile R satisfies *no indifference to the single set* if any agent is not indifferent between remaining single or being assigned to another agent of the other side of the market.[4] For instance, the following preference does not satisfy *no indifference to the single set.*

$$R_m : w_1, w_2, [w_3, m] .$$

We denote the marriage market with indifferences by (M, W, R). We say that $(m, w) \in M \times W$ is an **acceptable pair** if $m P_w w$ and $w P_m m$. Let $\boldsymbol{A}$ be the set of all acceptable pairs.

Definition 1 *A* ***matching*** μ *is a injective function* $\mu : M \cup W \to M \cup W$ *such that:*

1. $\mu(m) \neq m$ *implies* $\mu(m) \in W$.
2. $\mu(w) \neq w$ *implies* $\mu(w) \in M$.
3. $\mu(m) = w$ *if and only if* $\mu(w) = m$.[5]

Let $\mathcal{M}$ denote the set of all matchings. If $\mu(m) = w$, then man m and woman w are said to be matched to each other. If $\mu(i) = i$, then agent i is said to be single or unmatched. Given a preference R_i of agent i, we extend these binary relations to the set of matchings in a natural way. Let μ and μ' be two matchings, $\mu R_i \mu'$ if and only if $\mu(i) R_i \mu'(i)$. Moreover, given a preference profile R and a subset of agents $X \subseteq M \cup W$, then $\mu R_X \mu'$ if and only if $\mu(i) R_i \mu'(i)$ for all $i \in X$. In a similar way, we extend the relations I_X and P_X.

A matching μ is **individually rational** if it is not blocked by any individual agent, i.e., for all $i \in M \cup W$ we have that $\mu(i) R_i i$. Given a preference profile R, we denote the set of individually rational matchings by $IR(R)$.

Definition 2 *Let* μ *be a matching and let* $m \in M$ *and* $w \in W$.

- *A pair* (m, w) *is said to form a* ***blocking pair*** *if* $m P_w \mu(w)$ *and* $w P_m \mu(m)$.
- *A pair* (m, w) *is said to form a* ***strongly blocking pair*** *if either* $m P_w \mu(w)$ *and* $w R_m \mu(m)$, *or* $m R_w \mu(w)$ *and* $w P_m \mu(m)$.

Definition 3 *Let* μ *be a matching for a marriage market* (M, W, R).

- μ *is* ***stable*** *if it is individually rational and if there is no blocking pair.*

[4]This assumption is commonly used in the literature; see Erdil and Ergin [6] and [7] and Biró and McBride [4].

[5]Item 3 is equivalent to say that μ is a homogeneous function of order two, i.e., $\mu^2(i) = i$, for all $i \in M \cup W$.

- μ *is* ***strongly stable*** *if it is individually rational and there is not a strongly blocking pair.*

Given a preference profile R, we denote the set of stable matchings by $S(R)$. Also, we denote the set of strongly stable matchings by $SS(R)$.

Notice that, from Definition 2 and 3 it follows that $SS(R) \subseteq S(R)$.

Given a matching μ, we can define the **incidence vector** $x^{\mu} \in \{0,1\}^{|M|\times|W|}$, as follows: the entry $x^{\mu}_{m,w} = 1$ if and only if $\mu(m) = w$ and the entry $x^{\mu}_{m,w} = 0$ otherwise. We identify each matching with its incidence vector.

Let $C_{IR(R)}$ be the convex polytope generated by the following inequalities:

$$\sum_{j \in W} x_{m,j} \leq 1 \quad \text{for all } m \in M \tag{1}$$

$$\sum_{i \in M} x_{i,w} \leq 1 \quad \text{for all } w \in W \tag{2}$$

$$x_{m,w} \geq 0 \quad \text{for all } (m,w) \in A \tag{3}$$

$$x_{m,w} = 0 \quad \text{for all } (m,w) \in (M \times W) \setminus A \tag{4}$$

Notice that inequalities (4) are called *individual rationality linear inequalities*. The extreme points of $C_{IR(R)}$ are exactly the individually rational matchings. This follows from the Birkhoff-von Neumann Theorem [3].

Rothblum [19] defines the convex polytope $C_{S(P)}$ generated by adding to the linear inequalities of $C_{IR(R)}$, the following linear inequalities:

$$\sum_{jP_m w} x_{m,j} + \sum_{iP_w m} x_{i,w} + x_{m,w} \geq 1 \quad \text{for all } (m,w) \in \ A \tag{5}$$

Linear inequalities (5) assure that there is no blocking pair. These linear inequalities are called *stability linear inequalities*.

For the marriage market with strict preferences, Rothblum [19] characterizes the stable matchings as integer solutions of the linear inequality system (1)–(5).

3. A Polyhedral Approach

3.1. Stable Marriage Polytope

For the marriage market with indifferences (M, W, R), we introduce modifications to the convex polytope $C_{S(P)}$. These modifications will characterize the stable matchings as the integer extreme points of a new convex polytope.

For $m \in M$ and $w \in W$, we define

$$R_m(w) = \{w' : w' R_m w \text{ and } w' \neq w\}$$

and

$$R_w(m) = \{m' : m' R_w m \text{ and } m' \neq m\}.$$

Let $C_{S(R)}$ be the convex polytope generated by (1)–(4) and

$$\sum_{j \in R_m(w)} x_{m,j} + \sum_{i \in R_w(m)} x_{i,w} + x_{m,w} \geq 1 \quad \text{for all } (m,w) \in A \tag{6}$$

Notice that, the only linear inequalities modified from the marriage market with strict preferences are the ones that represent the stability restrictions. When preferences are strict, linear inequalities (5) and (6) are equivalent.

We define a **stable fractional matching** to be a (not necessarily integer) solution of (1)–(4) and (6).

The following theorem characterizes stable matchings as integer extreme points of the convex polytope $C_{S(R)}$.

Theorem 1 *Let (M, W, R) be a marriage market with indifferences. A matching μ is stable, if and only if its incidence vector is an integer point of $C_{S(R)}$.*

Proof.

$\Longrightarrow$) Assume that $\mu \in S(R)$. Let x^μ be its incidence vector. Is easy to check that x^μ satisfies linear inequalities (1)–(4). Assume that x^μ does not satisfy (6); that is, there is a pair $(m, w) \in A$ such that,

$$\sum_{j \in R_m(w)} x^\mu_{m,j} + \sum_{i \in R_w(m)} x^\mu_{i,w} + x^\mu_{m,w} < 1.$$

Notice that each entry of x^μ is either zero or one. Then,

$$\sum_{j \in R_m(w)} x^\mu_{m,j} = 0, \sum_{i \in R_w(m)} x^\mu_{i,w} = 0 \text{ and } x^\mu_{m,w} = 0. \tag{7}$$

Since $x^\mu_{m,w} = 0$, we have that m and w are not matched.

Now we consider the following cases:

(I) $\sum\limits_{j \in W} x^\mu_{m,j} = \sum\limits_{i \in M} x^\mu_{i,w} = 0.$

Since $(m, w) \in A$, $wP_m m = \mu(m)$ and $mP_w w = \mu(w)$. Then (m, w) is a blocking pair of μ. This is a contradiction to the assumption that μ is stable.

(II) $\sum\limits_{j \in W} x^\mu_{m,j} = 0$ and there is $m' \in M$, $m' \neq m$ such that $x^\mu_{m',w} = 1$.

Observe that $\sum_{i \in R_w(m)} x^\mu_{i,w} = 0$. Then $mP_w m' = \mu(w)$. Since $x_{m,j} = 0$ for each $j \in W$ and $(m, w) \in A$, then $wP_m m = \mu(m)$. That is, (m, w) is a blocking pair of μ. This is a contradiction to the assumption that μ is stable.

(III) $\sum\limits_{i \in M} x^\mu_{i,w} = 0$ and there is $w' \in W$, $w' \neq w$ such that $x_{m,w'} = 1$.

This case is similar to case (ii), and we omit the proof.

(IV) There is $m' \in M$, $m' \neq m$ and $w' \in W$, $w' \neq w$ such that $x^\mu_{m',w} = 1$ and $x^\mu_{m,w'} = 1$.

Observe that $\sum_{i \in R_w(m)} x^\mu_{i,w} = 0$ and $\sum_{j \in R_m(w)} x^\mu_{m,j} = 0$, then $m' \notin R_m(w)$ and $w' \notin R_w(m)$. Since $x_{m,w} = 0$, then $mP_w m' = \mu(w)$ and $wP_m w' = \mu(m)$.

Thus, the pair (m, w) is a blocking pair of μ. This is a contradiction to the assumption that μ is stable.

$\Longleftarrow$) Let x be an integer point of $C_{S(R)}$. Since x satisfies (1)–(4), from Birkhoff-von Neumann Theorem we have that x is the incidence vector of an individually rational matching, i.e., there is an individually rational matching μ such that $x = x^\mu$. We will prove that $\mu \in S(R)$.

Assume that there is a pair $(m, w) \in A$ that blocks μ, i.e., $wP_m\mu(m)$ and $mP_w\mu(m)$. This implies that

$$x^\mu_{m,w} = 0 \tag{8}$$

and

$$\sum_{i \notin R_w(m)} x_{i,w} = x^{\mu}_{\mu(m),w} = 1 \text{ and } \sum_{j \notin R_m(w)} x_{m,j} = x^{\mu}_{m,\mu(w)} = 1.$$

Since each entry of x^μ is either zero or one, it holds that:

$$\sum_{i \in R_w(m)} x^{\mu}_{i,w} = 0 \text{ and } \sum_{j \in R_m(w)} x^{\mu}_{m,j} = 0. \tag{9}$$

Then, (8) and (9) imply that linear inequalities (6) fails for the pair $(m, w) \in A$. This is a contradiction to the assumption that x^μ is an integer point of $C_{S(R)}$.

□

The following example shows that the convex polytope $C_{S(R)}$ may have non-integer extreme points.

Example 1 *Let (M, W, R) be a marriage market with indifferences. Let $M = \{m_1, m_2, m_3\}$, $W = \{w_1, w_2, w_3\}$ and the preference profile R be such that*

$$\begin{array}{ll} R_{m_1} : [w_2, w_1], w_3. & R_{w_1} : m_1, m_2, m_3. \\ R_{m_2} : [w_2, w_1], w_3. & R_{w_2} : m_3, m_1, m_2. \\ R_{m_3} : w_1, [w_2, w_3]. & R_{w_3} : m_1, m_2, m_3. \end{array}$$

There are only three stable matchings:

$$x^{\mu_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad x^{\mu_2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad x^{\mu_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

But the stable fractional matching

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

is also a vertex of the convex polytope $C_{S(R)}$.

3.2. Strongly Stable Marriage Polytope

For marriage market with indifferences (M, W, R), Kunysz [13] presents a linear inequality system to characterize the strongly stable matching as the extreme points of the convex polytope generated by these linear inequalities. Each extreme point of this convex polytope is an integer point, and the extreme points coincide with the strongly stable matchings. In this section, we present an alternative proof of Theorem 13 in Kunysz [13].

Let $C_{SS(R)}$ be the convex polytope generated by (1)–(4) and

$$\sum_{jP_m w} x_{m,j} + \sum_{iP_w m} x_{i,w} + \sum_{iI_w m} x_{i,w} \geq 1 \quad \text{for each } (m, w) \in A \tag{10}$$

$$\sum_{jP_m w} x_{m,j} + \sum_{iP_w m} x_{i,w} + \sum_{jI_m w} x_{m,j} \geq 1 \quad \text{for each } (m, w) \in A \tag{11}$$

Notice that, the only linear inequalities modified from the marriage market with strict preferences are the ones that represent the stability restrictions. When preferences are strict, linear inequalities (10) and (11) are equivalent to (5).

We define a **strongly stable fractional matching** to be a (not necessarily integer) solution of (1)–(4), (10) and (11).

The following lemma is taken from Kunysz [13] (Lemma 12).

Lemma 1 ***Kunysz [13]*** *Let x be a strongly stable fractional matching. Then, for each $(m, w) \in M \times W$ the following hold:*

$x_{m,w} > 0 \Rightarrow \sum_{jP_mw} x_{m,j} + \sum_{iP_wm} x_{i,w} + \sum_{jI_mw} x_{m,j} = 1$
$x_{m,w} > 0 \Rightarrow \sum_{jP_mw} x_{m,j} + \sum_{iP_wm} x_{i,w} + \sum_{iI_wm} x_{i,w} = 1$
$x_{m,w} > 0 \Rightarrow \sum_{j\in W} x_{m,j} = 1$
$x_{m,w} > 0 \Rightarrow \sum_{j\in W} x_{m,j} = 1$

Remark 1 *Note that for each feasible solution x if $x_{m,w} > 0$ then $\sum_{jI_mw} x_{m,j} = \sum_{iI_wm} x_{i,w}$. This implies that, if there is $w' \in W$ such that, $x_{m,w'} > 0$ and wI_mw', then there is at least $m' \neq m$ with $x_{m',w} > 0$ and there is at least $\bar{m} \neq m$ with $x_{\bar{m},w'} > 0$ (where m' and $\bar{m}$ may or may not be the same man). This means that if $x_{m,w} > 0$ then, $|\{j \in W : jI_mw, x_{m,j} > 0\}| \geq 2$ if and only if $|\{i \in M : iI_wm, x_{i,w} > 0\}| \geq 2$.*

Given a strongly stable fractional matching x, we define for each $m \in M$ the optimal class of indifference within those women that fulfill $x_{m,w} > 0$. Formally,

$$[\mu_x](m) = \{w \in W : x_{m,w} > 0 \text{ and } wR_mj \text{ for each } j \in W \text{ with } x_{m,j} > 0\}.$$

Analogously, we define for each $w \in W$ the pessimal class of indifference within those women that fulfill $x_{m,w} > 0$. Formally,

$$[\mu_x](w) = \{m \in M : x_{m,w} > 0 \text{ and } iR_wm \text{ for each } i \in M \text{ with } x_{i,w} > 0\}.$$

Lemma 2 *Let x be a strongly stable fractional matching. Then,*

(I) $w \in [\mu_x](m)$ if and only if $m \in [\mu_x](w)$.

(II) If $w \in [\mu_x](m)$ and $|[\mu_x](m)| \geq 2$, then $|[\mu_x](w)| \geq 2$.

(III) If $m \in [\mu_x](w)$ and $|[\mu_x](w)| \geq 2$, then $|[\mu_x](m)| \geq 2$.

Proof. Let x be a strongly stable fractional matching.

(I) Let $w \in [\mu_x](m)$. Then, $\sum_{jP_mw} x_{m,j} = 0$. Since $x_{m,w} > 0$, by Lemma 1 we have that,

$$\sum_{iP_wm} x_{i,w} + \sum_{iI_wm} x_{i,w} = 1.$$

This means that $\sum_{mP_wi} x_{i,w} = 0$. Since $x_{m,w} > 0$ we have that $m \in [\mu_x](w)$.

Let $m \in [\mu_x](w)$. This means that

$$\sum_{iP_wm} x_{i,w} + \sum_{iI_wm} x_{i,w} = 1.$$

Since $x_{m,w} > 0$, by Lemma 12 we have that $\sum_{jP_mw} x_{m,j} = 0$. Since $x_{m,w} > 0$ we have that $w \in [\mu_x](m)$.

The proof of items (ii) and (iii) is straightforward using Remark 1.

□

Given a strongly stable fractional matching, and a set of men $\tilde{M} \subseteq M$. We have that $\tilde{M}$ is partitioned in two subsets as follows:

$$\tilde{M}_1 = \{m \in \tilde{M} : |[\mu_x](m)| = 1\} \text{ and } \tilde{M}_2 = \{m \in \tilde{M} : |[\mu_x](m)| \geq 2\},$$

that is,

$$\tilde{M} = \tilde{M}_1 \cup \tilde{M}_2 \tag{12}$$

If $\tilde{M}_2 \neq \emptyset$ we define a cycle of agents as follows. Let $m_1 \in \tilde{M}_2$ and $C_{m_1} = \{m_1, w_1, \ldots, w_{k-1}, m_k, w_k\}$ with $m_i \neq m_j$, $w_i \neq w_j$ and

(I) $w_i \in [\mu_x](m_i)$ with $i = 1, \ldots, k$.

(II) $m_i \in [\mu_x](w_{i-1})$ with $i = 2, \ldots, k$.

(III) $m_1 \in [\mu_x](w_k)$.

Define $\tilde{W}_2 = \{w \in W : \text{ there is } m \in \tilde{M}_2, w \in [\mu_x](m)\}$.

Given $m' \in \tilde{M}_2$, we denote by $\tilde{M}_2(C_{m'}) = \{m \in \tilde{M}_2 : m \in C_{m'}\}$ and $\tilde{W}_2(C_{m'}) = \{w \in \tilde{W}_2 : w \in C_{m'}\}$.

The following lemma is used to prove that given a strongly stable fractional matching, there is always a strongly stable matching that assign to each man a woman in the optimal class, and to each woman a man in the pessimal class.

Lemma 3 *If $\tilde{M}_2 \neq \emptyset$, then there are $m' \in \tilde{M}_2$ and a cycle of agents C generated by m' such that $\tilde{M}_2(C) \subseteq \tilde{M}_2$ and $\tilde{W}_2(C) \subseteq \tilde{W}_2$.*

Proof. Let $m_1 \in \tilde{M}_2$, then there is $w_1 \in [\mu_x](m_1)$. Then, by Lemma 2 (i) and (ii) we have that $m_1 \in [\mu_x](w_1)$ and $|[\mu_x](w_1)| \geq 2$. Then, there is $m_2 \neq m_1$ such that $m_2 \in [\mu_x](w_1)$. Lemma 2 (i) implies that $w_1 \in [\mu_x](m_2)$. By Lemma 2 (iii) we have that $|[\mu_x](m_2)| \geq 2$ and there is $w_2 \neq w_1$ such that $w_2 \in [\mu_x](m_2)$. Lemma 2 (ii) implies that $|[\mu_x](w_2)| \geq 2$ and there is $m_3 \neq m_2$ such that $m_3 \in [\mu_x](w_2)$. If $m_3 = m_1$ we are done and $C = \{m_1, w_1, m_2, w_2\}$. If not, by Lemma 2 (i) and (iii) there is $w_3 \neq w_2$ such that $w_3 \in [\mu_x](m_3)$. If $w_3 = w_1$ we are done and $C = \{m_2, w_2, m_3, w_1\}$. If not, we continue this procedure until we have that the cycle is closed by the finiteness of $\tilde{M}_2$. By construction we have that $\tilde{M}_2(C) \subseteq \tilde{M}_2$ and $\tilde{W}_2(C) \subseteq \tilde{W}_2$.

□

Procedure to construct cycles in the men optimal indifference class:

Given M and W be the set of men and women respectively.

Step 1: We have that $M = M^1 = M_1^1 \cup M_2^1$ by decomposition (12). Denote by $W_2^1 = \{w \in W : \text{ there is } m \in M_2^1, w \in [\mu_x](m)\}$ and $W_1^1 = W \setminus W_2^1$.

If $M_2^1 = \emptyset$ the procedure stops.

If $M_2^1 \neq \emptyset$, Lemma 3 implies that there are $m' \in M_2^1$ and a cycle of agents C^1 generated by m' such that $M_2^1(C^1) \subseteq M_2^1$ and $W_2^1(C^1) \subseteq W_2^1$.

Let $M^2 = M_2^1 \setminus M_2^1(C^1)$.

Step $\boldsymbol{k > 1}$: We have that $M^k = M_1^k \cup M_2^k$ by decomposition (12). Denote by $W_2^k = \{w \in W : \text{ there is } m \in M_2^k, w \in [\mu_x](m)\}$ and $W_1^k = W \setminus W_2^k$.

If $M_2^k = \emptyset$ the procedure stops.

If $M_2^k \neq \emptyset$, by Lemma 3 we have that there are $m' \in M_2^k$ and a cycle of agents C^k generated by m' such that $M_2^k(C^k) \subseteq M_2^k$ and $W_2^k(C) \subseteq W_2^k$.

Let $M^{k+1} = M_2^k \setminus M_2^k(C^1)$.

Remark 2 *By the finiteness of the set of men M, there is a step $\tilde{k}$ such that the procedure stops.*

Given a strongly stable fractional matching x, $[\mu_x](m)$ for each $m \in M$ and the procedure to construct cycles in the optimal class for men, we define μ_x as follows

$$\mu_x(m) = \begin{cases} w & \text{if } m \in M_2(C^k),\ w \in [\mu_x](m) \text{ and } w \in W_2(C^k) \text{ for each } k = 1, \ldots, \tilde{k} \\ w & \text{if } m \in M_1^k,\ w \in [\mu_x](m) \text{ and } w \in W^k \text{ for each } k = 1, \ldots, \tilde{k} \\ m & \text{otherwise} \end{cases}$$

In the following lemma we prove that the assignment μ_x is a strongly stable matching.

Lemma 4 *Let x be a fractional solution of LP. Then, μ_x defined before is a strongly stable matching such that $\mu_x(m) \in [\mu_x](m)$ for each $m \in M$.*

Proof. Notice that by construction and Lemma 1, it is straightforward that x^{μ_x} fulfills inequalities (1), (2) and (3), then μ_x is a matching. Also, by construction, we have that $\mu_x(m) \in [\mu_x](m)$ for each $m \in M$. Now we prove that x^{μ_x} fulfills inequalities (4) and (5). Assume that (m, w) is a strongly blocking pair for μ_x. Then, either $mP_w\mu_x(w)$ and $wR_m\mu_x(m)$, or $mR_w\mu_x(w)$ and $wP_m\mu_x(m)$.

Case 1: Assume that $mR_w\mu_x(w)$ and $wP_m\mu_x(m)$. By construction of μ_x we have that

$$\sum_{jP_mw} x_{m,j} + \sum_{jI_mw} x_{m,j} = 0.$$

Then, since x satisfies inequalities (1) and (4) we have that

$$\sum_{iP_wm} x_{i,w} = 1.$$

Hence, we have that $\mu_x(w)P_wm$. Therefore, the pair (m, w) can not be a strongly blocking pair.

Case 2: Assume that $mP_w\mu_x(w)$ and $wR_m\mu_x(m)$. By construction of μ_x we have that

$$\sum_{jP_mw} x_{m,j} = 0.$$

Then, since x satisfies inequalities (1) and (5) we have that

$$\sum_{iP_wm} x_{i,w} + \sum_{iI_wm} x_{i,w} = 1.$$

Hence, we have that $\mu_x(w)R_wm$. Therefore, the pair (m, w) can not be a strongly blocking pair.

Hence, μ_x is a strongly stable matching. □

The following theorem states that the extreme points of the convex polytope coincide with the strongly stable matchings.

Theorem 2 *The extreme points of $C_{SS(R)}$ are exactly the strongly stable matchings.*

The proof follows similarly to the proof of Theorem 13 in Roth *et al.* [18]. *Proof.* It is straightforward that every integer solution of $C_{SS(R)}$ is an extreme point of $C_{IR(R)}$.[6] Then, strongly stable matchings are an extreme point of $C_{SS(R)}$. We only need to prove that each extreme point of LP is a strongly stable matching. Let x be a fractional solution of LP that is not a matching. Let x^{μ_x} be the incidence vector of μ_x. Since x is not a matching, then $x \neq x^{\mu_x}$. We show that x is not an extreme point of LP. Then consider for $0 < \alpha < 1$, the following vector

$$y^{\alpha} = \frac{x - \alpha x^{\mu_x}}{1 - \alpha}.$$

Since, $x = \alpha x^{\mu_x} + (1 - \alpha)y^{\alpha}$ for each $0 < \alpha < 1$, we only need to prove that y^{α} is a solution of $C_{SS(R)}$. That is,

[6] The extreme points of $C_{IR(R)}$ are exactly the individually rational matchings.

Inequality (1): If $m \in M$ has $x_{m,j} = 0$ for each $j \in W$, then $x^{\mu_x}_{m,j} = 0$ for each $j \in W$ and $y^{\alpha}_{m,j} = 0$ for each $j \in W$, assuring that y^{α} satisfies (1). If $m \in M$ has $\sum_{j \in W} x_{m,j} > 0$, then $\mu_x(m) \neq m$. Hence, $\sum_{j \in W} x^{\mu_x}_{m,j} = 1$. As x satisfies (1), then it follows that y^{α} satisfies inequality (1) for all $0 < \alpha < 1$.
Inequality (2): Analogous to inequality (1).
Inequality (3): Observe that $x^{\mu_x}_{m,w} = 0$ whenever $x_{m,w} = 0$. Hence, for a sufficient small positive α we have that y^{α} satisfies inequality 3.
Inequality (4): If $(m,w) \in (M \times W) \setminus A$, then we have that $x_{m,w} = 0$. Hence, we have that $y^{\alpha}_{m,w} = x^{\mu_x}_{m,w} = 0$ for each $0 < \alpha < 1$. Therefore, y^{α} satisfies inequality (4).
Inequality (10): Now we show that for a small positive α, y^{α} satisfies inequality (10). As

$$\sum_{jP_mw} y^{\alpha}_{m,j} + \sum_{iP_wm} y^{\alpha}_{i,w} + \sum_{iI_wm} y^{\alpha}_{i,w}$$

$$= \frac{1}{1-\alpha}\left[\left(\sum_{jP_mw} x_{m,j} + \sum_{iP_wm} x_{i,w} + \sum_{iI_wm} x_{i,w}\right) - \alpha\left(\sum_{jP_mw} x^{\mu_x}_{m,j} + \sum_{iP_wm} x^{\mu_x}_{i,w} + \sum_{iI_wm} x^{\mu_x}_{i,w}\right)\right]$$

for each pair (m,w) and as x satisfies inequality (10), it suffices to prove that whenever x satisfies inequality (10) as an equality so does x^{μ_x}. Then, assume that for (m,w), x satisfies inequality (10) as an equality. As μ_x is a strongly stable matching, then x^{μ_x} satisfies inequality (10). Further, x^{μ_x} satisfies inequality (10) strictly for (m,w) if and only if one of the following two cases may happen: either $\mu_x(m)P_mw$ and $\mu_x(w)P_wm$, or $\mu_x(m)P_mw$ and $\mu_x(w)I_wm$.

If $\mu_x(m)P_mw$ and $\mu_x(w)P_wm$, by construction of μ_x we have that

$$\sum_{jP_mw} x_{m,j} > 0 \text{ and } \sum_{iP_wm} x_{i,w} = 1.$$

If $\mu_x(m)P_mw$ and $\mu_x(w)I_wm$, by construction of μ_x we have that

$$\sum_{jP_mw} x_{m,j} > 0 \text{ and } \sum_{iP_wm} x_{i,w} + \sum_{iI_wm} x_{i,w} = 1.$$

Both cases, contradict the assertion that x satisfies inequality (10) for the pair (m,w) as an equality. This contradiction proves that whenever x satisfies inequality (10) as an equality, so does x^{μ_x}. Therefore, we have that for a small positive α, y^{α} satisfies inequality (10).
Inequality (11): Analogous to inequality (10).

□

4. Men-Optimal Matchings

Gale and Shapley [8] showed the existence of optimal stable matchings for the marriage market with strict preferences. The Deferred Acceptance algorithm is a mechanism that computes the men-optimal stable matching (μ_M) when men offer, and it computes the women-optimal stable matching (μ_W) when women offer. The stable matching μ_M is men-optimal in the sense that there is no other stable matching $\mu \neq \mu_M$ such that assigns to each man m a partner that m prefers to the agent assigned by μ_M. This is also the case for the women-optimal stable matching μ_W.

If we consider the marriage market with indifferences, we say that μ' is a **men-optimal stable matching** if there is no other stable matching μ, that assigns to each man m a partner *weakly preferred* to what μ' assigns to him, and there is at least one man that strictly prefers μ' to any other μ. That is, there is no $\mu \in S(R)$ such that $\mu P_M \mu'$.

Given a preference profile with indifferences R, we can define P to be a strict preference profile obtained from R by some tie-breaking. That is, P is a preference profile in which each agent replaces indifferences by some strict order.

Notice that there are many ways of breaking the indifferences. Then, we define $L(R)$, as the set of all linear orders that can be obtained from R by a tie-breaking. It is well known that $S(R)$ can be computed by finding all stable matchings for any associated matching market (M, W, P), where P is obtained from R by some tie-breaking. That is

$$S(R) = \cup_{P \in L(R)} S(P).$$

See Roth and Sotomayor [17] for details.

The following example shows that the men-optimal stable matching may not be unique. It also shows that when we choose different ways to break indifferences and apply the deferred acceptance algorithm with men offering, we may obtain a stable matching that is not men-optimal on the original preference profile R.

Example 2 *Let* $M = \{m_1, m_2, m_3, m_4\}$, $W = \{w_1, w_2, w_3, w_4\}$ *and the preference profile* R *be such that*

$$\begin{array}{ll} R_{m_1} : w_1, [w_2, w_3]. & R_{w_1} : m_3, [m_1, m_2]. \\ R_{m_2} : w_1, w_4. & R_{w_2} : [m_1, m_4]. \\ R_{m_3} : w_4, w_1. & R_{w_3} : [m_1, m_4]. \\ R_{m_4} : [w_2, w_3]. & R_{w_4} : m_2, m_3. \end{array}$$

$S(R) = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ *is given by:*

$$x^{\mu_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad x^{\mu_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix};$$

$$x^{\mu_3} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \quad x^{\mu_4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Notice that the men-optimal stable matching is not unique. The matchings μ_1 *and* μ_2 *are men-optimal stable matchings.*

Let us consider a tie-breaking such that the strict profile P *associated is as follows:*

$$\begin{array}{ll} P_{m_1} : w_1, w_2, w_3. & P_{w_1} : m_3, m_1, m_2. \\ P_{m_2} : w_1, w_4. & P_{w_2} : m_1, m_4. \\ P_{m_3} : w_4, w_1. & P_{w_3} : m_1, m_4. \\ P_{m_4} : w_2, w_3. & P_{w_4} : m_2, m_3. \end{array}$$

Then, the set of stable matchings for the preference profile P *is:* $S(P) = \{\mu_3, \mu_4\}$ *and the men-optimal stable matching for the preference profile* P *is* μ_3*, but this matching is not a men-optimal stable one for the original market* (M, W, R)*.*

Next, to find an optimal stable matching, we use linear programming tools. Roth *et al.* [18] present a linear program in which, the constraints are the inequalities (1)–(5). They introduce an objective function such that the optimal solutions of the linear program (LP) are precisely the extreme points of the convex polytope $C_{S(P)}$. That is, the stable matchings are the integer solutions of the following linear program:

$$\begin{array}{lll} LP & \text{máx} & \sum_{(i,j)\in A} x_{i,j} \\ & st: & x \in C_{S(P)}. \end{array}$$

Observe that using McVitie and Wilson's Theorem [16], the number of couples matched in each stable matching is the same. Therefore, the incidence vectors have the same amount of entries equal to one. Hence, all stable matchings generate the same value of the objective function. So, this linear program does not distinguish among any stable matching.

To compute a men-optimal stable matching for the marriage market with indifferences (M, W, R), we present a new linear program with weights in the objective function, which depends on men's preferences.[7]

Given a preference profile R, for each pair $(m, w) \in M \times W$, we define a **weight** $\alpha_{m,w} \in \mathbb{R}$ that satisfies the following conditions:

1. $\alpha_{m,w} > \alpha_{m,w'}$ when $wP_m w'$.
2. $\alpha_{m,w} = \alpha_{m,w'}$ when $wI_m w'$.

We denote by LPS the following linear program:

$$\begin{array}{lll} LPS & \text{máx} & \sum_{(i,j)\in A} \alpha_{i,j} x_{i,j} \\ & st: & x \in C_{S(R)}. \end{array}$$

In Section 3, we show that the convex polytope $C_{S(R)}$ can have fractional stable matchings as extreme points. The following example shows that the solution for LPS can be a stable fractional matching.

Example 3 *Continuing with Example 1: Consider the following associated weight matrix for the preference R is:*

$$\alpha = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

The linear program LPS is:

$$\text{máx} \quad 2x_{11} + 2x_{12} + x_{13} + 2x_{21} + 2x_{22} + x_{23} + 2x_{31} + x_{32} + x_{33}$$
$$st{:}x \in C_{S(R)}.$$

The value of the objective function for the stable matching μ_1 is 5.

$$x^{\mu_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Nevertheless, consider the value of the objective function at the stable fractional matching

$$x = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

It is also equal to 5. *So, there are at least two solutions, one integral and one fractional, that have the same value of the objective function.*

[7] Symmetrically, we can define these weights depending on women's preferences.

The previous example shows that, to reach a men-optimal stable matching we need to define the following integer linear program,

$$IPS \qquad \text{máx} \sum_{(i,j)\in A} \alpha_{i,j}x_{i,j}$$
$$st: \quad x \in C_{S(R)}, \quad x \in \{0,1\}.$$

The following proposition is used in the proof of Theorem 3.

Proposition 1 *Let (M,W,R) be a marriage market with indifferences. Let μ_1 and μ_2 be two stable matchings such that $\mu_1 P_M \mu_2$. Then,*

$$\sum_{(i,j)\in M\times W} \alpha_{j,i}x_{j,i}^{\mu_1} > \sum_{(i,j)\in M\times W} \alpha_{j,i}x_{j,i}^{\mu_2}.$$

Lemma 5 *If $\mu_1 R_m \mu_2$ for some $m \in M$, then*

$$\sum_{j\in W} \alpha_{m,j}x_{m,j}^{\mu_1} \geq \sum_{j\in W} \alpha_{m,j}x_{m,j}^{\mu_2}.$$

Moreover, if there exists $m \in M$ such that $\mu_1 P_m \mu_2$, then

$$\sum_{j\in W} \alpha_{m,j}x_{m,j}^{\mu_1} > \sum_{j\in W} \alpha_{m,j}x_{m,j}^{\mu_2}.$$

Proof.

For the marriage market with indifferences (M,W,R), let μ_1 and μ_2 be stable matchings.

Given $m \in M$ such that

$$w' = \mu_1(m) R_m \mu_2(m) = w'',$$

using the definition of $\alpha_{m,w}$, we have

$$\alpha_{m,w'} \geq \alpha_{m,w''}. \tag{13}$$

It also holds that

$$\sum_{j\in W} \alpha_{m,j}x_{m,j}^{\mu_1} = \alpha_{m,w'} \text{ and } \sum_{j\in W} \alpha_{m,j}x_{m,j}^{\mu_2} = \alpha_{m,w''}.$$

Therefore, condition (13) assures that

$$\sum_{j\in W} \alpha_{m,j}x_{m,j}^{\mu_1} \geq \sum_{j\in W} \alpha_{m,j}x_{m,j}^{\mu_2},$$

giving us the desired property.

For the case when there is $m \in M$ such that $\mu_1 P_m \mu_2$, the proof is analogous to the above. □

Proof of Proposition 1.

If $\mu_1 P_M \mu_2$, for all $m \in M$, it holds that

$$\mu_1 R_m \mu_2$$

and by Lemma 5, we have that for all $m \in M$

$$\sum_{j\in W} \alpha_{m,j}x_{m,j}^{\mu_1} \geq \sum_{j\in W} \alpha_{m,j}x_{m,j}^{\mu_2} \tag{14}$$

and there is at least one $m' \in M$ such that

$$\mu_1 P_{m'} \mu_2.$$

By Lemma 5, we have

$$\sum_{j\in W} \alpha_{m',j} x^{\mu_1}_{m',j} > \sum_{j\in W} \alpha_{m',j} x^{\mu_2}_{m',j}. \tag{15}$$

Therefore, conditions (14) and (15) imply that

$$\sum_{i\in M}\sum_{j\in W} \alpha_{i,j} x^{\mu_1}_{m,j} > \sum_{i\in M}\sum_{j\in W} \alpha_{i,j} x^{\mu_2}_{i,j}.$$

That is,

$$\sum_{(i,j)\in M\times W} \alpha_{i,j} x^{\mu_1}_{i,j} > \sum_{(i,j)\in M\times W} \alpha_{i,j} x^{\mu_2}_{i,j}.$$

This concludes the proof. □

Theorem 3 *Let (M, W, R) be a marriage market with indifferences. A solution for IPS is a men-optimal stable matching.*

Proof. For a marriage market with indifferences (M, W, R), let $\bar{x}$ be a solution of the integer linear program IPS. Let $\bar{\mu}$ be the stable matching associated to $\bar{x}$, $(\bar{x} = x^{\bar{\mu}})$. Assume that the stable matching $\bar{\mu}$ is not optimal for M. That is, there exists $\mu' \in S(R)$ such that $\mu' P_M \bar{\mu}$. By Proposition 1, we have

$$\sum_{(i,j)\in M\times W} \alpha_{i,j} x^{\mu'}_{i,j} > \sum_{(i,j)\in M\times W} \alpha_{i,j} x^{\bar{\mu}}_{i,j}.$$

Then $\bar{x}$ is not a solution for the linear program IPS.

□

If a marriage market with indifferences (M, W, R) has at least two men-optimal stable matchings, the solution of IPS depends on the selection of the weights $(\alpha_{m,w})$. That is, for different selections of weights on the objective function of IPS, the integer linear program may yield different men-optimal stable matchings as solutions. For more details, see Example 4 in the Appendix.

To compute a men-optimal strongly stable matching, let $LPSS$ be the following linear program. To compute a men-optimal strongly stable matching, we only use linear programming. This is because the extreme point of the convex polytope $C_{SS(R)}$ are exactly the strongly stable matchings.

$$LPSS \quad \text{máx} \sum_{(i,j)\in A} \alpha_{i,j} x_{i,j}$$
$$st: \quad x \in C_{SS(R)}.$$

Theorem 4 *Let (M, W, R) be a marriage market with indifferences. A solution for $LPSS$ is a men-optimal strongly stable matching.*

The proof is analogous to that of Theorem 3 .

4.1. Strict Preferences

The weights $\alpha_{i,j}$, defined in this Section, depend on the preferences of men (here the preferences are strict, so item (2) of definition of α does not apply)[8]. When preferences are strict, we will see that the linear program LPS computes the unique men-optimal stable matching. The following corollary characterizes the men-optimal stable matching for the marriage market with strict preferences as the unique solution of a linear program.

[8]2. $\alpha_{m,w} = \alpha_{m,w'}$ when $w I_m w'$.

Corollary 1 *Let (M, W, P) be a marriage market with strict preferences. The unique solution for LPS is the incidence vector of μ_M. That is,*

$$\text{máx} \sum_{(i,j)\in A} \alpha_{i,j} x_{i,j} = \sum_{(i,j)\in A} \alpha_{i,j} x_{i,j}^{\mu_M}.$$

Proof. The proof that x^{μ_M} is the solution of LPS is a particular case of Theorem 3.

Now we will prove that x^{μ_M} is a unique solution. To this end, assume that x^{μ_M} is not the unique optimal solution, that is, assume that $\bar{x}$ is also an optimal solution. Then, there is a extreme point of the convex polytope (1)–(5) such that $\bar{x}$ is the incidence vector of a stable matching different from μ_M. Denote by $x^{\mu} = \bar{x}$. Since $\mu \neq \mu_M$, by the optimality of μ_M, we have $\mu_M P_M \mu$. By Proposition 1,

$$\sum_{(i,j)\in A} \alpha_{i,j} x_{i,j}^{\mu_M} > \sum_{(i,j)\in A} \alpha_{i,j} x_{i,j}^{\mu}.$$

That is, x^{μ} is not an optimal solution of LPS. This proves that x^{μ_M} is the unique solution of LPS.

□

For a marriage market with strict preferences (M, W, P), the men-optimal stable matching is always unique, unlike the marriage market with indifferences (M, W, R); so, the solution of the LPS does not depend on the selection of the weights. If we define $\beta_{i,j}$ as the weight associated to the preferences of women, the linear program that maximizes $\sum_{(i,j)\in A} \beta_{i,j} x_{i,j}$ computes the incidence vector of μ_W. From the optimality results obtained by Gale and Shapley [8] and Knuth [12], μ_M (μ_W) is the preferred stable matching for men (women) and also the less preferred stable matching for women (men). In this way, the linear programs with the objective functions "mín $\sum_{(i,j)\in A} \alpha_{i,j} x_{i,j}$.ªnd "mín $\sum_{(i,j)\in A} \beta_{i,j} x_{i,j}$çompute the incidence vector of μ_W and μ_M, respectively. Notice that this does not happen when we allow for indifferences in preferences. See Example 5 in the Appendix.

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Appendix

The following example shows how the selection of weights influence the optimal solution reached.

Example 4 *Let $M = \{m_1, m_2\}$, $W = \{w_1, w_2, w_3, w_4\}$ and the preference profile R:*

$$\begin{array}{ll} R_{m_1} : [w_1, w_2], w_3, w_4. & R_{w_1} : [m_1, m_2]. \\ R_{m_2} : w_1, w_3, w_2. & R_{w_2} : m_2. \\ & R_{w_3} : m_2. \\ & R_{w_4} : m_1. \end{array}$$

$S(R) = \{\mu_1, \mu_2\}$ is given by

$$x^{\mu_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad x^{\mu_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Both matchings are men-optimal.
Consider two different selections of weights:

$$\alpha = \begin{pmatrix} 3 & 3 & 2 & 1 \\ 3 & 1 & 2 & 0 \end{pmatrix} \text{ and } \alpha^{\star} = \begin{pmatrix} 3 & 3 & 2 & 1 \\ 30 & 10 & 20 & 0 \end{pmatrix}.$$

If we compute the IPS, the solution will be a men-optimal stable matching for any selection of weights. Despite this, the solution of IPS with the weights $\alpha_{i,j}$ will be the stable matching μ_1. Meanwhile, the solution of $IPS^{\star}$ with the weights $\alpha^{\star}_{i,j}$ will be the stable matching μ_2. So, the solution depends on the selection of the weights.

The following example shows that for a marriage market with indifferences, if we change the linear program from a maximization problem to a minimization problem, the new linear program does not compute one of the women-optimal stable matchings.

Example 5 *Let $M = \{m_1, m_2, m_3\}$, $W = \{w_1, w_2, w_3\}$ and the preference profile R be such that*

$$\begin{array}{ll} R_{m_1} = [w_1, w_2], w_3. & R_{w_1} = m_1, m_2, m_3. \\ R_{m_2} = [w_1, w_2], w_3. & R_{w_2} = [m_1, m_2, m_3]. \\ R_{m_3} = w_1, w_2, w_3. & R_{w_3} = m_1, m_2, m_3. \end{array}$$

$S(R) = \{\mu_1, \mu_2, \mu_3\}$ is given by

$$x^{\mu_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ x^{\mu_2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ x^{\mu_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Notice that μ_3 is the unique women-optimal stable matching.
Consider the selection of weights:

$$\alpha = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 30 & 20 & 1 \end{pmatrix}.$$

If we compute the IPS, (minimization)

$$\begin{array}{lll} IPS & \text{mín} & \sum_{(i,j)\in A} \alpha_{i,j} x_{i,j} \\ & \text{st:} & x \in C^*, \ x \in \{0,1\}. \end{array}$$

The solutions of IPS are the stable matchings μ_1 and μ_2, which are not woman-optimal.