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Lattice structure of the random stable set in many-to-many matching markets*

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Abstract

We study the lattice structure of the set of random stable matchings for a many-to-many matching market. We define a partial order on the random stable set and present two natural binary operations for computing the least upper bound and the greatest lower bound for each side of the matching market. Then we prove that with these binary operations the set of random stable matchings forms two distributive lattices for the appropriate partial order, one for each side of the market. Moreover, these lattices are dual.

JEL classification: C71, C78, D49.

Keywords: Lattice Structure, Random Stable Matching markets, Many-to-many Matching Markets.

1 Introduction

There have been studies of matching markets for several decades, beginning with Gale and Shapley's pioneering paper (Gale and Shapley, 1962). They introduce the notion of stable matchings for a marriage market and provide an algorithm for finding them.

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Since then, a considerable amount of work has been carried out on both theory and applications of stable matchings. A matching is stable if all agents have acceptable partners and there is no pair of agents, one from each side of the market, that would prefer to be matched to each other rather than to remain with their current partners. Unfortunately, the set of many-to-one stable matchings may be empty. Substitutability is the weakest condition that has so far been imposed on agents' preferences under which the existence of stable matchings is guaranteed. An agent has *substitutable preference* if she wants to continue being partnering agents from the other side of the market even if other agents become unavailable (see Kelso and Crawford, 1982; Roth, 1984, for more detail).

One of the most important results in the literature on matching is that the set of stable matchings has a distributive, dual lattice structure. This is important for at least two reasons: First, it shows that even if agents from the same side of the market compete for agents from the other side the conflict is attenuated since, in the set of stable matchings, agents on the same side of the market have a coincidence of interests. Second, many algorithms that find the full set of stable matchings are based on this lattice structure.

In this paper, we study the lattice structure of the random stable set for a general matching market, many-to-many matching markets with substitutable preferences, and markets which satisfy the *law of aggregated demand* (L.A.D.). Random stable matchings are very useful for at least two reasons: First, randomization allows for a much richer space of possible outcomes and may be essential to achieve fairness and anonymity. Second, the framework of random stable matchings admits fractional matchings that capture time-sharing arrangements (see Rothblum, 1992; Roth et al., 1993; Teo and Sethuraman, 1998; Sethuraman et al., 2006; Baïou and Balinski, 2000; Doğan and Yıldız, 2016; Neme and Oviedo, 2019a,b, among others).

Roth et al. (1993) define binary operations to compute the *least upper bound (l.u.b.)* and the *greatest lower bound (g.l.b.)* for random stable matchings in the marriage market. To that end, they use first-order stochastic dominance as the partial order for random stable matchings. This partial order cannot be applied when agents' preferences are for subsets of agents from the other side of the market in a substitutable manner. We present a partial order –a natural extension of first-order stochastic dominance– for the random stable set of a matching market when agents' preferences are substitutable and satisfy the L.A.D.. Moreover, we prove that these partial orders (one for each side of the market) respect the polarization of interests. That is, if one random stable matching is unanimously preferred to another for one side of the market, the other side unanimously prefers the opposite. In general, any random stable matching can have many different representations. Despite this, we prove that there is a unique way to represent a random stable matching fulfilling a special property: The stable matchings involved in the lottery are completely ordered by the unanimous order of all firms, which we refer to from now on as *ordered representation*. In this way, the partial order is independent of the representations of the random stable matching. The process for constructing this ordered representation for each random stable matchings is presented via an algorithm.

Our main contribution in this paper is to define two natural binary operations (pointing functions) that compute the *l.u.b.* and *g.l.b.* for random stable matchings, by which the set of those matchings has a dual lattice structure. Moreover, the lattices are distributive. In other words, as long as the set of (deterministic) stable matchings has a lattice structure where binary operations are computed via pointing functions, the set of random stable matchings also has a lattice structure. The paper illustrates the successive results with numerical examples.

Related literature

The lattice structure of the set of stable matchings is introduced by Knuth (1976) for the marriage market. Given two stable matchings, he defines the *l.u.b.* for men by matching each man with the better of two partners, and the *g.l.b.* for men by matching each man with the less preferred of the two partners; these are usually called the *pointing functions* relative to a partial order. Roth (1985) shows that these binary operations (pointing functions) used in Knuth (1976) do not work in the more general many-to-many and many-to-one matching markets even under substitutable preferences. Roth and Sotomayor (1990) present a natural extension of Knuth's result for a specific many-to-one matching market with q-responsive preferences, the so-called college admission problem. Martínez et al. (2001) further extend the results proved by Roth and Sotomayor (1990). They identify a weaker condition than q-responsiveness, called q-separability, and propose two natural binary operations that give a dual lattice structure to the set of stable matchings in a many-to-one matching market with substitutable and q-separable preferences. Such binary operations are similar to those of Knuth. Pepa Risma (2015) generalizes the result of Martínez et al. (2001) by showing that their binary operations work well in many-to-one matching markets where the preferences of the agents satisfy substitutability and the *law of aggregate demand* (a less restrictive than q-separability). This paper is set in many-to-one matching markets with contracts. Manasero (2019) extends the result in Pepa Risma (2015) to the many-to-many marching market, where one side has substitutable preferences satisfying the law of aggregate demand and the other side has q-responsive preferences. Alkan (2002) considers a market with multiple partners on both sides. For that market, preferences are given by rather general *path-independent choice functions* that do not necessarily respect any ordering of individuals and satisfy the law of aggregate demand.[1] He shows that the set of stable matchings in any two-sided market with path-independent choice functions and preferences satisfying the law of aggregate demand has a lattice structure under the common preferences of all agents on either side of the market. Li (2014) presents an alternative proof for Alkan's result. The main distinction between Li (2014) and Alkan (2002) lies in the conditions as regards preferences: Li (2014) assumes agents with complete preferences, whereas Alkan (2002) assumes agents with incompletely revealed preferences. All of these papers share natural definitions of binary operations via pointing functions.

In another direction, there is an extensive literature that proves that the set of stable matchings has a lattice structure by using fixed points theorems, but does not compute binary operation (see Blair, 1988; Adachi, 2000; Fleiner, 2003; Echenique and Oviedo, 2004, 2006; Hatfield and Milgrom, 2005; Ostrovsky, 2008; Wu and Roth, 2018, among others).

In the related literature concerning lattice structures of random stable sets, Roth et al. (1993) define two binary operations for random stable matchings in marriage markets. For these very particular markets, they prove that the set of random stable matchings[2] endowed with a partial order (first-order stochastic dominance) has a lattice structure. They also present a natural extension of pairwise stability for random stable matchings –a random stable matching may be blocked by a pair in a fractional way– called the strong stability condition. Neme and Oviedo (2019a) prove for the marriage market that the strongly stable fractional matching set (random stable matchings that fulfill the strong stability condition), endowed with the same partial order (first-order stochastic dominance), has a lattice structure. The binary operations defined in Roth et al. (1993) and also used by Neme and Oviedo (2019a) cannot be extended to more general markets: Not even to the college admission problem with q-responsive preferences.

The rest of this paper is organized as follows. Section 2 introduces the matching market and preliminary results. Section 3 proves that each random stable matching has a unique strictly ordered representation (Theorem 1). Section 4 presents a partial order for random matchings when agents' preferences are substitutable and satisfy the L.A.D. (Proposition 1). Section 5 defines binary operations and proves that natural binary operations compute the *l.u.b.* and *g.l.b.* for each side of the market (Proposition 4). Moreover, these binary operations satisfy a distributive property (Proposition 5). The main result of the paper is also presented: The random stable set has a distributive

[1] Alkan (2002)) calls the law of aggregate demand "cardinal monotonicity".

[2] They prove that the "stable fractional matching set" coincides with the random stable matching set.

dual lattice structure (Theorem 2). Section 6 contains concluding remarks. Finally, Appendix A contains proofs for the ordered representation and Appendix B contains proofs of the partial order and the proof of the main theorem.

2 Preliminaries

We consider many-to-many matching markets where there are two disjoint sets of agents: The set of *firms* F and the set of *workers* W. Each firm has an antisymmetric, transitive, and complete preference relation $(>_f)$ over the set of all subsets of W. Each worker also has an antisymmetric, transitive, and complete preference relation $(>_w)$ over the set of all subsets of F. We denote by P the preference profile for all agents: Firms and workers. A many-to-many matching market is denoted by (F, W, P). Given a set of firms $S \subseteq F$, each worker $w \in W$ can determine which subset of S would most prefer to hire her. This is called w's choice set from S and is denoted by $Ch(S, >_w)$. Formally,

$$Ch(S, >_w) = \max_{>_w}\{T : T \subseteq S\}.$$

Symmetrically, given a set of workers $S \subseteq W$, let $Ch(S, >_f)$ denote firm f's most preferred subset of S according to her preference relation $>_f$. Formally,

$$Ch(S, >_f) = \max_{>_f}\{T : T \subseteq S\}.$$

Definition 1 *A **matching** μ is a function from the set $F \cup W$ into $2^{F \cup W}$ such that for each $w \in W$ and for each $f \in F$:*

1. $\mu(w) \subseteq F$,
2. $\mu(f) \subseteq W$,
3. $w \in \mu(f) \Leftrightarrow f \in \mu(w)$.

Agent $a \in F \cup W$ is said to be matched if $\mu(a) \neq \emptyset$, otherwise she is unmatched.

A matching μ is blocked by agent a if $\mu(a) \neq Ch(\mu(a), >_a)$. A matching is said to be individually rational if it is not blocked by any individual agent. A matching μ is blocked by a worker-firm pair (w, f) if $w \notin \mu(f), w \in Ch(\mu(f) \cup \{w\}, >_f)$, and $f \in Ch(\mu(w) \cup \{f\}, >_w)$. A matching μ is **stable** if it is not blocked by any individual agent or any worker-firm pair. The set of stable matchings is denoted by $\mathcal{S}(\mathcal{P})$.[3] A **random stable matching** is a lottery over stable matchings, and $\mathcal{RS}(\mathcal{P})$ denotes the random stable set for the many-to-many matching market (F, W, P).

[3]In the rest of the paper we use either μ or ν to denote a stable matching.

Given an agent a's preference relation ($>_a$) and two stable matchings μ and μ', we denote that $\mu(a) \geq_a^B \mu'(a)$ when $\mu(a) = Ch(\mu(a) \cup \mu'(a), >_a)$. It can be said that $\mu(a) >_a^B \mu'(a)$ if $\mu(a) \geq_a^B \mu'(a)$ and $\mu(a) \neq \mu'(a)$. Given a preference profile P and two stable matchings μ and μ', let $\mu >_F^B \mu'$ denote the case in which all firms like μ at least as well as μ', with at least one firm preferring μ to μ' outright. Let $\mu \geq_F^B \mu'$ denote that either $\mu >_F^B \mu'$ or that $\mu = \mu'$. Similarly, define $>_W^B$ and $\geq_W^B$. Notice that $\geq_F^B$ and $\geq_W^B$ are known as Blair's partial orders over the set of stable matchings (see Blair, 1988, for more detail). Moreover, there is a bridge between these two partial orders that is known as **polarization of interests**, and states that if $\mu, \mu' \in \mathcal{S}(\mathcal{P})$ then $\mu \geq_F^B \mu'$ if and only if $\mu' \geq_W^B \mu$ (see Blair, 1988; Alkan, 2002; Li, 2014, among others).

An agent a's preference relation satisfies **substitutability** if, for each subset S of the opposite set (for instance, if $a \in F$ then $S \subseteq W$) that contains agent b, $b \in Ch(S, >_a)$ implies that $b \in Ch(S' \cup \{b\}, >_a)$ for each $S' \subseteq S$. Moreover, if agent a's preference relation is substitutable then it holds that $Ch(S \cup S', >_a) = Ch(Ch(S, >_a) \cup S', >_a)$ for each subset S and S' of the opposite set. An agent a's preference relation ($>_a$) is said to satisfy the **law of aggregate demand (L.A.D.)** if for all subsets S of the opposite set and all $S' \subseteq S$, $|Ch(S', >_a)| \leq |Ch(S, >_a)|$. [4]

For a matching market (F, W, P) where the preference relation of each agent satisfies substitutability and the L.A.D., Alkan (2002)[5] proves that the set of stable matchings has a lattice structure. Given two stable matchings μ and μ', *l.u.b.* for firms is denoted by $\boldsymbol{\mu \vee_F \mu'}$ and *g.l.b.* for firms is denoted by $\boldsymbol{\mu \wedge_F \mu'}$. Similarly, *l.u.b.* for workers is denoted by $\boldsymbol{\mu \vee_W \mu'}$ and *g.l.b.* for workers is denoted by $\boldsymbol{\mu \wedge_W \mu'}$. The binary operations are defined as follows (see Blair, 1988; Alkan, 2002; Li, 2014, among others).

$$\mu \vee_F \mu'(f) := Ch(\mu(f) \cup \mu'(f), >_f), \text{ for each firm } f \in F,$$

$$\mu \vee_F \mu'(w) := \{f : w \in Ch(\mu(f) \cup \mu'(f), >_f)\}, \text{ for each worker } w \in W.$$

Similarly,

$$\mu \vee_W \mu'(w) := Ch(\mu(w) \cup \mu'(w), >_w), \text{ for each worker } w \in W,$$

$$\mu \vee_W \mu'(f) := \{w : f \in Ch(\mu(w) \cup \mu'(w), >_w)\}, \text{ for each firm } f \in F.$$

$\mu \vee_F \mu'$, $\mu \vee_F \mu'$, $\mu \vee_W \mu'$, and $\mu \vee_W \mu'$ are stable matchings (for more detail see Blair, 1988; Alkan, 2002; Li, 2014, among others). From the polarization of interests of Blair's partial orders ($\geq_F^B$ and $\geq_W^B$), it follows that

$$\mu \vee_F \mu' = \mu \wedge_W \mu',$$

[4] $|S|$ denotes the number of agents in S.

[5] Li (2014) presents an alternative proof for Alkan's result Li (2014) assumes agents with complete preferences, whereas Alkan (2002) assumes agents with incomplete preferences.

$$\mu \wedge_F \mu' = \mu \vee_W \mu',$$

Therefore, $(\mathcal{S}(\mathcal{P}), \vee_F, \wedge_F, \geq_F^B)$ and $(\mathcal{S}(\mathcal{P}), \vee_W, \wedge_W, \geq_W^B)$ are dual lattices. Moreover, these binary operations satisfy $\mu \wedge_F (\mu' \vee_F \mu'') = (\mu \wedge_F \mu') \vee_F (\mu \wedge_F \mu'')$ and $\mu \vee_F (\mu' \wedge_F \mu'') = (\mu \vee_F \mu') \wedge_F (\mu \vee_F \mu'')$ for each $\mu, \mu', \mu'' \in \mathcal{S}(\mathcal{P})$, and this implies that the lattices are distributive. The distributiveness of $\vee_W$ and $\wedge_W$ is analogous. Given $\mu, \mu', \mu'' \in \mathcal{S}(\mathcal{P})$, computing the *l.u.b.* (*g.l.b.*) among three stable matchings is equivalent to computing the *l.u.b.* (*g.l.b.*) between two of them, say μ and μ', and then computing the *l.u.b.* (*g.l.b.*) between the resulting stable matching and μ''. Formally,

$$(\mu \vee_F \mu'(f)) \vee_F \mu''(f) = Ch(Ch(\mu(f) \cup \mu'(f), >_f) \cup \mu''(f), >_f), \text{ for each firm } f \in F.$$

Given that for an agent a's preference relation it holds that $Ch(S \cup S', >_a) = Ch(Ch(S, >_a) \cup S', >_a)$ for each subset S and S' of the opposite set, it holds that

$$Ch(Ch(\mu(f) \cup \mu'(f), >_f) \cup \mu''(f), >_f) = Ch((\mu \cup \mu' \cup \mu'')(f), >_f), \text{ for each firm } f \in F.$$

Therefore, $(\mu \vee_F \mu'(f)) \vee_F \mu''(f) = \mu \vee_F \mu' \vee_F \mu''(f)$. This is called the associative property of $\vee_F$. The associative properties of $\vee_W$, $\wedge_W$, and $\wedge_F$ are analogous.

Let $T \subseteq \mathcal{S}(\mathcal{P})$. Denote

$$\mu_T^{\vee}(f) = Ch\left(\bigcup_{\mu \in T} \mu(f), >_f\right)$$

and

$$\mu_T^{\wedge}(f) = \left\{ w : f \in Ch\left(\bigcup_{\mu \in T} \mu(w), >_w\right)\right\}$$

for each $f \in F$. From the associative property, it follows that $\mu_T^{\vee}$ and $\mu_T^{\wedge}$ are stable matchings which are the *l.u.b.*$_{\geq_F}$ and *g.l.b.*$_{\geq_F}$ among the stable matchings in T respectively.

3 Random stable matchings: Representations

This section presents a result that is of interest in itself and is used in the following sections to define a partial order for random stable matchings and prove the main result of the paper.

To describe the representation of random stable matchings, we first need to define an incidence vector. Then, given a stable matching μ, a vector $x^{\mu} \in \{0,1\}^{|F| \times |W|}$ is an **incidence vector** where $x_{i,j}^{\mu} = 1$ if and only if $j \in \mu(i)$ and $x_{i,j}^{\mu} = 0$ otherwise. Hence, a random stable matching is represented as a lottery over the incidence vectors of stable matchings. That is,

$$x = \sum_{\nu \in \mathcal{M}} \lambda_\nu x^\nu$$

where $0 < \lambda_\nu \leq 1$, $\sum_{\nu \in \mathcal{S}(\mathcal{P})} \lambda_\nu = 1$, and $\mathcal{M} \subseteq \mathcal{S}(\mathcal{P})$.

This paper, only considers representations where each scalar, λ, is positive. Despite this, each random stable matching may have several representations. Notice that the incidence vector of a random stable matching is a vector fulfilling the requirements that $x \in [0,1]^{|F| \times |W|}$. Each entry $x_{i,j}$ can also be represented as the probability of i being matched with j. The following example illustrates this.

Example 1 *Let (F, W, P) be a many-to-one matching market instance where $F = \{f_1, f_2, f_3, f_4\}$, $W = \{w_1, w_2, w_3, w_4\}$ and the preference profile is given by*

$$>_{f_1} = \{w_1, w_2\}, \{w_1, w_3\}, \{w_2, w_4\}, \{w_3, w_4\}, \{w_1\}, \{w_2\}, \{w_3\}, \{w_4\}.$$
$$>_{f_2} = \{w_3, w_4\}, \{w_2, w_4\}, \{w_1, w_3\}, \{w_1, w_2\}, \{w_3\}, \{w_4\}, \{w_1\}, \{w_2\}.$$
$$>_{f_3} = \{w_1, w_3\}, \{w_3, w_4\}, \{w_1, w_2\}, \{w_2, w_4\}, \{w_1\}, \{w_3\}, \{w_2\}, \{w_4\}.$$
$$>_{f_4} = \{w_2, w_4\}, \{w_1, w_2\}, \{w_3, w_4\}, \{w_1, w_3\}, \{w_2\}, \{w_4\}, \{w_1\}, \{w_3\}.$$
$$>_{w_1} = \{f_2, f_4\}, \{f_2, f_3\}, \{f_1, f_4\}, \{f_1, f_3\}, \{f_2\}, \{f_4\}, \{f_3\}, \{f_1\}.$$
$$>_{w_2} = \{f_2, f_3\}, \{f_1, f_3\}, \{f_2, f_4\}, \{f_1, f_4\}, \{f_2\}, \{f_3\}, \{f_1\}, \{f_4\}.$$
$$>_{w_3} = \{f_1, f_4\}, \{f_2, f_4\}, \{f_1, f_3\}, \{f_2, f_3\}, \{f_1\}, \{f_4\}, \{f_2\}, \{f_3\}.$$
$$>_{w_4} = \{f_1, f_3\}, \{f_1, f_4\}, \{f_2, f_3\}, \{f_2, f_4\}, \{f_1\}, \{f_3\}, \{f_4\}, \{f_2\}.$$

It is easy to check that these preference relations are substitutable and satisfy the L.A.D.. The set of stable matchings $\{\nu_1, \nu_2, \nu_3, \nu_4\}$, is represented in Table 1 and its lattice for the partial order $\geq_F^B$ is represented in Figure 1.

	f_1	f_2	f_3	f_4
ν_1	$\{w_1, w_2\}$	$\{w_3, w_4\}$	$\{w_1, w_3\}$	$\{w_2, w_4\}$
ν_2	$\{w_1, w_3\}$	$\{w_2, w_4\}$	$\{w_3, w_4\}$	$\{w_1, w_2\}$
ν_3	$\{w_2, w_4\}$	$\{w_1, w_3\}$	$\{w_1, w_2\}$	$\{w_3, w_4\}$
ν_4	$\{w_3, w_4\}$	$\{w_1, w_2\}$	$\{w_2, w_4\}$	$\{w_1, w_3\}$

Table 1

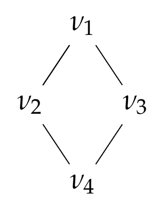

Figure 1

Let $x^1 = \frac{3}{4} x^{\nu_2} + \frac{1}{4} x^{\nu_3}$ be a random stable matching. The incidence vector associated is the following

$$x^1 = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

Note that x^1 is also represented by the following lottery

$$x^1 = \tfrac{1}{4}x^{\nu_1} + \tfrac{1}{2}x^{\nu_2} + \tfrac{1}{4}x^{\nu_4}.$$

Given a representation of a random stable matching x, say $x = \sum_{r=1}^{R} \lambda_r x^{\nu_r}$; $0 < \lambda_r \leq 1$, $\sum_{r=1}^{R} \lambda_r = 1$, $A = \{\nu_1, \ldots, \nu_R\}$ is defined as the set of all stable matchings involved in the representation. It can also be said that a random stable matching has a **weakly (strictly) ordered representation** if $\nu_r \geq_F^B \nu_{r+1}$ $\left(\nu_r >_F^B \nu_{r+1}\right)$ for each $r = 1, \ldots, R-1$. Notice that in Example 1 the latter representation is a strictly ordered representation of x^1. Given any representation of a random stable matching, we show that there is a unique strictly ordered representation.

Theorem 1 *If x is a random stable matching then x has a unique strictly ordered representation.*

Proof. See proof in Appendix A. □

The proof of Theorem 1 is constructive. Algorithm 1 presents the construction of the strictly ordered representation. But before the algorithm is formally presented a brief explanation of how the procedure goes is provided. Let x be a random stable matching with a representation. Let A be the set of stable matchings involved in the representation of x. Let B_1 be the set of stable matchings that form the minimum sublattice, concerning the partial order $\geq_F^B$, which contains the set A.[6] Set $x^1 := x$, and B_1 as the input. First, we compute μ_1 as the stable matching which is the $l.u.b._{\geq_F}$ of set B_1. Thus, μ_1 is the stable matching of the first term of the strictly ordered representation. Then define the first scalar as follows: set α_1 as the minimum positive probability assigned to pairs of agents in x^1 that are also matched in μ_1. Thus, $\alpha_1 x^{\mu_1}$ is the first term of the strictly ordered representation. Now, some stable matchings of the set B_1 must be eliminated, because once the first term of the strictly ordered representation is established, no other term can be allowed to share with μ_1 the entry of x^1 in which the minimum probability (α_1) is obtained. To do this, set $\mathcal{L}_1$ as the set of pairs of agents, which are assigned in μ_1, and have probability α_1 at x^1. Then eliminate from B_1 each stable matching that matches a pair of agents that belong to $\mathcal{L}_1$. If the resulting set is empty, the algorithm stops, which means that $x^1 = x^{\mu_1}$. If not, to complete the output of the first step, set x^2 such that $x^1 = \alpha_1 x^{\mu^1} + (1-\alpha_1)x^2$, and the algorithm continues to Step 2 with input x^2 and the resulting set of stable matchings.

Now, the algorithm is formally presented. Set $B_1 = \{\nu_T^{\vee} : T \subseteq A\} \bigcup \{\nu_T^{\wedge} : T \subseteq A\}$.

[6] Given the set of stable matchings, a lattice formed by a subset of stable matchings is called a sublattice (see Birkhoff, 1940, for more detail on lattice theory).

Algorithm 1:

Step $k=1$ `INPUT:` x^1 and B_1.

Set $\mu_1 := \nu^{\vee}_{B_1}$.

$\alpha_1 := \min\{x^1_{i,j} : x^{\mu_1}_{i,j} = 1\}$.

$\mathcal{L}_1 := \{(i,j) \in F \times W : x^1_{i,j} = \alpha_1 \text{ and } x^{\mu_1}_{i,j} = 1\}$.

$C_1 := \bigcup_{(i,j)\in\mathcal{L}_1} \{\nu \in B_1 : x^{\nu}_{i,j} = 1\}$.

$B_2 := B_1 \setminus C_1$.

`IF` $B_2 = \emptyset$,

`THEN`, the procedure stops.

`ELSE` set x^2 such that $x^1 = \alpha_1 x^{\mu_1} + (1-\alpha_k)x^2$.

`OUTPUT:` x^2 and B_2. Continue to Step 2.

Step $k>1$ `INPUT:` x^k and B_k.

Set $\mu_k := \nu^{\vee}_{B_k}$.

$\alpha_k := \min\{x^k_{i,j} : x^{\mu_k}_{i,j} = 1\}$.

$\mathcal{L}_k := \{(i,j) \in F \times W : x^k_{i,j} = \alpha_k \text{ and } x^{\mu_k}_{i,j} = 1\}$.

$C_k := \bigcup_{(i,j)\in\mathcal{L}_k} \{\nu \in B_k : x^{\nu}_{i,j} = 1\}$.

$B_{k+1} := B_k \setminus C_k$.

`IF` $B_{k+1} = \emptyset$,

`THEN`, the procedure stops.

`ELSE` set x^{k+1} such that $x^k = \alpha_k x^{\mu_k} + (1-\alpha_k)x^{k+1}$.

`THEN`, $x^1 = \alpha_1 x^{\mu_1} + \sum_{s=2}^{k} \prod_{\ell=1}^{s-1}(1-\alpha_\ell)\alpha_s x^{\mu_s} + \prod_{\ell=1}^{k}(1-\alpha_\ell)x^{k+1}$.

`OUTPUT:` x^{k+1} and B_{k+1}. Continue to Step $k+1$.

The following example illustrates Algorithm 1.

Example 1 (Continued) *Let $x^1 = \frac{3}{4}x^{\nu_2} + \frac{1}{4}x^{\nu_3}$ be a random stable matching. Now, obtain the strictly ordered representation of x^1 as follows. Recall that*

$$x^1 = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

Thus, $A = \{\nu_2, \nu_3\}$. The non-empty subsets of A are $T_1 = \{\nu_2\}, T_2 = \{\nu_3\}$ and $T_3 = \{\nu_2, \nu_3\}$. Thus, the elements of B_1 are $\nu^{\vee}_{T_1} = \nu_2$, $\nu^{\vee}_{T_2} = \nu_3$, $\nu^{\vee}_{T_3} = \nu_1$, $\nu^{\wedge}_{T_1} = \nu_2$, $\nu^{\wedge}_{T_2} = \nu_3$ and $\nu^{\wedge}_{T_3} = \nu_4$. Hence, $B_1 = \{\nu_1, \nu_2, \nu_3, \nu_4\}$.

Step 1 `INPUT:` *x^1 and B_1. Set $\mu_1 := \nu_1 = \nu^{\vee}_{B_1}$, $\alpha_1 = \frac{1}{4}$, and $C_1 = \{\nu_1, \nu_3\}$.*

Since $B_2 = B_1 \setminus C_1 = \{\nu_2, \nu_4\} \neq \emptyset$, set

$$x^2 := \frac{x - \frac{1}{4}x^{\mu_1}}{1 - \frac{1}{4}} = \begin{pmatrix} \frac{2}{3} & 0 & 1 & \frac{1}{3} \\ \frac{1}{3} & 1 & 0 & \frac{2}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

Thus, $x^1 = \frac{1}{4}x^{\mu_1} + (1 - \frac{1}{4})x^2$.
`OUTPUT:` *x^2 and B_2. Continue to Step 2.*
Step 2 `INPUT:` *x^2 and B_2. Set $\mu_2 := \nu_2 = \nu^{\vee}_{B_2}$, $\alpha_2 = \frac{2}{3}$, and $C_2 = \{\nu_2\}$.*
Since $B_3 = B_2 \setminus C_2 = \{\nu_4\} \neq \emptyset$, set

$$x^3 = \frac{x - \frac{2}{3}x^{\mu_2}}{1 - \frac{2}{3}} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Thus, $x^1 = \frac{1}{4}x^{\mu_1} + (1 - \frac{1}{4})(\frac{2}{3})x^{\mu_2} + (1 - \frac{1}{4})(1 - \frac{2}{3})x^3$.
`OUTPUT:` *x^3 and B_3. Continue to Step 3.*
Step 3 `INPUT:` *x^3 and B_3. Set $\mu_3 := \nu_4 = \nu^{\vee}_{B_3}$, $\alpha_3 = 1$, and $C_3 = \{\nu_4\}$.*
Since $B_4 = B_3 \setminus C_3 = \emptyset$, the procedure stops.
The output of Algorithm 1 is

$$x^1 = \tfrac{1}{4}x^{\mu_1} + (1 - \tfrac{1}{4})(\tfrac{2}{3})x^{\mu_2} + (1 - \tfrac{1}{4})(1 - \tfrac{2}{3})(1)x^{\mu_3}$$

$$= \tfrac{1}{4}x^{\mu_1} + \tfrac{1}{2}x^{\mu_2} + \tfrac{1}{4}x^{\mu_3}.$$

Since $\mu_1 = \nu_1$, $\mu_2 = \nu_2$ and $\mu_3 = \nu_4$, the ordered representation of x^1 is the following:

$$x^1 = \tfrac{1}{4}x^{\nu_1} + \tfrac{1}{2}x^{\nu_2} + \tfrac{1}{4}x^{\nu_4}.$$

As shown in Figure 1, the stable matchings of the representation lottery fulfill $\nu_1 >^B_F \nu_2 >^B_F \nu_4$.

4 Partial order for random stable matchings

This section defines a partial order for the random stable set in a many-to-many matching market with substitutable preferences satisfying the L.A.D.. This partial order is a generalization of the first-order stochastic dominance presented in Roth et al. (1993) for the random stable set in a marriage market (one-to-one matching market). For a given marriage market (M, W, P), they define the following partial order: With x and y being two random stable matchings, they say that x weakly dominates* y for man m, (here denoted by $x \succeq^\star_m y$) if

$$\sum_{j \succeq_m w} x_{m,j} \geq \sum_{j \succeq_m w} y_{m,j}$$

for each $w \in W$. They also say that $x \succeq^{\star}_M y$ if $x \succeq^{\star}_m y$ for each $m \in M$. The partial order $\succeq^{\star}_W$ is defined analogously. Notice that the partial orders $\succeq^{\star}_M$ and $\succeq^{\star}_W$ are defined over single agents. These relations cannot order random stable matchings when agents have preferences over subsets of agents on the other side of the market in a substitutable manner. Therefore, for the setting considered in this paper, a new partial order is defined. A desired property of that partial order is that it be independent of the representation of the random stable matchings. To that end it is defined for strictly ordered representations, since they are unique. Formally,

Definition 2 *Let x and y be two random stable matchings. Let $x = \sum_{i=1}^{I} \alpha_i x^{\mu_i^x}$ and $y = \sum_{j=1}^{J} \beta_j x^{\mu_j^y}$ their strictly ordered representations. It is said that **x weakly dominates y** for the firm f, ($x \succeq_f y$), if and only if for each $\mu_j^y(f)$*

$$\sum_{\left\{i:\mu_i^x(f) \succeq_f^B \mu_j^y(f)\right\}} \alpha_i \geq \sum_{\left\{l:\mu_l^y(f) \succeq_f^B \mu_j^y(f)\right\}} \beta_l.$$

It can also be said that ***x* strongly dominates *y*** for the firm f, ($x \succ_f y$), if the above inequalities hold with at least one strict inequality for any $\mu_j^y(f)$. That is, $x \succ_f y$ when $x \succeq_f y$ and $x \neq y$ for the firm f. Furthermore, if $x \succeq_f y$ for each $f \in F$ it can be said that $x \succeq_F y$. Relations $\succeq_w$, $\succ_w$ and $\succeq_W$ are defined analogously. Notice that if each random stable matching has a weakly ordered representation, the domination relations applies in the same way. Furthermore, if $x_{f,w}$ is interpreted as the probability that f is matched with w, then $x \succeq_f y$ if $x_{f,\cdot}$ stochastically dominates $y_{f,\cdot}$.

Remark 1 *For the particular case of a marriage market, the domination relation $\succeq_F$ ($\succeq_W$) coincides with $\succeq^{\star}_M$ ($\succeq^{\star}_W$).*

The following proposition states that the domination relation $\succeq_F$ ($\succeq_W$) is a partial order. Formally,

Proposition 1 *The domination relation $\succeq_F$ ($\succeq_W$) is a partial order.*

Proof. See proof in Appendix B. □

Before presenting the main results, we present a special representation of random stable matchings. This special representation is crucial in defining binary operations in a natural way. Given x and y two random stable matchings with their strictly ordered representations, these random stable matchings are said to have **Ordered Equally Scalar (OES)**-representations if and only if

$$x = \sum_{\ell=1}^{\tilde{k}} \gamma_\ell \mu_\ell \quad \text{and} \quad y = \sum_{\ell=1}^{\tilde{k}} \gamma_\ell \mu'_\ell,$$

with $\mu_\ell \geq_F^B \mu_{\ell+1}$ and $\mu'_\ell \geq_F^B \mu'_{\ell+1}$ for each $\ell = 1, \ldots, \tilde{k} - 1$. That is, both are weakly ordered representations that also have the same numbers of terms and the same scalar, term to term.

The following two subsections show how to construct OES-representations. The first subsection constructs the OES-representation for rational random stable matchings (each scalar is a rational number). The second subsection presents the general construction of OES-representations.

4.1 OES-representations: rational random stable matchings

This subsection, shows how to construct OES-representations for random stable matchings where each scalar of the strictly ordered representations is a *rational number*. These random stable matchings are called **rational random stable matchings**.

Let x and y be two rational random stable matchings, so that their strictly ordered representations are as follows:

$$x = \sum_{i=1}^{I} \alpha_i \mu_i^x, \text{ and } y = \sum_{j=1}^{J} \beta_j \mu_j^y. \tag{1}$$

α_i and β_j are positive rational numbers, so for each α_i there are natural numbers a_i, b_i such that $\alpha_i = \frac{a_i}{b_i}$. Similarly, for each β_j there are natural numbers c_j, d_j such that $\beta_j = \frac{c_j}{d_j}$.

Denote by e the *least common multiple* (***lcm***) of all denominators b_i, d_j for each $i = 1, \ldots, I$ and for each $j = 1, \ldots, J$. That is,

$$e = lcm(b_1, \ldots, b_I, d_1, \ldots, d_J).$$

It is then possible to write $\alpha_i = \frac{a_i}{b_i} = \frac{a_i \frac{e}{b_i}}{e}$ and $\beta_i = \frac{c_j}{d_j} = \frac{c_j \frac{e}{d_j}}{e}$ for each $i = 1, \ldots, I$ and for each $j = 1, \ldots, J$. Hence, all the scalars α and β ca be written with the same denominator.

Define

$$\tilde{\boldsymbol{\mu}}_{\boldsymbol{k}}^{\boldsymbol{x}} := \begin{cases} \mu_1^x & \text{for } k = 1, \ldots, \frac{a_1}{b_1} e \\ \mu_2^x & \text{for } k = \frac{a_1}{b_1} e + 1, \ldots, \left(\frac{a_2}{b_2} + \frac{a_1}{b_1} \right) e \\ \vdots & \vdots \\ \mu_I^x & \text{for } k = \left(\sum_{n=1}^{I-1} \frac{a_n}{b_n} \right) e + 1, \ldots, \left(\sum_{n=1}^{I} \frac{a_n}{b_n} \right) e \end{cases}$$

$$\tilde{\boldsymbol{\mu}}_{\boldsymbol{k}}^{\boldsymbol{y}} := \begin{cases} \mu_1^y & \text{for } k = 1, \ldots, \frac{c_1}{d_1} e \\ \mu_2^y & \text{for } k = \frac{c_1}{d_1} e + 1, \ldots, \left(\frac{c_2}{d_2} + \frac{c_1}{d_1} \right) e \\ \vdots & \vdots \\ \mu_J^y & \text{for } k = \left(\sum_{m=1}^{J-1} \frac{c_m}{d_m} \right) e + 1, \ldots, \left(\sum_{m=1}^{J} \frac{c_m}{d_m} \right) e \end{cases}$$

Then,

$$x = \sum_{i=1}^{I} \alpha_i \mu_i^x = \sum_{i=1}^{I} \frac{a_i}{b_i} \mu_i^x = \sum_{i=1}^{I} \frac{a_i \frac{e}{b_i}}{e} \mu_i^x = \sum_{k=1}^{e} \frac{1}{e} \tilde{\mu}_k^x. \tag{2}$$

Analogously,

$$y = \sum_{j=1}^{J} \beta_j \mu_j^y = \sum_{j=1}^{J} \frac{c_j}{d_j} \mu_j^y = \sum_{j=1}^{J} \frac{c_j \frac{e}{d_j}}{e} \mu_j^y = \sum_{k=1}^{e} \frac{1}{e} \tilde{\mu}_k^y. \tag{3}$$

Therefore, the OES-representations of x and y are the following,

$$x = \sum_{k=1}^{e} \frac{1}{e} \tilde{\mu}_k^x, \text{ and } y = \sum_{k=1}^{e} \frac{1}{e} \tilde{\mu}_k^y.$$

The following example illustrates this construction.

Example 1 (Continued) *Let x and y be two random stable matchings with their strictly ordered representations,*

$$x = \tfrac{1}{4} x^{\nu_1} + \tfrac{1}{2} x^{\nu_2} + \tfrac{1}{4} x^{\nu_4},$$

$$y = \tfrac{1}{6} x^{\nu^1} + \tfrac{1}{2} x^{\nu^3} + \tfrac{1}{3} x^{\nu^4}.$$

Let $e = lcm(2,3,4,6) = 12$. Thus, the random stable matchings x and y can be represented as:

$$x = \tfrac{1}{12} x^{\nu_1} + \tfrac{1}{12} x^{\nu_1} + \tfrac{1}{12} x^{\nu_1} + \tfrac{1}{12} x^{\nu_2} + \tfrac{1}{12} x^{\nu_2} + \tfrac{1}{12} x^{\nu_2} + \tfrac{1}{12} x^{\nu_2} + \tfrac{1}{12} x^{\nu_2}$$

$$+ \tfrac{1}{12} x^{\nu_2} + \tfrac{1}{12} x^{\nu_4} + \tfrac{1}{12} x^{\nu_4} + \tfrac{1}{12} x^{\nu_4},$$

$$y = \tfrac{1}{12} x^{\nu_1} + \tfrac{1}{12} x^{\nu_1} + \tfrac{1}{12} x^{\nu_3} + \tfrac{1}{12} x^{\nu_3} + \tfrac{1}{12} x^{\nu_3} + \tfrac{1}{12} x^{\nu_3} + \tfrac{1}{12} x^{\nu_3} + \tfrac{1}{12} x^{\nu_3}$$

$$+ \tfrac{1}{12} x^{\nu_4} + \tfrac{1}{12} x^{\nu_4} + \tfrac{1}{12} x^{\nu_4} + \tfrac{1}{12} x^{\nu_4}.$$

4.2 OES-representation: The general case

This subsection briefly explains the procedure for constructing weakly ordered OES-representations for a general case. That is, the scalar of the strictly ordered representations of x or y are not necessarily rational numbers. This procedure is formalized by Algorithm 2 in Appendix B. First, take two random stable matchings x and y and represent them by their strictly ordered representations. That is, $x = \sum_{i=1}^{I} \alpha_i x^{\mu_i^x}$ and $y = \sum_{j=1}^{J} \beta_j x^{\mu_j^y}$ with $\mu_i^x >_F^B \mu_{i+1}^x$ for each $i = 1, \dots, I-1$ and $\mu_j^y >_F^B \mu_{j+1}^y$ for each $j = 1, \dots, J-1$. Thus, the OES-representation procedure goes as follows: Let $\gamma_1 = \min\{\alpha_1, \beta_1\}$. W.l.o.g. assume that $\gamma_1 = \alpha_1$. Thus,

$$x = \gamma_1 \mu_1^x + \sum_{i=2}^{I} \alpha_i x^{\mu_i^x}$$

$$y = \gamma_1 \mu_1^y + (\beta_1 - \gamma_1) \mu_1^y + \sum_{j=2}^{J} \beta_j x^{\mu_j^y}.$$

Notice that the first term of each new representation has the same scalar. Now, take the second scalar of each representation and set $\gamma_2 = \min\{\alpha_2, \beta_1 - \gamma_1\}$. If $\gamma_2 = \alpha_2$, then

$$x = \gamma_1 \mu_1^x + \gamma_2 \mu_2^x + \sum_{i=3}^{I} \alpha_i x^{\mu_i^x}$$

$$y = \gamma_1 \mu_1^y + \gamma_2 \mu_1^y + (\beta_1 - \gamma_1 - \gamma_2) \mu_1^y + \sum_{j=2}^{J} \beta_j x^{\mu_j^y}.$$

If $\gamma_2 = \beta_1 - \gamma_1$, then

$$x = \gamma_1 \mu_1^x + \gamma_2 \mu_2^x + (\alpha_2 - \gamma_2) \mu_2^x + \sum_{i=3}^{I} \alpha_i x^{\mu_i^x}$$

$$y = \gamma_1 \mu_1^y + \gamma_2 \mu_1^y + \sum_{j=2}^{J} \beta_j x^{\mu_j^y}.$$

The first two terms of each new representation have the same scalars. Now take the third scalar of each representation and set either $\gamma_3 = \min\{\alpha_3, \beta_1 - \gamma_1 - \gamma_2\}$ or $\gamma_3 = \min\{\alpha_2 - \gamma_2, \beta_2\}$, and so on.[7] Notice that the OES-representations are weakly ordered representations.

The OES-representation general procedure is illustrated by the following example.

Example 1 (Continued) *Let* $x = \frac{1}{4} x^{\nu_1} + \frac{1}{2} x^{\nu_2} + \frac{1}{4} x^{\nu_4}$ *and* $y = \frac{1}{6} x^{\nu^1} + \frac{1}{2} x^{\nu^3} + \frac{1}{3} x^{\nu^4}$. *Notice that both random stable matchings have their strictly ordered representations. If* $\gamma_1 = \min\{\frac{1}{4}, \frac{1}{6}\} = \frac{1}{6}$, *then*

[7] Notice that the procedure for constructing the OES-representations for more than two random stable matchings is analogous.

$$x = \frac{1}{6}x^{\nu_1} + (\frac{1}{4} - \frac{1}{6})x^{\nu_1} + \frac{1}{2}x^{\nu_2} + \frac{1}{4}x^{\nu_4},$$

$$y = \frac{1}{6}x^{\nu^1} + \frac{1}{2}x^{\nu^3} + \frac{1}{3}x^{\nu^4}$$

Notice that the first term of each new representation has the same scalar $\frac{1}{6}$. *If* $\gamma_2 = \min\{\frac{1}{4} - \frac{1}{6}, \frac{1}{2}\} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$, *then*

$$x = \frac{1}{6}x^{\nu_1} + \frac{1}{12}x^{\nu_1} + \frac{1}{2}x^{\nu_2} + \frac{1}{4}x^{\nu_4},$$

$$y = \frac{1}{6}x^{\nu^1} + \frac{1}{12}x^{\nu_3} + (\frac{1}{2} - \frac{1}{12})x^{\nu^3} + \frac{1}{3}x^{\nu^4}.$$

Notice that the second term of each new representation also has the same scalar $\frac{1}{12}$. *If* $\gamma_3 = \min\{\frac{1}{2}, \frac{1}{2} - \frac{1}{12}\} = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}$, *then*

$$x = \frac{1}{6}x^{\nu_1} + \frac{1}{12}x^{\nu_1} + \frac{5}{12}x^{\nu^2} + (\frac{1}{2} - \frac{5}{12})x^{\nu_2} + \frac{1}{4}x^{\nu_4},$$

$$y = \frac{1}{6}x^{\nu^1} + \frac{1}{12}x^{\nu_3} + \frac{5}{12}x^{\nu^3} + \frac{1}{3}x^{\nu^4}.$$

Notice that the third term of each new representation also has the same scalar $\frac{5}{12}$. *If* $\gamma_4 = \min\{\frac{1}{2} - \frac{5}{12}, \frac{1}{3}\} = \frac{1}{2} - \frac{1}{12} = \frac{1}{12}$, *then*

$$x = \frac{1}{6}x^{\nu_1} + \frac{1}{12}x^{\nu_1} + \frac{5}{12}x^{\nu^2} + \frac{1}{12}x^{\nu_2} + \frac{1}{4}x^{\nu_4},$$

$$y = \frac{1}{6}x^{\nu^1} + \frac{1}{12}x^{\nu_3} + \frac{5}{12}x^{\nu^3} + \frac{1}{12}x^{\nu_4} + (\frac{1}{3} - \frac{1}{12})x^{\nu^4}.$$

Notice that the fourth term of each new representation also has the same scalar $\frac{1}{12}$. *If* $\gamma_4 = \min\{\frac{1}{4}, \frac{1}{3} - \frac{1}{12}\} = \min\{\frac{1}{4}, \frac{1}{4}\} = \frac{1}{4}$, *then*

$$x = \frac{1}{6}x^{\nu_1} + \frac{1}{12}x^{\nu_1} + \frac{5}{12}x^{\nu^2} + \frac{1}{12}x^{\nu_2} + \frac{1}{4}x^{\nu_4},$$

$$y = \frac{1}{6}x^{\nu^1} + \frac{1}{12}x^{\nu_3} + \frac{5}{12}x^{\nu^3} + \frac{1}{12}x^{\nu_4} + \frac{1}{4}x^{\nu^4}.$$

Notice that the fifth term of each new representation also has the same scalar $\frac{1}{4}$. *Now, once the OES-representation procedure is complete, both x and y have five terms in each representation. Moreover, both lotteries have the same scalar, term to term.*

In Appendix B the same example is used to illustrate the OES-representation procedure using Algorithm 2, detailing the procedure step by step.

It can now be stated that any two random stable matchings will have OES-representations.

Proposition 2 *If x and y are two random stable matchings then x and y have OES-representations.*

Proof. See proof in Appendix B. □

The following proposition presents a bridge between the partial order $\succeq_F \; (\succeq_W)$ over random stable matchings and the partial order $\geq_F^B \; (\geq_W^B)$ over the stable matchings involved in their OES-representations.

Consider x and y two random stable matchings with their OES-representations, i.e.

$$x = \sum_{\ell=1}^{\tilde{k}} \gamma_\ell \tilde{\mu}_\ell^x \quad \text{and} \quad y = \sum_{\ell=1}^{\tilde{k}} \gamma_\ell \tilde{\mu}_\ell^y,$$

where for each $\ell = 1, \ldots, \tilde{k}$, $0 < \gamma_\ell \leq 1$, $\sum_{\ell=1}^{\tilde{k}} \gamma_\ell = 1$, and for each $\ell = 1, \ldots, \tilde{k} - 1$ $\tilde{\mu}_\ell^x \geq_F^B \tilde{\mu}_{\ell+1}^x$ and $\tilde{\mu}_\ell^y \geq_F^B \tilde{\mu}_{\ell+1}^y$.

Proposition 3 *It is said that x weakly dominates y for all firms ($x \succeq_F y$) if and only if $\tilde{\mu}_\ell^x \geq_F^B \tilde{\mu}_\ell^y$ for each $\ell = 1, \ldots, \tilde{k}$. Analogously, $x \succeq_W y$ if and only if $\tilde{\mu}_\ell^x \geq_W^B \tilde{\mu}_\ell^y$ for each $\ell = 1, \ldots, \tilde{k}$.*

Proof. See proof in Appendix B. □

As a consequence of Proposition 3 and the polarization of interest of partial orders $\geq_F^B$ and $\geq_W^B$, the partial orders for random stable matching also have the property of polarization of interests. Formally,

Lemma 1 *If x and y are two random stable matchings, then*

$$x \succeq_F y \Longleftrightarrow y \succeq_W x.$$

Proof. See proof in Appendix B. □

5 Main result

This section presents the main result of the paper, proving that the random stable set of a many-to-many matching market endowed with the partial orders ($\succeq_F$ and $\succeq_W$) has a distributive, dual lattice structure. Moreover, given two random stable matchings, natural binary operations are defined for computing the *l.u.b.* and *g.l.b.* for each side of the market.

Recall that $\vee_W$, $\wedge_W$, $\vee_F$ and $\wedge_F$ are the binary operations relative to the partial orders $\geq_F^B$ and $\geq_W^B$ defined between two (deterministic) stable matchings. These binary operations are now extended to random stable matchings.

Given x and y two random stable matchings with their OES-representations,

$$x = \sum_{\ell=1}^{\tilde{k}} \gamma_\ell \tilde{\mu}_\ell^x \quad \text{and} \quad y = \sum_{\ell=1}^{\tilde{k}} \gamma_\ell \tilde{\mu}_\ell^y.$$

This study defines

$$\boldsymbol{x} \,\underline{\vee}_{\boldsymbol{F}}\, \boldsymbol{y} := \sum_{\ell=1}^{\tilde{k}} \gamma_\ell(\tilde{\mu}_\ell^x \vee_F \tilde{\mu}_\ell^y) \ , \quad \boldsymbol{x} \,\overline{\wedge}_{\boldsymbol{F}}\, \boldsymbol{y} := \sum_{\ell=1}^{\tilde{k}} \gamma_\ell(\tilde{\mu}_\ell^x \wedge_F \tilde{\mu}_\ell^y),$$

and

$$\boldsymbol{x} \,\underline{\vee}_{\boldsymbol{W}}\, \boldsymbol{y} := \sum_{\ell=1}^{\tilde{k}} \gamma_\ell(\tilde{\mu}_\ell^x \vee_W \tilde{\mu}_\ell^y) \ , \quad \boldsymbol{x} \,\overline{\wedge}_{\boldsymbol{W}}\, \boldsymbol{y} := \sum_{\ell=1}^{\tilde{k}} \gamma_\ell(\tilde{\mu}_\ell^x \wedge_W \tilde{\mu}_\ell^y).$$

Observe that, for each $\ell = 1, \ldots, \tilde{k}$, $0 < \gamma_\ell \leq 1$, $\sum_{\ell=1}^{\tilde{k}} \gamma_\ell = 1$, $\tilde{\mu}_\ell^x, \tilde{\mu}_\ell^y \in \mathcal{S}(\mathcal{P})$, and for each $\ell = 1, \ldots, \tilde{k}-1$, $\tilde{\mu}_\ell^x \geq_F \tilde{\mu}_{\ell+1}^x$ and $\tilde{\mu}_\ell^y \geq_F \tilde{\mu}_{\ell+1}^y$.

For each $\ell = 1, \ldots, \tilde{k}$ it follows that $\tilde{\mu}_\ell^x \wedge_F \tilde{\mu}_\ell^y$, $\tilde{\mu}_\ell^x \wedge_W \tilde{\mu}_\ell^y$, $\tilde{\mu}_\ell^x \vee_W \tilde{\mu}_\ell^y$, and $\tilde{\mu}_\ell^x \vee_F \tilde{\mu}_\ell^y$ are stable matchings, so $x \,\underline{\vee}_F\, y$, $x \,\overline{\wedge}_F\, y$, $x \,\underline{\vee}_W\, y$ and $x \,\overline{\wedge}_W\, y$ are random stable matchings. Now, it can be stated that these binary operations defined for random stable matchings are actually the *l.u.b.* and *g.l.b.* for each side of the market. Formally,

Proposition 4 *If x and y are two random stable matchings then for $X \in \{F, W\}$ it follows that*

$$x \,\underline{\vee}_X\, y = l.u.b._{\succeq_X}(x, y) \quad \text{and} \quad x \,\overline{\wedge}_X\, y = g.l.b._{\succeq_X}(x, y).$$

Also,

$$x \,\underline{\vee}_F\, y = x \,\overline{\wedge}_W\, y \quad \text{and} \quad x \,\underline{\vee}_W\, y = x \,\overline{\wedge}_F\, y.$$

Proof. See proof in Appendix B. □

The following proposition states that the binary operations for random stable matchings are distributive.

Proposition 5 *If x, y, and z are random stable matchings then for $X \in \{F, W\}$ it follows that*

$$x \,\underline{\vee}_X\, (y \,\overline{\wedge}_X\, z) = (x \,\underline{\vee}_X\, y) \,\overline{\wedge}_X\, (x \,\underline{\vee}_X\, z), \text{ and } x \,\overline{\wedge}_X\, (y \,\underline{\vee}_X\, z) = (x \,\overline{\wedge}_X\, y) \,\underline{\vee}_X\, (x \,\overline{\wedge}_X\, z).$$

Proof. See proof in Appendix B. □

Furthermore, with Propositions 4 and 5, it is possible to state the main result.

Theorem 2 *$(\mathcal{RS}(\mathcal{P}), \succeq_F, \underline{\vee}_F, \overline{\wedge}_F)$ and $(\mathcal{RS}(\mathcal{P}), \succeq_W, \underline{\vee}_W, \overline{\wedge}_W)$ are distributive and dual lattices.*

Remark 2 *Notice that if x and y are two rational random stable matchings (i.e. each α and each β in (1) are rational numbers) then for $X \in \{F, W\}$ it follows that*

$$x \,\underline{\vee}_X\, y = \sum_{k=1}^{e} \frac{1}{e}(\tilde{\mu}_k^x \vee_X \tilde{\mu}_k^y) \quad \text{and} \quad x \,\overline{\wedge}_X\, y = \sum_{k=1}^{e} \frac{1}{e}(\tilde{\mu}_k^x \wedge_X \tilde{\mu}_k^y).$$

The following example illustrates how to compute the binary operations for two random stable matchings.

Example 1 (Continued) *Given x and y represented by the OES-representation, $x \veebar_F y$ and $x \barwedge_F y$ are computed as follows (the other two binary operations are similar):*

$$x = \tfrac{1}{6}x^{\nu_1} + \tfrac{1}{12}x^{\nu_1} + \tfrac{5}{12}x^{\nu_2} + \tfrac{1}{12}x^{\nu_2} + \tfrac{1}{4}x^{\nu_4},$$

$$y = \tfrac{1}{6}x^{\nu_1} + \tfrac{1}{12}x^{\nu_3} + \tfrac{5}{12}x^{\nu_3} + \tfrac{1}{12}x^{\nu_4} + \tfrac{1}{4}x^{\nu_4}.$$

$$\begin{aligned} x \veebar_F y &= \tfrac{1}{6}x^{\nu_1 \vee_F \nu_1} + \tfrac{1}{12}x^{\nu_1 \vee_F \nu_3} + \tfrac{5}{12}x^{\nu_2 \vee_F \nu_3} + \tfrac{1}{12}x^{\nu_2 \vee_F \nu_4} + \tfrac{1}{4}x^{\nu_4 \vee_F \nu_4} \\ &= \tfrac{1}{6}x^{\nu_1} + \tfrac{1}{12}x^{\nu_1} + \tfrac{5}{12}x^{\nu_1} + \tfrac{1}{12}x^{\nu_2} + \tfrac{1}{4}x^{\nu_4} \\ &= \tfrac{2}{3}x^{\nu_1} + \tfrac{1}{12}x^{\nu_2} + \tfrac{1}{4}x^{\nu_4}. \\ x \barwedge_F y &= \tfrac{1}{6}x^{\nu_1 \wedge_F \nu_1} + \tfrac{1}{12}x^{\nu_1 \wedge_F \nu_3} + \tfrac{5}{12}x^{\nu_2 \wedge_F \nu_3} + \tfrac{1}{12}x^{\nu_2 \wedge_F \nu_4} + \tfrac{1}{4}x^{\nu_4 \wedge_F \nu_4} \\ &= \tfrac{1}{6}x^{\nu_1} + \tfrac{1}{12}x^{\nu_3} + \tfrac{5}{12}x^{\nu_4} + \tfrac{1}{12}x^{\nu_4} + \tfrac{1}{4}x^{\nu_4} \\ &= \tfrac{1}{6}x^{\nu_1} + \tfrac{1}{12}x^{\nu_3} + \tfrac{3}{4}x^{\nu_4}. \end{aligned}$$

6 Concluding remarks

Martınez et al. (2004) present an algorithm for computing the full set of many-to-many stable matchings. They consider many-to-many matching markets where agents have substitutable preferences. The algorithm obtains a finite sequence of stable matchings, starting from the *firm-optimal* stable matching (μ_F) and ending with the *worker-optimal* stable matching (μ_W) (which is also the firm-pessimal stable matching). First, they use the deferred acceptance algorithm to compute μ_F and μ_W. Then the algorithm identifies all firm–worker pairs (f, w) where firm f is matched with worker w in μ_F but not in μ_W. Successively, for each of these pairs, it modifies the preference of firm f by declaring all subsets of workers that contain worker w unacceptable but leaving the orderings of all subsets that do not contain w unchanged. For each new preference profile, they compute the firm-optimal stable matching and thus construct the sequence of stable matchings. The algorithm stops when there is no firm–worker pair (f, w) where firm f is matched with worker w in the firm-optimal stable matching (relative to the "reduced" preference profile) but not in μ_W. Notice that each sequence describes a "path" –from μ_F to μ_W– over the lattice of stable matchings relative to the partial order $\geq_F^B$. Thus, the full set of stable matchings is described by the set of these possible sequences. Moreover, each sequence shares the same property with our strictly ordered representation of a random stable matching. That is, the stable matchings of the sequence are strictly ordered in regard to the partial order $\geq_F^B$. Thus, each possible a lottery from among the stable matchings of a sequence –allowing a scalar of the representation to be zero– is a strictly ordered representation of random stable matching. Furthermore, Theorem 1 enables the full set of random stable matchings to be described as all possibles lotteries over each sequence. This considerably reduces the set of possible lotteries needed to describe the full set of random stable matchings.

Recall that the binary operations for random stable matchings are defined for the OES-representation of the random matchings. Notice that the OES-representation is also a lottery over stable matchings in a sequence. Now, given that each firm's preference relation ($>_f$) is substitutable, it emerges that $Ch(S \cup S', >_f) = Ch(Ch(S, >_f) \cup S', >_f)$ for each S and S' subsets of W. Thus, for each $f \in F$

$$
\begin{aligned}
&Ch\left((\mu_\ell^x \vee_F \mu_\ell^y)(f) \cup (\mu_{\ell+1}^x \vee_F \mu_{\ell+1}^y)(f), >_f\right) \\
&= Ch\left(Ch(\mu_\ell^x(f) \cup \mu_\ell^y(f)), >_f) \cup Ch(\mu_{\ell+1}^x(f) \cup \mu_{\ell+1}^y(f), >_f), >_f\right) \\
&= Ch\left(\mu_\ell^x(f) \cup \mu_\ell^y(f) \cup \mu_{\ell+1}^x(f) \cup \mu_{\ell+1}^y(f), >_f\right) \\
&= Ch\left(\mu_\ell^x(f) \cup \mu_{\ell+1}^x(f) \cup \mu_\ell^y(f) \cup \mu_{\ell+1}^y(f), >_f\right) \\
&= Ch\left(Ch(\mu_\ell^x(f) \cup \mu_{\ell+1}^x(f)), >_f) \cup Ch(\mu_\ell^y(f) \cup \mu_{\ell+1}^y(f), >_f), >_f\right) \\
&= Ch\left(\mu_\ell^x(f) \cup \mu_\ell^y(f, >_f), >_f\right) = (\mu_\ell^x \vee_F \mu_\ell^y)(f).
\end{aligned}
$$

Thus, $\mu_\ell^x \vee_F \mu_\ell^y \geq_F^B \mu_{\ell+1}^x \vee_F \mu_{\ell+1}^y$ for each $\ell = 1, \ldots, \tilde{k} - 1$. The proof for $\mu_\ell^x \wedge_F \mu_\ell^y \geq_F^B \mu_{\ell+1}^x \wedge_F \mu_{\ell+1}^y$ is analogous. Therefore, the *l.u.b.* and *g.l.b.* of two random stable matchings also have OES-representations. That is, *l.u.b.* and *g.l.b.* of two lotteries over the elements of a sequence, are also lotteries over elements of a sequence. Thus, some how this states that the set of sequences, when the lotteries over the stable matchings of each sequence are considered, also has a lattice structure.

Another fact that follows from this analysis is that firm-optimal random stable matching is the degenerated lottery that coincides with μ_F (the deterministic firm-optimal stable matching). The same goes for the worker-optimal random stable matching.

For more general matching markets, e.g. markets that only satisfy substitutability (not L.A.D.), the binary operations between (deterministic) stable matchings are computed as fixed points. Thus, the lattice structure of the set of random stable matchings for these markets is still an open problem and is left for future research.

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Appendix

A The ordered representation

The following technical results are used in the proof of Theorem 1.

Lemma 2 *$\mu_k \in B_k$ for each $k = 1, \ldots, \tilde{k}$.*

Proof. Let $\mu_1 = \nu^{\vee}_{B_1}$. Then, by definition of B_1, $\mu_1 \in B_1$. Let $\mu_2 = \nu^{\vee}_{B_2}$. Assuming that $\mu_2 \notin B_2$, then $\mu_2 \in C_2$. Since μ_2 is computed via pointing functions, there is $\nu' \in B_2$ such that $\nu' \in C_1$, which is a contradiction, so $\mu_2 \in B_2$. Similar arguments proves that $\mu_k \in B_k$ for each $k = 1, \ldots, \tilde{k}$, where $\tilde{k}$ is the last step of Algorithm 1. □

Lemma 3 *If $B_k \neq \emptyset$, then $B_{k+1} \subset B_k$.*

Proof. By definition of μ_k and C_k, $\mu_k \in B_k \cap C_k$. Thus, $B_{k+1} = B_k \setminus C_k \subset B_k$. □

Lemma 4 *Let $\tilde{\nu} = \nu^{\wedge}_{B_1}$ and $\tilde{k}$ be the step of Algorithm 1 in which $B_{\tilde{k}} \neq \emptyset$ and $B_{\tilde{k}+1} = \emptyset$. Thus, $\tilde{\nu} \in B_{\tilde{k}}$.*

Proof. Let $\tilde{\nu} = \nu^{\wedge}_{B_1}$ and $\tilde{k}$ be the step of Algorithm 1 in which $B_{\tilde{k}} \neq \emptyset$ and $B_{\tilde{k}+1} = \emptyset$. By definition of B_1, $\tilde{\nu} \in B_1$. Assume that $\tilde{\nu} \notin B_{\tilde{k}}$, there is thus a Step $k' < \tilde{k}$ such that $\tilde{\nu} \in B_{k'}$ and $\tilde{\nu} \notin B_{k'+1}$. Thus, $\tilde{\nu} \in C_{k'}$. Hence, by definition of $C_{k'}$, there is a pair (i', j') such that $x^{k'}_{i',j'} = \alpha_{k'}$ and $x^{\tilde{\nu}}_{i',j'} = x^{\mu_{k'}}_{i',j'} = 1$. Notice that, by definition of $\mu_{k'}$, it holds that $j' \in \mu_{k'}(i') = Ch(\cup_{\nu \in B_{k'}} \nu(i'), >_{i'})$. Since the preferences relation $>_{i'}$ is substitutable and $\tilde{\nu} \in B_{k'}$, it follows that

$$j' \in Ch(\tilde{\nu}(i') \cup \{j'\}, >_{i'}). \tag{4}$$

By Lemma 3 and $k' < \tilde{k}$, it holds that $B_{k'+1} \neq \emptyset$. Thus, there is $\nu' \in B_{k'}$ such that $\nu' \notin C_{k'}$. We claim that $j' \notin \nu'(i')$. If $j' \in \nu'(i')$, for (i', j') then $x^{k'}_{i',j'} = \alpha_{k'}$ and $x^{\nu^{\tilde{k}}}_{i',j'} = x^{\mu_{k'}}_{i',j'} = 1$, then $\nu' \in C_{k'}$, which is a contradiction. Thus, $j' \notin \nu'(i')$. Since $\nu' \in B_{k'} \subseteq B_1$, then $\nu' \geq^{B}_{F} \tilde{\nu}$. That is, $\nu'(i') = Ch(\tilde{\nu}(i') \cup \nu'(i'), >_{i'})$. Now, given that $j' \in \tilde{\nu}(i') \setminus \nu'(i')$, it follows that $j' \notin Ch(\tilde{\nu}(i') \cup \{j'\}, >_{i'})$, which is a contradiction with (4). Therefore, $\tilde{\nu} \in B_{\tilde{k}}$. □

To prove the following lemma, first we need to state an important result of matching theory. The *Rural Hospital Theorem* (**RHT** from now on), is proven in different context by many authors (see McVitie and Wilson, 1970; Roth, 1984, 1985; Martínez et al., 2000; Alkan, 2002; Kojima, 2012, among others). The version of this theorem for a many-to-many matching market where all agents have substitutable preferences satisfying the L.A.D. presented in Alkan (2002) states that each agent is matched with the same number of partners in every stable matching. That is, $|\mu(a)| = |\mu'(a)|$ for each $\mu, \mu' \in \mathcal{S}(\mathcal{P})$ and for each $a \in F \cup W$.

Lemma 5 *Let $\mu \in \mathcal{S}(\mathcal{P})$, let x^1 be a random stable matching, and $x^k = \frac{x^{k-1} - \alpha_{k-1} x^{\mu_{k-1}}}{1-\alpha_{k-1}}$ be the matrix constructed by Algorithm 1 in Step k. Then, for each k, $\sum_{i \in F} x^k_{i,j} = |\mu(j)|$ for each $j \in W$, and $\sum_{j \in W} x^k_{i,j} = |\mu(i)|$ for each $i \in F$.*

Proof. We proceed by an inductive process. Let $\mu \in \mathcal{S}(\mathcal{P})$ and let $k = 1$ be the first step of Algorithm 1. If $B_2 = \emptyset$, then $B_1 = C_1$. That is, $\tilde{\nu} \in C_1$. Hence, there is $(i,j) \in \mathcal{L}_1$ such that $x^{\mu_1}_{i,j} = 1, x^{\tilde{\nu}}_{i,j} = 1$ and $x^1_{i,j} = \alpha_1$. Thus, for each $\nu \in B_1$ such that $\mu_1 \geq^B_F \nu \geq^B_F \tilde{\nu}$ it follows that $x^\nu_{i,j} = 1$. Hence, $\alpha_1 = 1$. Since $\{(i,j) : x^{\mu_1}_{i,j} > 0\} \subseteq \{(i,j) : x^1_{i,j} > 0\}$ and $\alpha_1 = \min\{x^1_{i,j} : x^{\mu_1}_{i,j} = 1\}$, then $x^1 = x^{\mu_1}$. Thus, by Theorem (RHT) and definition of incidence vector, $\sum_{i \in F} x^{\mu_1}_{i,j} = |\mu(j)|$ for each $j \in W$, and $\sum_{j \in W} x^{\mu_1}_{i,j} = |\mu(i)|$ for each $i \in F$.

Assume that $B_2 \neq \emptyset$ and $\sum_{i \in F} x^{k-1}_{i,j} = |\mu(j)|$ for each $j \in W$. Thus, then by Theorem (RHT) and definition of x^k

$$\sum_{i \in F} x^k_{i,j} = \frac{\sum_{i \in F} x^{k-1}_{i,j} - \alpha_{k-1} \sum_{i \in F} x^{\mu_{k-1}}_{i,j}}{1 - \alpha_{k-1}} = \frac{|\mu(j)| - \alpha_{k-1}|\mu(j)|}{1 - \alpha_{k-1}} = |\mu(j)|.$$

Therefore, $\sum_{i \in F} x^k_{i,j} = |\mu(j)|$ for each $j \in W$ and for each $k = 1, \ldots, \tilde{k}$. Similarly, it can be proved that $\sum_{j \in W} x^k_{i,j} = |\mu(i)|$ for each $i \in F$ and for each $k = 1, \ldots, \tilde{k}$. □

Lemma 6 *$B_{k+1} \neq \emptyset$ if and only if $\alpha_k < 1$.*

Proof. ($\Longrightarrow$) Let $B_{k+1} \neq \emptyset$. Thus $B_k \neq C_k$. Hence, $|B_k| > 1$. By Lemma 4, $\tilde{\nu} \in B_k$. Also, by definition of μ_k, it follows that $\tilde{\nu} \neq \mu_k$. Thus, by Theorem (RHT), there are at least three agents $i' \in F$ and $\tilde{j}, j' \in W$ such that:

$$x^k_{i',j'} > 0, \;\; x^k_{i',\tilde{j}} > 0, \;\; x^{\mu_k}_{i',j'} = 1, \;\; x^{\mu_k}_{i',\tilde{j}} = 0, \;\; x^{\tilde{\nu}}_{i',j'} = 0, \text{ and } \;\; x^{\tilde{\nu}}_{i',\tilde{j}} = 1.$$

By Lemma 5, $\sum_{j \in W} x^k_{i',j} = |\mu_k(i')| = |\tilde{\nu}(i')|$. Since $\{(i,j) : x^{\mu_k}_{i,j} > 0\} \subset \{(i,j) : x^k_{i,j} > 0\}$, and $\{(i,j) : x^{\tilde{\nu}}_{i,j} > 0\} \subset \{(i,j) : x^k_{i,j} > 0\}$, it follows that $|\{j \in W : x^k_{i',j} > 0\}| > |\mu_k(i')|$. There is thus an agent $\hat{j} \in W$ such that $x^{\mu_k}_{i',\hat{j}} = 1$ and $0 < x^k_{i',\hat{j}} < 1$. Thus, $\alpha_k = \min\{x^k_{i,j} : x^{\mu_k}_{i,j} = 1\} \leq x^k_{i',\hat{j}} < 1$.

($\Longleftarrow$) Let $\alpha_k < 1$. Thus, there is a pair (i', j') such that $x^{\mu_k}_{i',j'} = 1$ and $x^k_{i',j'} = \alpha_k < 1$. Then, by Lemma 5 there is a pair $(i', \tilde{j})$ such that $x^{\mu_k}_{i',\tilde{j}} = 0$, $x^k_{i',\tilde{j}} > 0$ and $x^{\tilde{\nu}}_{i',\tilde{j}} = 1$. Hence, for each pair (i,j) such that $x^{\mu_k}_{i,j} = 1$ and $x^{\tilde{\nu}}_{i,j} = 1$, by Theorem (RHT) it follows that $x^\nu_{i,j} = 1$ for each $\nu \in B_k$. Thus, $x^k_{i,j} \geq x^k_{i',\tilde{j}} + x^k_{i',j'} = x^k_{i',\tilde{j}} + \alpha_k > \alpha_k$, Then, $\tilde{\nu} \notin C_k$. Therefore, $B_{k+1} \neq \emptyset$. □

Corollary 1 *If $\alpha_k = 1$, then $x^k = x^{\mu_k}$.*

Proof. Let $\mathcal{L}_k = \{(i,j) \in F \times W : x^k_{i,j} = \alpha_k \text{ and } x^{\mu_k}_{i,j} = 1\}$ and recall that by definition of μ_k, it follows that $\{(i,j) : x^{\mu_k}_{i,j} > 0\} \subseteq \{(i,j) : x^k_{i,j} > 0\}$. If $\alpha_k = 1$, then $\mathcal{L}_k = \{(i,j) : x^{\mu_k}_{i,j} > 0\}$. By Lemma 5, $\sum_{i \in F} x^k_{i,j} = |\mu_k(j)|$ for each $i \in F$, so $\{(i,j) : x^k_{i,j} > 0\} = \{(i,j) : x^{\mu_k}_{i,j} > 0\}$. Therefore, $x^k = x^{\mu_k}$. □

Proof of Theorem 1. Let x be a random stable matching. The output of Algorithm 1 is

$$x = \alpha_1 x^{\mu_1} + \sum_{s=2}^{k} \prod_{\ell=1}^{s-1} (1-\alpha_\ell)\alpha_s x^{\mu_s} + \prod_{\ell=1}^{k}(1-\alpha_\ell) x^{k+1}.$$

By Lemma 3, $B_{k+1} \subset B_k$. By the finiteness of the set of stable matchings, there is a step of Algorithm 1, say Step $\tilde{k}$, such that $B_{\tilde{k}+1} = \emptyset$. Then the algorithm stops. Hence, by Lemma 6 $\alpha_{\tilde{k}} = 1$. Therefore, by Corollary 1, $x^{\tilde{k}} = x^{\mu_{\tilde{k}}}$. Thus, the output of Algorithm 1 is

$$x = \alpha_1 x^{\mu_1} + \sum_{s=2}^{\tilde{k}} \prod_{\ell=1}^{s-1} (1-\alpha_\ell)\alpha_s x^{\mu_s}.$$

Recall that $\mu_k = \nu^{\vee}_{B_k}$, and $\mu_{k+1} = \nu^{\vee}_{B_{k+1}}$. By Lemma 3, $B_{k+1} \subset B_k$, by Lemma 2 $\mu_k \in B_k$, and by definition of C_k, $\mu_k \in C_k$. Hence, $\mu_k \notin B_{k+1}$. Thus, $\mu_k >^B_F \mu_{k+1}$.

To simplify the notation, set $\beta_1 = \alpha_1,\ \beta_2 = (1-\alpha_1)\alpha_2,\ \beta_3 = (1-\alpha_1)(1-\alpha_2)\alpha_3, \ldots,$ and $\beta_{\tilde{k}} = \prod_{k=1}^{\tilde{k}-1}(1-\alpha_k)$.

Now we prove that $\sum_{k=1}^{\tilde{k}} \beta_k = 1$.

$$\sum_{k=1}^{\tilde{k}} \beta_k = \sum_{k=1}^{\tilde{k}-1} \beta_k + \beta_{\tilde{k}} = \sum_{k=1}^{\tilde{k}-1} \prod_{\ell=1}^{k-1}(1-\alpha_\ell) + \prod_{\ell=1}^{\tilde{k}-1}(1-\alpha_\ell).$$

Note that

$$\beta_{\tilde{k}-1} + \beta_{\tilde{k}} = \prod_{\ell=1}^{\tilde{k}-2}(1-\alpha_\ell)\alpha_{\tilde{k}-1} + \prod_{\ell=1}^{\tilde{k}-1}(1-\alpha_\ell) = \prod_{\ell=1}^{\tilde{k}-2}(1-\alpha_\ell)(\alpha_{\tilde{k}-1} + (1-\alpha_{\tilde{k}-1})) = \prod_{\ell=1}^{\tilde{k}-2}(1-\alpha_\ell).$$

Also,

$$\beta_{\tilde{k}-2} + \beta_{\tilde{k}-1} + \beta_{\tilde{k}} = \prod_{\ell=1}^{\tilde{k}-3}(1-\alpha_\ell)\alpha_{\tilde{k}-2} + \prod_{\ell=1}^{\tilde{k}-3}(1-\alpha_\ell)(1-\alpha_{\tilde{k}-2}) = \prod_{\ell=1}^{\tilde{k}-3}(1-\alpha_\ell).$$

Continuing this inductive process, $\beta_2 + \cdots + \beta_{\tilde{k}} = (1-\alpha_1)$. Thus,

$$\sum_{k=1}^{\tilde{k}} \beta_k = \beta_1 + \sum_{k=2}^{\tilde{k}} \beta_k = \alpha_1 + (1-\alpha_1) = 1.$$

Therefore,

$$x = \sum_{k=1}^{\tilde{k}} \beta_k x^{\mu_k}$$

where $0 < \beta_k \leq 1$, $\sum_{k=1}^{\tilde{k}} \beta_k = 1$, and $\mu_k >_F^B \mu_{k+1}$ for each $k = 1, \ldots, \tilde{k} - 1$.

Uniqueness: Assume that x has two different representations:

$$x = \sum_{\nu \in A} \lambda_\nu x^\nu = \sum_{\nu' \in A'} \lambda'_{\nu'} x^{\nu'}$$

where $0 < \lambda_\nu \leq 1$, $0 < \lambda'_{\nu'} \leq 1$, $\sum_{\nu \in A} \lambda_\nu = 1$, $\sum_{\nu' \in A'} \lambda'_{\nu'} = 1$, and $\nu, \nu' \in \mathcal{S}(\mathcal{P})$.

Since, $\bigcup_{\nu \in A} \nu(i) = \{j : x_{i,j} > 0\} = \bigcup_{\nu' \in A'} \nu'(i)$, so $\mu_1(i) = Ch(\bigcup_{\nu \in B_1} \nu(i), >_i) = Ch(\bigcup_{\nu' \in B'_1} \nu'(i), >_i) = \mu'_1(i)$ for each $i \in F$. Therefore, $\mu_1 = \mu'_1$.

Let $k > 1$ such that $\mu_1 = \mu'_1, \ldots, \mu_{k-1} = \mu'_{k-1}$. Then, $x^k = \frac{x^{k-1} - \alpha_{k-1} x^{\mu_{k-1}}}{1 - \alpha_{k-1}} = \frac{x^{k-1} - \alpha_{k-1} x^{\mu'_{k-1}}}{1 - \alpha_{k-1}}$.

We claim that $\{(i,j) : x^k_{i,j} > 0\} = \{(i,j) : \bigcup_{\nu \in B_k} x^\nu_{i,j} = 1\}$ (and $\{(i,j) : x^k_{i,j} > 0\} = \{(i,j) : \bigcup_{\nu' \in B'_k} x^{\nu'}_{i,j} = 1\}$). If not, there is a pair (i,j) such that $x^k_{i,j} > 0$ and $x^\nu_{i,j} = 0$ for each $\nu \in B_k$. Thus, $x^{\tilde{k}}_{i,j} > 0$ and there is no $\nu \in B_{\tilde{k}}$ such that $x^\nu_{i,j} = 1$. This contradicts the fact that $x^{\tilde{k}}_{i,j} = x^{\mu_{\tilde{k}}}_{i,j}$. Hence, $\{(i,j) : \bigcup_{\nu \in B_k} x^\nu_{i,j} = 1\} \supseteq \{(i,j) : x^k_{i,j} > 0\}$.

Assume that there is $\nu \in B_k$ such that $x^\nu_{i,j} = 1$ and $x^k_{i,j} = 0$. Since $x^\nu_{i,j} = 1$, $x_{i,j} > 0$. Hence, there is $k' < k$ such that $x^{k'}_{i,j} > 0$ and $x^{k'+1}_{i,j} = 0$. Since $x^{k'+1}_{i,j} = \frac{x^{k'}_{i,j} - \alpha_{k'} x^{\mu_{k'}}_{i,j}}{1 - \alpha_{k'}} = 0$ it follows that $x^{k'}_{i,j} = \alpha_{k'} x^{\mu_{k'}}_{i,j}$. Hence, $(i,j) \in \mathcal{L}_{k'}$ and $\nu \in C_{k'}$ because we assume that $x^\nu_{i,j} = 1$. Thus, $\nu \notin B_{k'+1}$, and since $k' + 1 \leq k$ it follows that $B_{k'+1} \subseteq B_k$. This in turns implies that $\nu \notin B_k$, which is a contradiction. Therefore, $\{(i,j) : \bigcup_{\nu \in B_k} x^\nu_{i,j} = 1\} \subseteq \{(i,j) : x^k_{i,j} > 0\}$.

Similar arguments prove that $\{(i,j) : x^k_{i,j} > 0\} = \{(i,j) : \bigcup_{\nu' \in B'_k} x^{\nu'}_{i,j} = 1\}$.

Since $\bigcup_{\nu \in B_k} \nu(i) = \{j : x^k_{i,j} > 0\} = \bigcup_{\nu' \in B'_k} \nu'(i)$, it follows that $\mu_k(i) = Ch(\bigcup_{\nu \in B_k} \nu(i), \geq_i) = Ch(\{j : x^k_{i,j} > 0\}, >_i) = Ch(\bigcup_{\nu' \in B'_k} \nu'(i), >_i) = \mu'_k(i)$ for each $i \in F$. Therefore, $\mu_k = \mu'_k$.

□

B Partial order and OES-representation

Proof of partial order

Proof of Proposition 1. Let x, y and z be random stable matchings with their strictly ordered representations:

$$x = \sum_{i=1}^{I} \alpha_i x^{\mu_i^x},\ y = \sum_{j=1}^{J} \beta_j x^{\mu_j^y} \text{ and } z = \sum_{k=1}^{K} \gamma_k x^{\mu_k^z}.$$

Reflexivity: $x \succeq_F x$.

By the uniqueness of the strictly ordered representation of x, it follows that for each μ_k^x

$$\sum_{\left\{i:\mu_i^x \geq_F^B \mu_k^x\right\}} \alpha_l \geq \sum_{\left\{i:\mu_i^x \geq_F^B \mu_k^x\right\}} \alpha_l.$$

Transitivity: If $x \succeq_F y$ and $y \succeq_F z$, then $x \succeq_F z$.

Since $y \succeq_F z$, it follows that $\sum_{\left\{l:\mu_l^y \geq_F^B \mu_k^z\right\}} \beta_l \geq \sum_{\left\{n:\mu_n^z \geq_F^B \mu_k^z\right\}} \gamma_n$ for each μ_k^z. Since $x \succeq_F y$, it follows that $\sum_{\left\{i:\mu_i^x \geq_F^B \mu_j^y\right\}} \alpha_i \geq \sum_{\left\{l:\mu_l^y \geq_F^B \mu_j^y\right\}} \beta_l$ for each μ_j^y. Recall that x, y and z are represented by the strictly ordered representations. Then, for each μ_k^z there is an unique $\mu_j^y = \min_{\geq_F^B}\{\mu_l^y : \mu_l^y \geq_F^B \mu_k^z\}$ such that

$\sum_{\left\{m:\mu_m^x \geq_F^B \mu_k^z\right\}} \alpha_m = \sum_{\left\{i:\mu_i^x \geq_F^B \mu_j^y\right\}} \alpha_i$	by $\{\mu_l^y : \mu_l^y \geq_F^B \mu_j^y\} = \{\mu_l^y : \mu_l^y \geq_F^B \mu_k^z\}$
$\sum_{\left\{i:\mu_i^x \geq_F^B \mu_j^y\right\}} \alpha_i \geq \sum_{\left\{l:\mu_l^y \geq_F^B \mu_j^y\right\}} \beta_l$	by $x \succeq_F y$
$\sum_{\left\{l:\mu_l^y \geq_F^B \mu_j^y\right\}} \beta_l = \sum_{\left\{l:\mu_l^y \geq_F^B \mu_k^z\right\}} \beta_l$	by $\{\mu_m^x : \mu_m^x \geq_F^B \mu_k^z\} = \{\mu_m^x : \mu_m^x \geq_F^B \mu_j^y\}$
$\sum_{\left\{l:\mu_l^y \geq_F^B \mu_k^z\right\}} \beta_l \geq \sum_{\left\{n:\mu_n^z \geq_F^B \mu_k^z\right\}} \gamma_n$	by $y \succeq_F z$

Hence, for each μ_k^z

$$\sum_{\left\{m:\mu_m^x \geq_F^B \mu_k^z\right\}} \alpha_m \geq \sum_{\left\{n:\mu_n^z \geq_F^B \mu_k^z\right\}} \gamma_n.$$

Therefore, $x \succeq_F z$.

Antisymmetry: If $x \succeq_F y$ and $y \succeq_F x$, then $x = y$.

Assume that $x \succeq_F y$ and $x \neq y$. We then prove that $y \not\succeq_F x$. By definition of $x \succeq_F y$ it follows that $x \succeq_f y$ for each $f \in F$. Since $x \neq y$, there is at least one $f' \in F$ such that $x \succ_{f'} y$. Hence, by definition of $x \succ_{f'} y$, there is $\mu_j^y(f')$ such that

$$\sum_{\left\{i:\mu_i^x(f') \geq_{f'}^B \mu_j^y(f')\right\}} \alpha_i > \sum_{\left\{l:\mu_l^y(f') \geq_{f'}^B \mu_j^y(f')\right\}} \beta_l.$$

Then, $y \not\succeq_{f'} x$, which in turns implies that $y \not\succeq_F x$.

Therefore, the domination relation $\succeq_F$ is a partial order. $\square$

Algorithm 2

Let x and y be two random stable matchings such that

$$x = \sum_{i=1}^{I} \alpha_i^0 \mu_i^x \qquad \text{and} \qquad y = \sum_{j=1}^{J} \beta_j^0 \mu_j^y.$$

where $0 < \alpha_i^0 \leq I$ for $i = 1, \ldots, I$, $0 < \beta_j^0 \leq J$ for $j = 1, \ldots, J$, $\sum_{i=1}^{I} \alpha_i^0 = 1$ and $\sum_{j=1}^{J} \beta_j^0 = 1$.

Let $I^0 = \{1, \ldots, I\}$ and $J^0 = \{1, \ldots, J\}$. Set $\Omega = \emptyset$.

Algorithm 2:

Step $k \geq 1$ `IF` $|I^{k-1}| = 1$ and $|J^{k-1}| = 1$,

`THEN`, the procedure stops.

Set, $\gamma_k = \alpha_1^{k-1} = \beta_1^{k-1}$, $\tilde{\mu}_k^x = \mu_I^x$, $\tilde{\mu}_k^y = \mu_J^y$.

Set $\Omega = \Omega \cup \{(\gamma_k, \tilde{\mu}_k^x, \tilde{\mu}_k^y)\}$.

`ELSE` ($|I^{k-1}| > 1$ or $|J^{k-1}| > 1$), the procedure continues as follows:

Set $\gamma_k = \min\{\alpha_1^{k-1}, \beta_1^{k-1}\}$.

`IF` $\gamma_k \neq \alpha_1^{k-1}$,

`THEN`, set $I^k := I^{k-1}$ and $\alpha_\ell^k := \begin{cases} \alpha_1^{k-1} - \gamma_k & \text{if } \ell = 1 \\ \alpha_\ell^{k-1} & \text{if } \ell > 1 \end{cases}$,

for each $\ell \in I^{k-1}$.

`ELSE` ($\gamma_k = \alpha_1^{k-1}$), set $I^k := I^{k-1} \setminus \{\max I^{k-1}\}$ and $\alpha_{\ell-1}^k = \alpha_\ell^{k-1}$

for each $\ell \in I^{k-1}$.

`IF` $\gamma_k \neq \beta_1^{k-1}$,

`THEN`, $J^k := J^{k-1}$ and $\beta_\ell^k := \begin{cases} \beta_1^{k-1} - \gamma_k & \text{if } \ell = 1 \\ \beta_\ell^{k-1} & \text{if } \ell > 1 \end{cases}$,

for each $\ell \in J^{k-1}$.

`ELSE` ($\gamma_k = \beta_1^{k-1}$), set $J^k := J^{k-1} \setminus \{\max J^{k-1}\}$ and $\beta_{\ell-1}^k = \beta_\ell^{k-1}$

for each $\ell \in J^{k-1}$.

Set $p = |I^0| - |I^{k-1}|$ and $r = |J^0| - |J^{k-1}|$.

Set $\tilde{\mu}_k^x = \mu_{p+1}^x$ and $\tilde{\mu}_k^y = \mu_{r+1}^y$.

Set $\Omega = \Omega \cup \{(\gamma_k, \tilde{\mu}_k^x, \tilde{\mu}_k^y)\}$, and continue to Step k+1.

Notice that the procedure for more than two random stable matching is analogous.

Lemma 7 *Algorithm 2 stops in a finite number of steps. That is, there is a $\tilde{k}$ such that $|I^{\tilde{k}-1}| = |J^{\tilde{k}-1}| = 1$ and $\alpha_1^{\tilde{k}} = \beta_1^{\tilde{k}}$.*

Proof. Note that in each step of Algorithm 2, $|I^k| = |I^{k-1}| - 1$ or $|J^k| = |J^{k-1}| - 1$. Moreover, in each Step k of the algorithm

$$\sum_{\ell \in I^k} \alpha_\ell^k = \sum_{\ell \in I^{k-1}} \alpha_\ell^{k-1} - \gamma_k \quad \text{and} \quad \sum_{\ell \in J^k} \beta_\ell^k = \sum_{\ell \in J^{k-1}} \beta_\ell^{k-1} - \gamma_k.$$

Hence,

$$\sum_{\ell \in I^k} \alpha_\ell^k = \sum_{\ell \in I^0} \alpha_\ell^0 - \sum_{t=1}^{k} \gamma_t = 1 - \sum_{t=1}^{k} \gamma_t.$$

Similarly,

$$\sum_{\ell \in J^k} \beta_\ell^k = \sum_{\ell \in J^0} \beta_\ell^0 - \sum_{t=1}^{k} \gamma_t = 1 - \sum_{t=1}^{k} \gamma_t.$$

That is, for each k it follows that

$$\sum_{\ell \in I^k} \alpha_\ell^k = \sum_{\ell \in J^k} \beta_\ell^k = 1 - \sum_{t=1}^{k} \gamma_t. \tag{5}$$

By the finiteness of the sets I^0 and J^0, and given that in each step of Algorithm 2 it holds that $|I^k| = |I^{k-1}| - 1$ or $|J^k| = |J^{k-1}| - 1$. We claim that there is a $\tilde{k}$ such that $|I^{\tilde{k}-1}| = |J^{\tilde{k}-1}| = 1$. Assume that there is a Step $k_1 - 1$ such that $|I^{k_1-1}| = 1$ and $|J^{k_1-1}| > 1$. By equality (5), $\alpha_1^{k_1-1} = \sum_{\ell \in J^{k_1-1}} \beta_\ell^{k_1-1}$. Hence, $\alpha_1^{k_1-1} > \beta_\ell^{k_1-1}$ for each $\ell \in J^{k_1-1}$, and $|I^{k_1}| = |I^{k_1-1}|$. Thus, $\alpha_{k_1} = \alpha_1^{k_1-1} - \gamma_{k_1} = \alpha_1^{k_1-1} - \beta_1^{k_1-1}$ and $J^{k_1} = J^{k_1-1} \setminus \{\max J^{k_1-1}\}$, and $\beta_\ell^{k_1} = \beta_{\ell+1}^{k_1-1}$ for each $\ell \in J^{k_1}$. Thus, $|I^{k_1-1}| = |I^{k_1}| = 1$ and $|J^{k_1}| = |J^{k_1-1}| - 1 \geq 1$. If $|J^{k_1}| > 1$, then proceed with Algorithm 2 until there is a step $\tilde{k}$ such that $|I^{\tilde{k}-1}| = |J^{\tilde{k}-1}| = 1$ and the procedure stops. Therefore, by equality (5), $\alpha_1^{\tilde{k}} = \beta_1^{\tilde{k}} = \gamma_{\tilde{k}}$. □

Proof of Proposition 2. First we prove that there is k_1 such that $\alpha_1^0 = \sum_{t=1}^{k_1} \gamma_t$. Since $\gamma_1 = \min\{\alpha_1^0, \beta_1^0\}$, we analyze two cases.

Case 1: $\gamma_1 = \alpha_1^0$. In this case $k_1 = 1$.

Case 2: $\gamma_1 < \alpha_1^0$. In this case $|I^0| = |I^1|$ and $\alpha_1^1 = \alpha_1^0 - \gamma_1$. Thus, in the next step, $\gamma_2 \leq \alpha_1^1$.

If $\gamma_2 = \alpha_1^1$, then $\alpha_1^0 = \gamma_1 + \gamma_2$.

If $\gamma_2 < \alpha_1^1$, then repeat this procedure until k_1 is found such that $\gamma_{k_1} = \alpha_1^{k_1-1}$. Then $\alpha_1^0 = \sum_{t=1}^{k_1} \gamma_t$. Note that $|I^0| = |I^1| = \ldots = |I^{k_1}|$. Then, we have that $\tilde{\mu}_t^x = \mu_1^x$ for $t = 1, \ldots, k_1$ and

$$\sum_{t=1}^{k_1} \gamma_t \tilde{\mu}_t^x = \sum_{t=1}^{k_1} \gamma_t \mu_1^x = \alpha_1^0 \mu_1^x.$$

Notice that $|I^{k_1}| = |I^{k_1-1}| - 1$. That is, $1 = p = |I^0| - |I^{k_1}|$ and $\tilde{\mu}^x_{k_1+1} = \mu^x_2$. Then, for each $k \geq k_1 + 1$ it emerges that $\tilde{\mu}^x_k \neq \mu^x_1$.

When k_1 is found this procedure must be repeated with each α^0_ℓ for $\ell \geq 2$.

The case for β is similar. □

We illustrate Algorithm 2 with two random matchings from Example 1.

Example 1 (Continued) *Let $x = \frac{1}{4}x^{\nu_1} + \frac{1}{2}x^{\nu_2} + \frac{1}{4}x^{\nu_4}$ and $y = \frac{1}{6}x^{\nu^1} + \frac{1}{2}x^{\nu^3} + \frac{1}{3}x^{\nu^4}$. Notice that both random stable matchings are represented as in Theorem 1. We use Algorithm 2 to obtain their OES-representations. Let $I^0 = \{1,2,3\}$ and $J^0 = \{1,2,3\}$. Set $\Omega = \emptyset$.*

Step 1 *Since $I^0 = \{1,2,3\}$ and $J^0 = \{1,2,3\}$, set $\gamma_1 = \min\{\frac{1}{4}, \frac{1}{6}\} = \frac{1}{6}$,*

$$\left| \begin{array}{l} \alpha^1_1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12} \\ \alpha^1_2 = \frac{1}{2} \\ \alpha^1_3 = \frac{1}{4} \end{array} \right. \qquad \left| \begin{array}{l} \beta^1_1 = \frac{1}{2} \\ \beta^1_2 = \frac{1}{3} \end{array} \right.$$

Then, $I^1 = \{1,2,3\}$, $J^1 = \{1,2\}$, $\tilde{\mu}^x_1 = \nu_1$ and $\tilde{\mu}^y_1 = \nu_1$. Set $\Omega = \Omega \cup \{(\nu_1, \nu_1, \frac{1}{6})\}$ and continue to Step 2.

Step 2 *Since $I^1 = \{1,2,3\}$, $J^1 = \{1,2\}$, set $\gamma_2 = \min\{\frac{1}{12}, \frac{1}{2}\} = \frac{1}{12}$,*

$$\left| \begin{array}{l} \alpha^2_1 = \frac{1}{2} \\ \alpha^2_2 = \frac{1}{4} \end{array} \right. \qquad \left| \begin{array}{l} \beta^2_1 = \frac{1}{2} - \frac{1}{12} = \frac{5}{12} \\ \beta^2_2 = \frac{1}{3} \end{array} \right.$$

Then, $I^2 = \{1,2\}$, $J^2 = \{1,2\}$, $\tilde{\mu}^x_2 = \nu_1$ and $\tilde{\mu}^y_2 = \nu_3$. Set $\Omega = \Omega \cup \{(\nu_1, \nu_2, \frac{1}{12})\}$ and continue to Step 3.

Step 3 *Since $I^2 = \{1,2\}$, $J^2 = \{1,2\}$, set $\gamma_3 = \min\{\frac{1}{2}, \frac{5}{12}\} = \frac{5}{12}$,*

$$\left| \begin{array}{l} \alpha^3_1 = \frac{1}{2} - \frac{5}{12} = \frac{1}{12} \\ \alpha^3_2 = \frac{1}{4} \end{array} \right. \qquad \left| \begin{array}{l} \beta^3_1 = \frac{1}{4} \end{array} \right.$$

Then, $I^3 = \{1,2\}$, $J^3 = \{1\}$, $\tilde{\mu}^x_3 = \nu_2$ and $\tilde{\mu}^y_3 = \nu_3$. Set $\Omega = \Omega \cup \{(\nu_2, \nu_3, \frac{5}{12})\}$ and continue to Step 4.

Step 4 *Since $I^3 = \{1,2\}$, $J^3 = \{1\}$, $\tilde{\mu}^x_3 = \nu_2$, set $\gamma_4 = \min\{\frac{1}{12}, \frac{1}{3}\} = \frac{1}{12}$,*

$$\left| \begin{array}{l} \alpha^4_1 = \frac{1}{4} \end{array} \right. \qquad \left| \begin{array}{l} \beta^4_1 = \frac{1}{3} - \frac{1}{12} = \frac{1}{4} \end{array} \right.$$

Then, $I^4 = \{1\}$, $J^4 = \{1\}$, $\tilde{\mu}^x_4 = \nu_2$ and $\tilde{\mu}^y_4 = \nu_4$. Set $\Omega = \Omega \cup \{(\nu_2, \nu_4, \frac{1}{12})\}$ and continue to Step 5.

Step 5 *Since $I^4 = \{1\}$, $J^4 = \{1\}$, so the procedure stops. Set $\gamma_6 = min\{\frac{1}{4}, \frac{1}{4}\} = \frac{1}{4}$, $\tilde{\mu}^x_6 = \nu_4$ and $\tilde{\mu}^y_6 = \nu_4$. Set $\Omega = \Omega \cup \{(\nu_4, \nu_4, \frac{1}{4})\}$*

Therefore, the random stable matchings x and y can be represented as follows:

$$x = \tfrac{1}{6}x^{\nu_1} + \tfrac{1}{12}x^{\nu_1} + \tfrac{5}{12}x^{\nu_2} + \tfrac{1}{12}x^{\nu_2} + \tfrac{1}{4}x^{\nu_4},$$

$$y = \tfrac{1}{6}x^{\nu_1} + \tfrac{1}{12}x^{\nu_3} + \tfrac{5}{12}x^{\nu_3} + \tfrac{1}{12}x^{\nu_4} + \tfrac{1}{4}x^{\nu_4}.$$

Observe that x and y have five terms in each representation. Moreover, both lotteries have the same scalar, term to term.

Proof of Proposition 3.
($\Longrightarrow$) Let x and y be two random stable matchings with their OES-representations. Assume that $x \succeq_F y$. Fix $f \in F$. We prove that $\tilde{\mu}^x_\ell(f) \geq^B_f \tilde{\mu}^y_\ell(f)$ for each $\ell = 1, \ldots, \tilde{k}$.

If $\tilde{\mu}^y_1(f) >^B_f \tilde{\mu}^x_1(f)$, then

$$0 = \sum_{\left\{\ell : \tilde{\mu}^x_\ell(f) \geq^B_f \tilde{\mu}^y_1(f)\right\}} \gamma_\ell \geq \sum_{\left\{\ell : \tilde{\mu}^y_\ell(f) \geq^B_f \tilde{\mu}^y_1(f)\right\}} \gamma_\ell = \gamma_1 > 0,$$

which is a contradiction. Thus, $\tilde{\mu}^x_1(f) \geq^B_f \tilde{\mu}^y_1(f)$. Assume that there is $k_1 \leq \tilde{k}$ such that for each $\ell < k_1$ there is $\tilde{\mu}^x_\ell(f) \geq^B_f \tilde{\mu}^y_\ell(f)$, and $\tilde{\mu}^x_{k_1}(f) <^B_f \tilde{\mu}^y_{k_1}(f)$.

Note that $\tilde{\mu}^y_\ell(f) \geq^B_f \tilde{\mu}^y_{\ell+1}(f)$ for each $\ell = 1, \ldots, \tilde{k} - 1$ implies that

$$\sum_{\ell=1}^{k_1} \gamma_\ell = \sum_{\left\{\ell : \tilde{\mu}^y_\ell(f) \geq^B_f \tilde{\mu}^y_{k_1}(f)\right\}} \gamma_\ell. \tag{6}$$

By hypothesis ($x \succeq_F y$), in particular for $w = \tilde{\mu}^y_{k_1}(m)$ it follows that

$$\sum_{\left\{\ell : \tilde{\mu}^y_\ell(f) \geq^B_f \tilde{\mu}^y_{k_1}(f)\right\}} \gamma_\ell \leq \sum_{\left\{\ell : \tilde{\mu}^x_\ell(f) \geq^B_f \tilde{\mu}^y_{k_1}(f)\right\}} \gamma_\ell.$$

Notice that for k_1, it follows that $\tilde{\mu}^x_{k_1-1}(f) \geq^B_f \tilde{\mu}^y_{k_1-1}(f)$ and $\tilde{\mu}^x_{k_1}(f) <^B_f \tilde{\mu}^y_{k_1}(f)$. Thus, $\tilde{\mu}^x_{k_1-1}(f) \geq^B_f \tilde{\mu}^y_{k_1-1}(f) \geq^B_f \tilde{\mu}^y_{k_1}(f) >^B_f \tilde{\mu}^x_{k_1}(f)$. Hence,

$$\sum_{\left\{\ell : \tilde{\mu}^x_\ell(f) \geq^B_f \tilde{\mu}^y_{k_1}(f)\right\}} \gamma_\ell = \sum_{\left\{\ell : \tilde{\mu}^x_\ell(f) \geq^B_f \tilde{\mu}^x_{k_1-1}(f)\right\}} \gamma_\ell = \sum_{\ell=1}^{k_1-1} \gamma_\ell. \tag{7}$$

Thus, by equalities (6) and (7), it follows that $\sum_{\ell=1}^{k_1} \gamma_\ell \leq \sum_{\ell=1}^{k_1-1} \gamma_\ell$, which is a contradiction since $\gamma_{k_1} > 0$. Thus, there is no k_1 such that for each $\ell < k_1$ it follows that $\tilde{\mu}^x_\ell(f) \geq^B_f \tilde{\mu}^y_\ell(f)$, and $\tilde{\mu}^x_{k_1}(f) <^B_f \tilde{\mu}^y_{k_1}(f)$. Thus, $\tilde{\mu}^x_\ell(f) \geq^B_f \tilde{\mu}^y_\ell(f)$ for each $\ell = 1, \ldots, \tilde{k}$.

($\Longleftarrow$) Recall that both x and y are represented by their OES-representations. That is, both representations have the same numbers of terms and the same scalar term to term. Moreover, $\tilde{\mu}^x_\ell \geq^B_F \tilde{\mu}^x_{\ell+1}$ and $\tilde{\mu}^y_\ell \geq^B_F \tilde{\mu}^y_{\ell+1}$ for each $\ell = 1, \ldots, \tilde{k} - 1$. Also, by hypothesis it follows that $\tilde{\mu}^x_\ell \geq^B_F \tilde{\mu}^y_\ell$ for each $\ell = 1, \ldots, \tilde{k}$. Fix ℓ', then

$$\left\{\gamma_\ell : \tilde{\mu}^x_\ell \geq^B_F \tilde{\mu}^x_{\ell'}\right\} = \left\{\gamma_\ell : \tilde{\mu}^y_\ell \geq^B_F \tilde{\mu}^y_{\ell'}\right\} \subseteq \left\{\gamma_\ell : \tilde{\mu}^x_\ell \geq^B_F \tilde{\mu}^y_{\ell'}\right\}.$$

Hence,

$$\sum_{\left\{\ell:\tilde{\mu}_\ell^y(f)\geq_f^B\tilde{\mu}_{\ell'}^y(f)\right\}}\gamma_\ell \leq \sum_{\left\{\ell:\tilde{\mu}_\ell^x(f)\geq_f^B\tilde{\mu}_{\ell'}^y(f)\right\}}\gamma_\ell$$

for each $f \in F$ and for each $\ell' = 1,\ldots,\tilde{k}$. Then, $x \succeq_F y$.

□

Proof of Lemma 1. Let x and y be two random stable matching with their OES-representations. Assume that $x \succeq_F y$. By Proposition 3, this is equivalent to $\tilde{\mu}_\ell^x \geq_F^B \tilde{\mu}_\ell^y$ for each $\ell = 1,\ldots,\tilde{k}$. By the polarization of interest of the partial orders $\geq_F^B$ and $\geq_W^B$, it emerges that $\tilde{\mu}_\ell^x \geq_F^B \tilde{\mu}_\ell^y$ if and only if $\tilde{\mu}_\ell^y \geq_W^B \tilde{\mu}_\ell^x$ for each $\ell = 1,\ldots,\tilde{k}$. Again, by Proposition 3, this is equivalent to $y \succeq_W x$. □

Proof of Proposition 4. We prove that $\boldsymbol{x \veebar_X y = l.u.b._{\succeq_X}(x,y)}$. Recall that both x and y are represented by their OES-representations.

(i) $\boldsymbol{x \veebar_X y \succeq_X x}$ **:**

Since $\tilde{\mu}_\ell^x \vee_X \tilde{\mu}_\ell^y \geq_X^B \tilde{\mu}_\ell^x$ for each $\ell = 1,\ldots,\tilde{k}$, then $x \veebar_X y \succeq_X x$.

(ii) $\boldsymbol{x \veebar_X y \succeq_X y}$ **:**

Since $\tilde{\mu}_\ell^x \vee_X \tilde{\mu}_\ell^y \geq_X^B \tilde{\mu}_\ell^y$ for each $\ell = 1,\ldots,\tilde{k}$, then $x \veebar_X y \succeq_X y$.

(iii) **If $\boldsymbol{z \succeq_X x}$ and $\boldsymbol{z \succeq_X y}$, then $\boldsymbol{z \succeq_X x \veebar y}$:**

We have that $\tilde{\mu}_\ell^z \geq_X^B \tilde{\mu}_\ell^x$ and $\tilde{\mu}_\ell^z \geq_X^B \tilde{\mu}_\ell^y$ for each $\ell = 1,\ldots,\tilde{k}$. Since, $\tilde{\mu}_\ell^x \vee_X \tilde{\mu}_\ell^y$ is the $l.u.b._{\geq_X^B}(\tilde{\mu}_\ell^x, \tilde{\mu}_\ell^y)$, then $\tilde{\mu}_\ell^z \geq_X^B \tilde{\mu}_\ell^x \vee_X \tilde{\mu}_\ell^y$ for each $\ell = 1,\ldots,\tilde{k}$. Therefore, $z \succeq_X x \veebar_X y$.

The proof for $\boldsymbol{x \barwedge_X y = g.l.b._{\succeq_X}(x,y)}$ is analogous.

To prove that $\boldsymbol{x \veebar_F y = x \barwedge_W y}$, recall that the lattices of stable matchings are dual, that is, given $\mu,\ \mu' \in \mathcal{S}(\mathcal{P})$ $\mu \vee_F \mu' = \mu \wedge_W \mu'$. By definition of binary operations, it follows that if $0 < \gamma_\ell \leq 1$, $\sum_{\ell=1}^{\tilde{k}} \gamma_\ell = 1$, $\tilde{\mu}_\ell^x \in \mathcal{S}(\mathcal{P})$, $\tilde{\mu}_\ell^x \geq_F^B \tilde{\mu}_{\ell+1}^x$ and $\tilde{\mu}_\ell^y \geq_F^B \tilde{\mu}_{\ell+1}^y$, then

$$x \veebar_F y = \sum_{\ell=1}^{\tilde{k}} \gamma_\ell \left(\tilde{\mu}_\ell^x \vee_F \tilde{\mu}_\ell^y\right) = \sum_{\ell=1}^{\tilde{k}} \gamma_\ell \left(\tilde{\mu}_\ell^x \wedge_W \tilde{\mu}_\ell^y\right) = x \barwedge_W y.$$

The proof for $\boldsymbol{x \veebar_W y = x \barwedge_F y}$ is analogous. □

Proof of Proposition 5. Let x, y and z be random stable matchings with their OES-representations,

$$x = \sum_{\ell=1}^{\tilde{k}} \gamma_\ell \tilde{\mu}_\ell^x,\ \ y = \sum_{\ell=1}^{\tilde{k}} \gamma_\ell \tilde{\mu}_\ell^y, \text{ and } z = \sum_{\ell=1}^{\tilde{k}} \gamma_\ell \tilde{\mu}_\ell^z.$$

First we prove that $x \veebar_F (y \barwedge_F z) = (x \veebar_F y) \barwedge_F (x \veebar_F z)$.

$$x \veebar_F (y \barwedge_F z) = \sum_{\ell=1}^{\tilde{k}} \gamma_\ell \left(\tilde{\mu}_\ell^x \vee_F \left(\tilde{\mu}_\ell^y \wedge_F \tilde{\mu}_\ell^z\right)\right). \tag{8}$$

Then, by distributive of $\vee_F$ and $\wedge_F$, it follows that (8) is equal to

$$\sum_{\ell=1}^{\tilde{k}} \gamma_\ell \left((\tilde{\mu}_\ell^x \wedge_F \tilde{\mu}_\ell^y) \wedge_F (\tilde{\mu}_\ell^x \wedge_F \tilde{\mu}_\ell^z)\right) = (x \veebar_F y) \barwedge_F (x \veebar_F z).$$

Therefore, $x \veebar_F (y \barwedge_F z) = (x \veebar_F y) \barwedge_F (x \veebar_F z)$. The proof for $x \veebar_W (y \barwedge_W z) = (x \veebar_W y) \barwedge_W (x \veebar_W z)$, $x \barwedge_F (y \veebar_F z) = (x \barwedge_F y) \veebar_F (x \barwedge_F z)$, and $x \barwedge_W (y \veebar_W z) = (x \barwedge_W y) \veebar_W (x \barwedge_W z)$ are analogous. □