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Pablo Neme (Universidad Nacional de San Luis/CONICET)
Jorge Oviedo (Universidad Nacional de San Luis/CONICET)

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ON THE SET OF MANY-TO-ONE STRONGLY STABLE FRACTIONAL MATCHINGS

Pablo A. Neme* J. Oviedo*

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Abstract

For a many-to-one matching market where firms have strict and $\boldsymbol{q}$-responsive preferences, we give a characterization of the set of strongly stable fractional matchings as the union of the convex hull of all connected sets of stable matchings. We also prove that a strongly stable fractional matching is represented as a convex combination of stable matchings that are ordered in the common preferences of all firms.

Key words: Matching Markets; Many-to-one Matching Market; Strongly Stable Fractional Matchings; Linear Programming.
MSC2000 subject classification: Primary: 90C05; secondary: 91B68.

1 Introduction.

A large part of the matching literature studies many-to-one matching markets. The agents in these markets are divided into two disjoint sets: The *many*-side of the market, namely resident doctors, students, workers, etc, and the *one*-side, namely hospitals, colleges, firms, etc. The main property studied in the matching literature is stability. A matching is called stable if all agents have acceptable partners and there is no unmatched pair (hospital-doctor, college-student, firm-worker, etc.), where both agents would prefer to be matched to each other rather than staying with their current partners under the proposed matching. Each agent has a preference list that determines an

*Instituto de Matemática Aplicada San Luis, Universidad Nacional de San Luis and CONICET, San Luis, Argentina. RedNIE. Emails: pabloneme08@gmail.com (P.A. Neme) and joviedo@unsl.edu.ar (J. Oviedo).

order over the agents or sets of agents on the other side of the market, with the possibility of staying unmatched. In this paper, the agents on the many-side have $\boldsymbol{q}$-responsive and strict preferences.

Linear programming is a widely used mathematical tool in matching theory. Each matching can be represented by an assignment matrix called the *incidence vector* of the matching.

Vande Vate [21] and Rothblum [17] present a system of linear inequalities that characterizes the set of stable matchings of the marriage market for two different restrictions of the market. Both papers show that the set of stable matchings for the marriage market corresponds to the set of incidence vectors (integer solutions for linear inequalities). In other words, stable matchings are exactly the extreme points of the polytope generated by the system of linear inequalities. Roth *et al.* [16], for the marriage market, introduce a linear program that characterizes all stable matchings as the integer solutions.

Linear programming approaches have been developed for the theory of stable matching markets also by Abeledo and Rothblum [4] [3], Abeledo and Blum [1], Abeledo *et al.* [2], Fleiner [8], [9], Sethuraman *et al.* [19], Sethuraman and Teo [20], and many others.

Baïou and Balinski [5] present two characterizations of the convex polytope for the many-to-one matching market. We focus on one of these characterizations (the most general one).

Lotteries over stable matchings have been studied in many instances in the literature. For the marriage market, Roth *et al.* [16] studied lotteries over stable matchings via linear programming. When the extreme points of the convex polytope generated by the constraints of a linear program are exactly the stable matchings of the market (this is the case, for instance, in the marriage market) a random matching coincides with the concept of stable fractional matching. Roth *et al.* [16] define a *stable fractional matching* as a not necessarily integer solution of the linear program. When the extreme points are not all integer, these two concepts are not the same, for instance, in a many-to-one matching market with $\boldsymbol{q}$-responsive and strict preferences. That is to say, a random matching is always a stable fractional matching, but some stable fractional matchings cannot be written as a lottery over stable matchings. Example 1 shows a many-to-one matching market with an extreme point that is not a stable matching.

Each entry of an incidence vector of a stable fractional matching can be interpreted as the time that each agent spends with one agent on the other side of the market. For a stable fractional matching, it can happen that two agents, one of each side of the market, have an incentive to increase the time that they spend together at the expense of those matched agents that they like less than each other at a stable fractional matching. To study a "good" fractional solution, the idea is to avoid this and prevent that agents have incentives to "block" the stable fractional matching in a fractional way. For a marriage market, Roth *et al.* [16] define a *strongly stable fractional matching* as a stable fractional matching that fulfils non-linear equalities that represent this non-blocking condition mentioned above. In other words, a stable fractional matching that fulfils the non-linear equalities from Roth *et al.* [16], is a strongly stable fractional matching. Neme and Oviedo [13] give a characterization of the strongly stable fractional matching for the marriage market. Our work extends their result and provides a characterization for the set of many-to-one strongly stable fractional matchings. We extend the strong stability condition from Roth *et al.* [16] to a many-to-one matching market. Our first result states that a strongly stable fractional matching is represented by a convex combination among stable matching that are ordered in the eyes of all firms (Theorem 1). Since we focus on one of the characterizations due to Baïou and Balinski [5], a salient question now is, are the non-integer extreme points of this convex polytope strongly stable fractional matchings? A corollary of Theorem 1 answers negatively this question.

In the school choice set-up, strong stability for lotteries has been introduced by Kesten and Ünver [11] which they called ex-ante stability for lotteries. In this market, they deal with indifferences in the priority of the schools. Kesten and Ünver [11] also present a fractional deferred-acceptance algorithm that computes a unique strongly ex-ante stable random matching. Their paper analyses the strategy proofness and efficiency of this mechanism. Our characterization goes in another direction, we study the relationship among the stable matchings that are involved in the lotteries.

Bansal *et al.* [6] and Cheng *et al.* [7] study the concept of cycles in preferences and cyclic matchings for many-to-many and many-to-one matching markets, respectively. These papers are an extension of Irving and Leather [10]. To seek for cycles in preferences, these authors first reduce the preference lists of all agents. We present the reduction procedure for our market in the Appendix. This reduction procedure allows us to find cycles in preferences. Since the cycles of a reduced list are disjoint, we extend the definition of cyclic matching to a set of cycles in the reduced preference profile.

Following the extension of cyclic matching used by Bansal *et al.* [6] and Cheng *et al.* [7], we define a *connected set* generated by a stable matching μ as the set of all cyclic matchings of μ (including μ). Then, we characterize a strongly stable fractional matching as a lottery over stable matchings that belong to the same connected set (Theorem 2). Moreover, by Theorem 1, we prove that the stable matchings that belong to the same connected set, also have the decreasing order in the eyes of all firms. In this way, we characterize the set of all strongly stable fractional matchings as the union of the convex hulls of these connected sets (Corrolary 2).

Roth *et al.* [16], (in Corollary 21) proved a necessarily condition that states that in a strongly stable fractional matching, each agent is matched with at most two agents of the other side of the market. Schlegel [18] generalizes this necessarily condition for the school choice set-up with strict priorities (similar setting as ours). He shows that a strongly stable fractional matching fulfils that each worker has a positive probability to be matched to at most two distinct firms, and for each firm, all but possibly one position are assigned deterministically. For the one position that is assigned by a lottery, two workers have a positive probability of been matched to the firm (here stated as Corrolary 3). Further, although he proves that a strongly stable fractional matching is "almost" integral, he does not describes which agents are matched (there are several "almost" integral stable fractional matchings that are not strongly stable, Example 1 presents an "almost" integral stable fractional matching that is not strongly stable). Recall that our characterization gives a necessary and sufficient condition for a stable fractional matching to be strongly stable. As a particular case, our characterization gives an alternative proof for these two results, for the school choice set-up due to Schlegel [18] is straightforward, and for the marriage market due to Roth *et al.* [16], it's only necessary to set all quotas of all firms equal to one. Moreover, our characterization shows explicitly which are the matched agents in a strongly stable fractional matching, through the stable matching involved in the convex combination (*Cf.* (14) in proof of Theorem 2).

This paper is organized as follows. In Section 2 we formally introduce the market, preliminary results, and one of Baïou and Balinski's characterizations of stable matchings. In Section 3 we define a strongly stable fractional matching and prove that it can be represented by a convex combination over stable matchings that are ordered for all firms. We also discuss cycles and cyclic matching properties that we use in the characterization result. In Section 4, we present our characterization of a strongly stable fractional matching. The Appendix contains the reduction procedure, lemmas, and proofs of the lemmas needed for our characterization.

2 Preliminary Results.

The many-to-one matching market that we study, consists of two sets of agents, the set of firms $F = \{f_1, \ldots, f_n\}$ and the set of workers $W = \{w_1, \ldots, w_m\}$. Each worker w has an antisymmetric, transitive, and complete preference relation $\succ_w$ over $F \cup \{w\}$, and each firm f has an antisymmetric, transitive, and complete preference relation $\succ_f$ over the power set of workers, 2^W. Also, each firm f has a maximum number of positions to fill: its quota, denoted by q_f. Let $\boldsymbol{q} = (q_f)_{f \in F}$ be the vector of quotas. Given $W_0, W_1 \subseteq W$, we write $W_0 \succeq_f W_1$ to indicate that the firm f likes W_0 as much as W_1. Given the preference relation $\succ_f$, we say that $W_0 \succ_f W_1$ when $W_0 \succeq_f W_1$ and $W_0 \neq W_1$. Analogously, for each worker w, and any two firms, $f_0, f_1 \in F$, we write

$f_0 \succeq_w f_1$ and $f_0 \succ_w f_1$.

Preference profiles are $(n+m)-tuples$ of preference relations and they are denoted by $P = (\succ_{f_1}, \ldots, \succ_{f_n}, \succ_{w_1}, \ldots, \succ_{w_m})$. The matching market for the sets W and F with the preference profile P and vector of quotas $\boldsymbol{q}$ is denoted by $(F, W, P, \boldsymbol{q})$.

We say that a pair $(f, w) \in F \times W$ is an *acceptable pair* at P if w is acceptable for f, and f is acceptable for w, that is, $\{w\} \succ_f \emptyset$ and $f \succ_w w$. Let us denote by $A(P)$ the set of all acceptable pairs of the matching market $(F, W, P, \boldsymbol{q})$, (simply A, when no confusion arises).

The *assignment problem* consists of matching workers with firms keeping the bilateral nature of their relationship and allowing for the possibility that firms and workers remain unmatched. Formally,

Definition 1 *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. A **matching** μ is a mapping from the set $F \cup W$ into the set of all subsets of $F \cup W$ such that, for all $w \in W$ and $f \in F$:*

1. *$|\mu(w)| = 1$ and if $\mu(w) \neq \{w\}$, then $\mu(w) \subseteq F$.*
2. *$\mu(f) \in 2^W$ and $|\mu(f)| \leq q_f$.*
3. *$\mu(w) = \{f\}$ if and only if $w \in \mu(f)$.*

Usually we will omit the curly brackets, for instance, instead of condition 1. and 3., we will write: "1. $|\mu(w)| = 1$ and if $\mu(w) \neq w$, then $\mu(w) \subseteq F$." and "3. $\mu(w) = f$ if and only if $w \in \mu(f)$." Assume that each firm f gives its ranking of workers individually, and orders subsets of workers in a *responsive* manner. That is to say, adding "good" workers to a set leads to a better set, whereas adding "bad" workers to a set leads to a worse set. In addition, for any two subsets that differ in only one worker, the firm prefers the subset containing the most preferred worker. Formally,

Definition 2 *The preference relation $\succ_f$ over 2^W is **q-responsive** if it satisfies the following conditions:*

1. *For all $T \subseteq W$ such that $|T| > q_f$, we have that $\emptyset \succ_f T$.*
2. *For all $T \subseteq W$ such that $|T| < q_f$ and $w \notin T$, we have that*

$$T \cup \{w\} \succ_f T \text{ if and only if } w \succ_f \emptyset.$$

3. *For all $T \subseteq W$ such that $|T| < q_f$ and $w, w' \notin T$, we have*

$$T \cup \{w\} \succ_f T \cup \{w'\} \text{ if and only if } w \succ_f w'.$$

Let $\mu \succ_F \mu'$ denote that all firms like μ at least as well as μ' with at least one firm strictly preferring μ to μ', that is, $\mu(f) \succeq_f \mu'(f)$ for all $f \in F$ and $\mu(f') \succ_{f'} \mu'(f')$ for at least one firm $f' \in F$. We say that $\mu \succeq_F \mu'$ means that either $\mu \succ_F \mu'$ or $\mu = \mu'$. Analogously, define $\mu \succ_W \mu'$ and $\mu \succeq_W \mu'$.

We say that a matching μ is *individually rational* if $\mu(w) = f$ for some worker w and firm f, then (f, w) is an acceptable pair. Similarly, a pair (f, w) is a *blocking pair* for matching μ, if the worker w is not employed by the firm f, but they both prefer to be matched to each other. That is, a matching μ is blocked by a *firm-worker pair* (f, w):

1. If $|\mu(f)| = q_f, \mu(w) \neq f,\ f \succ_w \mu(w)$ and $w \succ_f w'$ for some $w' \in \mu(f)$.

2. If $|\mu(f)| < q_f, \mu(w) \neq f$ and $f \succ_w \mu(w)$ and $w \succ_f \emptyset$.

In that way, a matching μ is *stable* if it is individually rational and has no blocking pairs. We denote by $S(P)$ to the set of all stable matchings at the preference profile P.

An importat result of matching theory is the *Rural Hospital Theorem* (RHT). When firms have q-responsive preference and workers strict preference, the Rural Hospital Theorem states the following: (see Roth [14],[15] for more details)

Theorem (RHT) *The set of matched agents is the same under every stable matching. Moreover, each firm that does not fill its quota has the same set of agents matched under every stable matching.*

2.1 Linear Programming Approach.

For the marriage market, Rothblum [17] characterizes stable matchings as extreme points of a convex polytope generated by a system of linear inequalities. Baïou and Balinski [5] present two generalizations of the convex polytope for the many-to-one matching market $(F, W, P, \boldsymbol{q})$ with $\boldsymbol{q}$-responsive preferences. We focus on the most general one.

Given a matching μ, a vector $x^\mu \in \{0,1\}^{|F|\times|W|}$ is an **incidence vector** when $x^\mu_{f,w} = 1$ if and only if $\mu(w) = f$ and $x^\mu_{f,w} = 0$ otherwise. When no confusion arises, we identify each matching with its incidence vector.

Let CP be the convex polytope generated by the following linear inequalities:

$$\sum_{j \in W} x_{f,j} \leq q_f \qquad f \in F \tag{1}$$

$$\sum_{i \in F} x_{i,w} \leq 1 \qquad w \in W \tag{2}$$

$$x_{f,w} \geq 0 \qquad (f, w) \in F \times W \tag{3}$$

$$x_{f,w} = 0 \qquad (f, w) \in F \times W \setminus A \tag{4}$$

Notice that an integer solution of CP represents the incidence vector of a individually rational matching for the many-to-one matching market. The extreme points of this convex polytope are all integer points. This convex polytope is known as the polytope of the transportation problem. For more detail, see Luenberger and Ye [12]. A non-integer solution of CP is called a **fractional matching**.

Define a new convex polytope SCP, by adding to the convex polytope CP the following inequality:

$$\sum_{j \succ_f w} x_{f,j} + q_f \sum_{i \succ_w f} x_{i,w} + q_f x_{f,w} \geq q_f \qquad (f,w) \in A \tag{5}$$

Lemma 1 (Baïou and Balinski [5]) *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. μ is a stable matching for $(F, W, P, \boldsymbol{q})$ if and only if its incidence vector is an integer solution of SCP.*

We define a **stable fractional matching** as a not necessarily integer solution of the convex polytope SCP. For the marriage market, i.e. $q_f = 1$ for all $f \in F$, Rothblum [17] proves that the extreme points of the associated convex polytope, are the stable matchings. It is naturally expected that this result carries over to the more general case, a many-to-one matching market. But this is not true for the convex polytope SCP. Here, we present an example taken from Baïou and Balinski [5] that shows a many-to-one market, where the convex polytope has fractional extreme points. This also shows that a lottery over stable matchings is also a stable fractional matching. However, the opposite case does not always hold.

Example 1 *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. Let $F = \{f_1, f_2\}$, $W = \{w_1, w_2, w_3, w_4\}$, P is the following preference profile:*

$$\begin{array}{ll} \succ_{f_1} = w_1, w_2, w_3, w_4 & \succ_{w_1} = f_2, f_1 \\ \succ_{f_2} = w_4, w_3, w_2, w_1 & \succ_{w_2} = f_2, f_1 \\ & \succ_{w_3} = f_2, f_1 \\ & \succ_{w_3} = f_1, f_2, \end{array}$$

and $q_1 = q_2 = 2$. The only two stable matchings for this market are:

$$x^{\mu_F} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}; \qquad x^{\mu_W} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Baïou and Balinski observe that the stable fractional matching

$$x^1 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix},$$

is a vertex of the convex polytope SCP.

After observing that the convex polytope has fractional extreme points, Baïou and Balinski [5] present a second generalization for the many-to-one matching market. In this second generalization, the extreme points of the convex polytope, are exactly the stable matchings for the many-to-one market. This assures that this last convex polytope, is a subset of the convex polytope SCP. For that reason, our study is based on the convex polytope SCP.

3 The Strongly Stable Fractional Matchings.

Each entry of the vector that represents a stable fractional matching, $x_{f,w}$, can be interpreted as the time that firm f and worker w spend with each other. For a stable fractional matching x, it can happen that two agents, one from each side of the market, have an incentive to increase the time that they spend together at the expense of those they like less at a stable fractional matching x. The importance of a strongly stable fractional matching is to avoid this and prevent that agents have incentives to block the stable fractional matching in a fractional way. We formally present the definition of a strongly stable fractional matching for our market.

Definition 3 *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. A fractional matching $\bar{x}$ is* ***strongly stable*** *if, for each $(f, w) \in A$, $\bar{x}$ satisfies the strong stability condition*

$$\left[q_f - \sum_{j \succeq_f w} \bar{x}_{f,j}\right] \cdot \left[1 - \sum_{i \succeq_w f} \bar{x}_{i,w}\right] = 0. \tag{6}$$

The matching $\bar{x}$ satisfying the strong stability condition is known as a ***strongly stable fractional matching****.*

We denote by $SSF(P)$ to the set of all strongly stable fractional matchings at the preference profile P. Assume that for a pair $(f, w) \in A$, the fractional matching $\bar{x}$ does not fulfil condition (6). Then, we have that $q_f - \sum_{j \succeq_f w} \bar{x}_{f,j} > 0$ and $1 - \sum_{i \succeq_w f} \bar{x}_{i,w} > 0$. Meaning that there are at least two agents f' and w' such that, $f \succ_w f'$, $w \succ_f w'$, $\bar{x}_{f,w'} > 0$, $\bar{x}_{f',w} > 0$ and $\bar{x}_{f,w} < 1$. Hence, both f and w will have an incentive to increase the time that they spend together at the expense of w' and f' respectively. This means that $\bar{x}$ is blocked in a fractional way by the pair (f, w).

Example 1 (Continued) *Recall the stable fractional matching*

$$x^1 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$

Also, we have that $f_2 \succ_{w_3} f_1$ and $w_4 \succ_{f_2} w_3 \succ_{f_2} w_2$. Now, we compute condition (6) for the pair (f_2, w_3).

$$\left[q_{f_2} - \sum_{j \succeq_{f_2} w_3} \bar{x}_{f_2,j}\right] \cdot \left[1 - \sum_{i \succeq_{w_3} f_2} \bar{x}_{i,w_3}\right]$$
$$= \left[2 - \frac{3}{2}\right] \cdot \left[1 - \frac{1}{2}\right] \neq 0.$$

Hence, x^1 does not fulfils condition (6) for the pair (f_2, w_3). Moreover, since $x^1_{f_1,w_3} = \frac{1}{2} > 0$, $x^1_{f_2,w_2} = \frac{1}{2} > 0$ and $x^1_{f_2,w_3} = \frac{1}{2} < 1$, then agents f_2 and w_3 have incentive to increase the time that they spend together at expense of w_2 and f_1 respectively. Hence, x^1 is blocked in a fractional way by the pair (f_2, w_3). Therefore, x^1 is not strongly stable.

Remark 1 *The incidence vector of a stable matching, also fulfils condition (6).*

For the particular case where all quotas are equal to one, (the marriage market), and for a stable fractional matching x, Rothblum [17] defines a stable matching that assigns to each firm f the most preferred worker among those that $x_{f,w} > 0$, for all $w \in W$. Here we generalize this definition for the many-to-one matching market $(F, W, P, \boldsymbol{q})$. We denote $supp(x)$ to the support of the fractional matching x, that is, $supp(x) = \{(f, w) : x_{f,w} > 0\}$.

For a many-to-one matching market $(F, W, P, \boldsymbol{q})$, and for a given stable fractional matching x, we define the set of workers employed in the best q_f positions of f. Let $C^0_f(x) = \{w : (f, w) \in supp(x)\}$, and define $C^k_f(x) = \{w \in C^0_f(x) :$ there is no $w' \in C^0_f(x) \setminus C^{k-1}_f(x),\ w' \succ_f w\}$. In words, $C^k_f(x)$ is the set of the k-best workers in the $supp(x)$ for the firm f.

Now, we define the matching where each firm is matched to the best q_f workers in the $supp(x)$. Formally,

Definition 4 *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. Let x be a stable fractional matching. For each firm f, we define μ_x as:*

$$\mu_x(f) = C^{q_f}_f(x).$$

Remark 2 *If for some firm f we have that $|C^0_f(x)| \leq q_f$, then $x^{\mu_x}_{f,w} = 1$ for all $w \in C^0_f(x)$.*

The following lemma generalizes Lemma 12 of Roth *et al.* [16], and states that $\mu_{\bar{x}}$ is a stable matching whenever $\bar{x}$ is a strongly stable fractional matching.

Lemma 2 *Let* $(F, W, P, \boldsymbol{q})$ *be a many-to-one matching market. Let* $\bar{x}$ *be a strongly stable fractional matching. Then,* $\mu_{\bar{x}}$ *is a stable matching.*

Proof. See the Appendix. □

The following lemma is a technical result used further in Theorem 1. This lemma states that a strongly stable fractional matching $\bar{x}$ is always represented as a convex combination between the stable matching $\mu_{\bar{x}}$ and another strongly stable fractional matching.

Lemma 3 *Let* $(F, W, P, \boldsymbol{q})$ *be a many-to-one matching market. Let* $\bar{x}$ *be a strongly stable fractional matching and* $\bar{x} \neq x^{\mu_{\bar{x}}}$*. Let* $\alpha = min\{\bar{x}_{f,w} : x^{\mu_{\bar{x}}}_{f,w} = 1\}$*. Then,* y *defined as:*

$$y = \frac{\bar{x} - \alpha x^{\mu_{\bar{x}}}}{1 - \alpha}$$

is a strongly stable fractional matching, such that $supp(y) \subset supp(\bar{x})$*.*[1]

Proof. See the Appendix. □

The following theorem states that a strongly stable fractional matching can be represented by a particular convex combination of stable matchings. These stable matchings are all comparable in the eyes of all firms.

Theorem 1 *Let* $(F, W, P, \boldsymbol{q})$ *be a many-to-one matching market. Let* $\bar{x}$ *be a strongly stable fractional matching. Then, there are stable matchings* $\mu^1, \ldots, \mu^k$*, and real numbers* $\alpha_1, \ldots, \alpha_k$ *such that*

$$\bar{x} = \sum_{l=1}^{k} \alpha_l x^{\mu^l}, \; 0 < \alpha_l \leq 1, \; \sum_{l=1}^{k} \alpha_l = 1, \text{ and } \mu^1 \succ_F \mu^2 \succ_F \ldots \succ_F \mu^k. \tag{7}$$

Proof. Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market and let $\bar{x}$ be a strongly stable fractional matching. By Lemma 2, $\mu_{\bar{x}}$ is a stable matching. Denote by $\mu^1 = \mu_{\bar{x}}$.

If $\bar{x} = x^{\mu^1}$ (i.e., $\bar{x}$ is a stable matching), then $\bar{x}$ is represented as in (7) with $k = 1$ and $\alpha_1 = 1$.

If $\bar{x} \neq x^{\mu^1}$ (i.e., $\bar{x}$ is not a stable matching), then by Lemma 3, there is a strongly stable fractional matching, x^2, defined by

$$x^2 = \frac{\bar{x} - \alpha'_1 x^{\mu^1}}{1 - \alpha'_1},$$

for some $0 < \alpha'_1 < 1$, with $supp\left(x^2\right) \subset supp\left(\bar{x}\right)$. Then,

$$\bar{x} = \left(1 - \alpha'_1\right) x^2 + \alpha'_1 x^{\mu_1}. \tag{8}$$

[1]Notice that, here we use "⊂" to denote the strict inclusion; that is, $A \subset B$ means that A is a proper subset of B.

for $0 < \alpha_1' < 1$ and $supp\left(x^{\mu_1}\right) \subset supp\left(\bar{x}\right)$.

By Lemma 2, μ_{x^2} is a stable matching. Denote by $\mu^2 = \mu_{x^2}$. Notice that, since $supp\left(x^{\mu_1}\right) \subset supp\left(\bar{x}\right)$, $supp\left(x^2\right) \subset supp\left(\bar{x}\right)$, then by definitions of μ^1 and x^2, we have that $\mu^1 \succ_F \mu^2$.

If $x^2 = x^{\mu^2}$ (i.e., x^2 is a stable matching), then $\bar{x}$ is represented as in (7).

If $x^2 \neq x^{\mu^2}$ (i.e., x^2 is not a stable matching), again by Lemma 3, there is a strongly stable fractional matching x^3, defined by

$$x^3 = \frac{x^2 - \alpha_2' x^{\mu^2}}{1 - \alpha_2'},$$

for some $0 < \alpha_2' < 1$ with $supp\left(x^3\right) \subset supp\left(x^2\right)$. That is,

$$x^2 = \left(1 - \alpha_2'\right) x^3 + \alpha_2' x^{\mu^2}. \tag{9}$$

Since $0 < \alpha_2' < 1$, we have that $supp\left(x^{\mu_2}\right) \subset supp\left(x^2\right)$. By Lemma 2, μ_{x^3} is a stable matching. Denote by $\mu^3 = \mu_{x^3}$ Since $supp\left(x^3\right) \subset supp\left(x^2\right)$, we have that $\mu^2 \succ_F \mu^3$. Then, $\mu^1 \succ_F \mu^2 \succ_F \mu^3$.

If $x^3 = x^{\mu^3}$ (i.e., x^3 is a stable matching), from equalities (8) and (9) we have that

$$\begin{aligned}
\bar{x} &= \left(1 - \alpha_1'\right) x^2 + \alpha_1' x^{\mu_1} \\
&= \left(1 - \alpha_1'\right) \left(\left(1 - \alpha_2'\right) x^3 + \alpha_2' x^{\mu^2}\right) + \alpha_1' x^{\mu^1} \\
&= \left(1 - \alpha_1'\right) \left(1 - \alpha_2'\right) x^3 + \left(1 - \alpha_1'\right) \alpha_2' x^{\mu^2} + \alpha_1' x^{\mu^1}.
\end{aligned}$$

Then $\bar{x}$ is represented as in (7) with $k = 3$, $\alpha_1 = \alpha_1'$, $\alpha_2 = \left(1 - \alpha_1'\right) \alpha_2'$, and $\alpha_3 = \left(1 - \alpha_1'\right)\left(1 - \alpha_2'\right)$. Notice that $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

If $x^3 \neq x^{\mu_3}$ (i.e., x^3 is not a stable matching), then we continue this procedure. The finiteness of the $supp\left(\bar{x}\right)$ guarantees that this procedure ends by constructing a stable matching. This proves that $\bar{x}$ is represented as in (7) for some $k \geq 1$. □

Recall that the convex polytope SCP has extreme points that are not integer. A salient question now is, are these non-integer extreme points strongly stable fractional matchings? The following corollary answers this, and states that non-integer extreme points of the convex polytope SCP are not strongly stable fractional matchings.

Corollary 1 *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. Let x be a non-integer extreme point of the convex polytope SCP. Then, x is not a strongly stable fractional matching.*

Proof. Let x be a non-integer extreme point of the convex polytope SCP. Then, x cannot be represented as a convex combination of different extreme points of the same convex polytope. More precisely, x cannot be represented as a convex combination of different *integer* extreme points of the convex polytope SCP (stable matchings). Therefore, by Theorem 1, x is not a strongly stable fractional matching. □

3.1 Cycles in Preferences.

For the marriage market, Irving and Leather [10] define a cycle in preference and a cyclic matching in order to present an algorithm that finds all stable matchings. Bansal *et al.* [6] and Cheng *et al.* [7] extend the concept of cycles and cyclic matchings for many-to-many and many-to-one matching markets, respectively. We will state some properties of cycles that are taken from these authors. They refer to the cycles as rotations.

Given a stable matching μ for a many-to-one matching market $(F, W, P, \boldsymbol{q})$, we define a **reduced preference profile P^{μ}**, as the preference profile obtained after the reduction procedure. This reduction procedure is presented in the Appendix. The reduced preference list of firm f, is denoted by $\succ_f^{\mu}$. In the same way, the reduced preference list of worker w, is denoted by $\succ_w^{\mu}$.

Definition 5 *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. Given a stable matching μ, and the reduced preference profile P^{μ}, a set of firms $\sigma = \{e_1, \ldots, e_r\} \subseteq F$ defines a **cycle** if for $w_{e_1}, \ldots, w_{e_r} \in W$ we have that:*

1. *For each $d = 1, \ldots, r-1$, $w_{e_d} \in \mu(e_{d+1})$, $w_{e_d} \notin \mu(e_d)$ and $w_{e_d} \succeq_{e_d} w'$ for all $w' \notin \mu(e_d)$.*
2. *$w_{e_r} \notin \mu(e_r)$, $w_{e_r} \succeq_{e_r} w'$ for all $w' \notin \mu(e_r)$, and $w_{e_r} \in \mu(e_1)$.*

Given a cycle σ, we can define a cyclic matching as follows:

Definition 6 *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. Given a stable matching μ, and the reduced preference profile P^{μ}, let $\sigma = \{e_1, \ldots, e_r\}$ be a cycle in P^{μ}, and let $\{w_{e_1}, \ldots, w_{e_r}\}$ be the set of workers defined by the cycle σ. The **cyclic matching** of μ is defined as follows:*

$$\mu[\sigma] = \begin{cases} \mu[\sigma](e_1) = & \mu(e_1) \setminus \{w_{e_r}\} \cup \{w_{e_1}\}, \\ \mu[\sigma](e_d) = & \mu(e_d) \setminus \{w_{e_{d-1}}\} \cup \{w_{e_d}\} \quad \text{for } d = 2, \ldots, r-1, \\ \mu[\sigma](e_r) = & \mu(e_r) \setminus \{w_{e_{r-1}}\} \cup \{w_{e_r}\} \\ \mu[\sigma](f) = & \mu(f) \quad\quad\quad\quad\quad\quad\quad \text{for all } f \notin \sigma. \end{cases}$$

In the reduced preference lists of each firm that belongs to the cycle σ, we have that the preferred worker that is unmatched under μ is always matched to another firm in the same cycle. Think of each firm e_d in the cycle as being asked to hire its preferred worker that is unmatched in its preference list. Also, this new worker replaces the worker that other firm in the cycle wants to hire. The firms that do not belong to the cycle σ, will be matched to the same set of workers. Notice that if a firm f belongs to a cycle σ, this means that it has different sets of workers assigned in μ as well as in $\mu[\sigma]$. Then, by Theorem RHT, we have that $|\mu(f)| = q_f$.

Lemma 4 (Bansal *et al.* [6]) *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market.*

1. *Let μ be a stable matching and let σ be a cycle in P^μ. Then, the cyclic matching $\mu[\sigma]$ is a stable matching in the original preference profile.*

2. *A matching μ' is stable under P^μ if and only if μ' is stable under the original preference profile and $\mu \succeq_F \mu'$.*

Let $\boldsymbol{\Phi(\mu)}$ denote the set of cycles of the reduced preference profile P^μ. Now, we can extend the definition of a cyclic matching as follows.

Definition 7 *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. For a stable matching μ, and the reduced preference profile P^μ, let $K \subseteq \Phi(\mu)$, define the **cyclic matching $\boldsymbol{\mu[K]}$** as follows:*

1. *If $K = \emptyset$, then $\mu[K] = \mu$.*

2. *If $K \neq \emptyset$, and $K = \{\sigma_1, \ldots, \sigma_n\}$, then*

$$\mu[K](f) = \begin{cases} \mu[\sigma_h](f) & f \in \sigma_h,\ h = 1, \ldots, n \\ \mu(f) & \text{otherwise.} \end{cases}$$

Lemma 5 (Cheng *et al.* [7]) *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market, and let P^μ be the reduced preference profile at μ.*

1. *Let σ and σ' be two different cycles. Then, $\sigma \cap \sigma' = \emptyset$.*

2. *Let μ' be a stable matching in P^μ. If $\mu \neq \mu'$, then there is a cycle $\sigma \in P^\mu$ such that $\mu[\sigma] \succeq_F \mu'$.*

Remark 3 *Let $K \subseteq \Phi(\mu)$ be a subset of cycles of P^μ. By Lemma (5), we have that $\mu[K](f) = \mu[\sigma](f)$ for each $f \in \sigma$ with $\sigma \in K$.*

Notice that Lemmas 4 and 5 assure that the cyclic matching $\mu[K]$ from Definition 7 is stable under the original preference profile.

The following lemma states that, the matching obtained by applying different cycles is independent from the order in which they are applied.

Lemma 6 *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. Let P^μ be the reduced preference profile at μ, and let σ and σ' be two different cycles in $\Phi(\mu)$.*

1. *σ is a cycle of $P^{\mu[\sigma']}$.*

2. *$\mu[\sigma, \sigma'] = \mu[\sigma', \sigma]$.*

Proof. See the Appendix. ☐

4 A characterization of the set of strongly stable fractional matchings.

In this section, we present our main findings. Our aim is to describe the relationship among the stable matchings involved in the lottery that represents a strongly stable fractional matching. Our characterization of a strongly stable fractional matching is based on the idea of cyclic matchings. For a many-to-one matching market, finding all stable matchings via cycles in preference and cyclic matchings requires a polynomial time algorithm, (see Bansal *et al.* [6]). The importance of our characterization is to present an elegant and useful way to describe the strongly stable fractional matchings via the stable matchings involved in the lottery that generates them. Since our result is based on the idea of cycles and cyclic matchings, we need that a stable fractional matching that is strongly stable in a reduced preference profile, is also strongly stable in the original preference profile. This statement is proved on Lemma 8 in the Appendix.

In order to present our characterization, we define a connected set generated by a stable matching.

Definition 8 *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. A set of stable matchings $\mathcal{M}$ is **connected** if there is a stable matching μ and a set of cycles $K' \subseteq \Phi(\mu)$ such that*

$$\mathcal{M} = \mathcal{M}_{\mu}^{K'},$$

where $\mathcal{M}_{\mu}^{K'} = \{\mu[K] : K \subseteq K'\}$. Let us denote by $\mathcal{M}_{\mu} = \mathcal{M}_{\mu}^{\Phi(\mu)}$.

Notice that from the Definition 7, we can see that μ is also a cyclic matching of itself and, for each $K' \subseteq \Phi(\mu)$, we have that $\mu \in \mathcal{M}_{\mu}^{K'}$.

The main result of this paper states that $\bar{x}$ is a strongly stable fractional matching if and only if it belongs to the convex hull of a connected set. Formally:

Theorem 2 *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market and let $\bar{x}$ be a stable fractional matching in (F, W, P). Then, $\bar{x}$ is strongly stable if and only if there is a collection of connected stable matchings $\{\mu^1, \ldots, \mu^k\}$, such that $\bar{x} = \sum_{l=1}^{k} \alpha_l x^{\mu^l}$, $0 < \alpha_l \leq 1$, and $\sum_{l=1}^{k} \alpha_l = 1$.*

Proof. Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market.

($\Longrightarrow$) Let $\bar{x}$ be a strongly stable fractional matching. Theorem 1, assures that there are stable matchings $\mu^1, \ldots, \mu^k$ and real numbers $\alpha_1, \ldots, \alpha_k$ such that

$$\bar{x} = \sum_{l=1}^{k} \alpha_l x^{\mu^l}, \;\; 0 < \alpha_l \leq 1, \;\; \sum_{l=1}^{k} \alpha_l = 1, \text{ and } \mu^1 \succ_F \mu^2 \succ_F \ldots \succ_F \mu^k.$$

Denote by $Conv\left\{\mathcal{M}_{\mu^1}\right\}$ the convex hull of elements of $\mathcal{M}_{\mu^1}$. Assume, by way of contradiction, that $\bar{x} \notin Conv\left\{\mathcal{M}_{\mu^1}\right\}$. Then, there is μ^t of the convex combination of $\bar{x}$

such that $\mu^1, \ldots, \mu^{t-1} \in \mathcal{M}_{\mu^1}$ and $\mu^t \notin \mathcal{M}_{\mu^1}$. Let $K' \subseteq \Phi(\mu)$ be the set of cycles such that $\mu[K'] = \mu^{t-1}$. Notice that, $\mathcal{M}_{\mu^1}^{K'} \subseteq \mathcal{M}_{\mu^1}$ is the smallest connected set generated by μ^1 such that $\mu^1, \ldots, \mu^{t-1} \in \mathcal{M}_{\mu^1}^{K'}$ and $\mu^t \notin \mathcal{M}_{\mu^1}^{K'}$.

Then, by Lemma 4 item 2) and Lemma 5 item 2) there is a cycle $\sigma^\star \in \Phi\left(\mu^{t-1}\right)$ such that $\mu^{t-1}[\sigma^\star] \notin \mathcal{M}_{\mu^1}^{K'}$ and $\mu^{t-1}[\sigma^\star] \succeq_F \mu^t$. Notice that this implies that $\sigma^\star \notin K'$.

Here we analyse two cases:

Case 1: If there is $\sigma \in K'$ such that $\sigma^\star \cap \sigma \neq \emptyset$.

Notice that in this case, by Lemma 5 item 1), $\sigma^\star \notin \Phi(\mu^1)$. Then for any $\tilde{f} \in \sigma^\star \cap \sigma$, we have that, $\mu^1 \succ_{\tilde{f}} \mu^{t-1} \succ_{\tilde{f}} \mu^{t-1}[\sigma^\star] \succeq_{\tilde{f}} \mu^t$. If $t = 2$, we have that $\sigma^\star \in K'$, and since $\sigma \cap \sigma^\star \neq \emptyset$, then $\sigma = \sigma^\star$ which results in a contradiction. Therefore $t \geq 3$.

By Theorem RHT, there are $w^\star,\ w_1,\ w_2$ such that:

$$\begin{aligned} w_1 &\in \mu^1(\tilde{f}) - \left(\mu^{t-1}(\tilde{f}) \cup \mu^{t-1}[\sigma^\star](\tilde{f})\right), \\ w^\star &\in \mu^{t-1}(\tilde{f}) - \mu^1(\tilde{f}), \\ w_2 &\in \mu^{t-1}[\sigma^\star](\tilde{f}) - \left(\mu^{t-1}(\tilde{f}) \cup \mu^1(\tilde{f})\right), \end{aligned} \tag{10}$$

and $w_1 \succ_{\tilde{f}} w^\star \succ_{\tilde{f}} w_2$. Now, we prove that for the pair $(\tilde{f}, w^\star)$, condition (6) fails. That is,

$$\left[q_{\tilde{f}} - \sum_{j \succeq_{\tilde{f}} w^\star} \bar{x}_{\tilde{f},j}\right] \cdot \left[1 - \sum_{i \succeq_{w^\star} \tilde{f}} \bar{x}_{i,w^\star}\right] \neq 0.$$

We analyse the two factors separately:

Case 1.1: $q_{\tilde{f}} - \sum_{j \succeq_{\tilde{f}} w^\star} \bar{x}_{\tilde{f},j}$.

$$q_{\tilde{f}} - \sum_{j \succeq_{\tilde{f}} w^\star} \bar{x}_{\tilde{f},j} = q_{\tilde{f}} - \sum_{j \succeq_{\tilde{f}} w^\star} \left(\sum_{l=1}^{k} \alpha_l x_{\tilde{f},j}^{\mu^l}\right) =$$

$$q_{\tilde{f}} - \sum_{l=1}^{k} \alpha_l \left(\sum_{j \succeq_{\tilde{f}} w^\star} x_{\tilde{f},j}^{\mu^l}\right).$$

Since $\mu^t \preceq_{\tilde{f}} \mu^{t-1}[\sigma^\star]$, we have that

$$\sum_{j \succeq_{\tilde{f}} w} x_{\tilde{f},j}^{\mu^t} \leq \sum_{j \succeq_{\tilde{f}} w} x_{\tilde{f},j}^{\mu[K'][\sigma^\star]}$$

for all $w \in W$.

In particular for $w = w^\star$, and the fact that $w^\star \succ_{\tilde{f}} w^2$ with $w^2 \in \mu^{t-1}[\sigma^\star](\tilde{f})$, we have that

$$\sum_{j \succeq_{\tilde{f}} w^\star} x_{\tilde{f},j}^{\mu^t} \leq \sum_{j \succeq_{\tilde{f}} w^\star} x_{\tilde{f},j}^{\mu^{t-1}[\sigma^\star]} < q_{\tilde{f}}.$$

By (10) we also have that for each $l = 1, \ldots, k$

$$\sum_{j \succeq_{\tilde{f}} w^\star} x^{\mu^l}_{\tilde{f},j} \leq q_{\tilde{f}}.$$

Using the decreasing sequence of stable matchings of Theorem 1, we have that $\alpha_l > 0$ for each $l = 1, \ldots, k$. Then, we have that

$$q_{\tilde{f}} - \sum_{j \succeq_{\tilde{f}} w^\star} \bar{x}_{\tilde{f},j} = q_{\tilde{f}} - \sum_{l=1}^{k} \alpha_l \left(\sum_{j \succeq_{\tilde{f}} w^\star} x^{\mu^l}_{\tilde{f},j} \right) > q_{\tilde{f}} - \left(\sum_{l=1}^{k} \alpha_l q_{\tilde{f}} \right) = 0.$$

Case 1.2: $\mathbf{1 - \sum_{i \succeq_{w^\star} \tilde{f}} \bar{x}_{i,w^\star}}$.

$$1 - \sum_{i \succeq_{w^\star} \tilde{f}} \bar{x}_{i,w^\star} = 1 - \sum_{i \succeq_{w^\star} \tilde{f}} \left(\alpha_1 x^{\mu^1}_{i,w^\star} + \sum_{l=2}^{k} \alpha_l x^{\mu^l}_{i,w^\star} \right) =$$

$$\left[1 - \left(\alpha_1 \sum_{i \succeq_{w^\star} \tilde{f}} x^{\mu^1}_{i,w^\star} + \sum_{l=2}^{k} \alpha_l \sum_{i \succeq_{w^\star} \tilde{f}} x^{\mu^l}_{i,w^\star} \right) \right].$$

By (10), we have that $w^\star \notin \mu^1(\tilde{f})$. Since $\mu^1 \succ_{\tilde{f}} \mu^{t-1}$, we have that $\mu^{t-1}(w^\star) = \tilde{f} \succ_{w^\star} \mu^1(w^\star)$. Therefore,

$$\sum_{i \succeq_{w^\star} \tilde{f}} x^{\mu^1}_{i,w^\star} = 0.$$

Since $\alpha_l > 0$ for each $l = 1, \ldots, k$, then

$$1 - \left(\alpha_1 \sum_{i \succeq_{w^\star} \tilde{f}} x^{\mu^1}_{i,w^\star} + \sum_{l=2}^{t} \alpha_l \sum_{i \succeq_{w^\star} \tilde{f}} x^{\mu^l}_{i,w^\star} \right) =$$

$$1 - \sum_{l=2}^{t} \alpha_l \sum_{i \succeq_{w^\star} \tilde{f}} x^{\mu^l}_{i,w^\star} = 1 - \sum_{l=2}^{t} \alpha_l > 0.$$

Then from cases 1.1 and 1.2, we have that for the pair $(\tilde{f}, w^\star)$,

$$\left[q_{\tilde{f}} - \sum_{j \succeq_{\tilde{f}} w^\star} \bar{x}_{\tilde{f},j} \right] \cdot \left[1 - \sum_{i \succeq_{w^\star} \tilde{f}} \bar{x}_{i,w^\star} \right] \neq 0.$$

That is, for the pair $(\tilde{f}, w^\star)$ condition (6) fails.

Case 2: **If** $\boldsymbol{\sigma \cap \sigma^\star = \emptyset}$ **for all** $\boldsymbol{\sigma \in K'}$**.**

Notice that in this case, $\sigma^\star$ may or may not belong to $\Phi(\mu^1)$. Notice also that $\mu^{t-1}(f) \neq \mu^{t-1}[\sigma^\star](f)$ for all $f \in \sigma^\star$.

Case 2.1: $\sigma \notin \Phi(\mu^1)$.

We claim that there are $\bar{f} \in \sigma^\star$ and $\bar{w} \in W \setminus \{\mu^1(\bar{f})) \cup \mu^{t-1}(\bar{f})\}$, such that for $\mu^1(\bar{f}) = \{w_1^{\mu^1}, \ldots, w_{q_{\bar{f}}}^{\mu^1}\}$, $\mu^{t-1}(\bar{f}) = \{w_1^{\mu^{t-1}}, \ldots, w_{q_{\bar{f}}}^{\mu^{t-1}}\}$, we have that $w_1^{\mu^1} \succ_{\bar{f}} \bar{w} \succ_{\bar{f}} w_{q_{\bar{f}}}^{\mu^{t-1}}$.

If not, for all $f \in \sigma^\star$, we have that $\mu^1(f) \succ_f \mu^{t-1}[\sigma^\star](f) \succ_f w$, for each $w \notin \{\mu^1(f) \cup \mu^{t-1}[\sigma^\star](f)\}$.[2] Since $\sigma \cap \sigma^\star = \emptyset$, we have that $\mu^1(f) = \mu^{t-1}(f)$ for all $f \in \sigma^\star$. Then, let $\{w^f\} = \mu^{t-1}[\sigma^\star](f) \setminus \mu^1(f)$ for each $f \in \sigma^\star$. That is, w^f is the most preferred worker in the reduced preference list $P^{\mu^1}(f)$ such that it does not belong to $\mu^1(f)$. This implies that $\sigma^\star \in \Phi(\mu^1)$, and it is a contradiction since $\sigma^\star \notin \Phi(\mu^1)$.

Therefore, there are $\bar{f} \in \sigma^\star$ and $\bar{w} \in W \setminus \{\mu^1(\bar{f})) \cup \mu^{t-1}(\bar{f})\}$, such that

$$\mu^1(\bar{f}) = \mu^{t-1}(\bar{f}) \succ_{\bar{f}} \bar{w} \succ_{\bar{f}} \mu^{t-1}[\sigma^\star](\bar{f}) \tag{11}$$

Since $\sigma^\star \in \Phi(\mu^{t-1})$, in order to obtain the reduced preference lists $P^{\mu^{t-1}}$, $\bar{f}$ should have eliminated $\bar{w}$ by means of the third step of the reduction procedure. Then, we have that

$$\mu^t(\bar{w}) \succeq_{\bar{w}} \mu^{t-1}[\sigma^\star](\bar{w}) \succ_{\bar{w}} \mu^{t-1}(\bar{w}) \succ_{\bar{w}} \bar{f} \succ_{\bar{w}} \mu^1(\bar{w}). \tag{12}$$

Since $\bar{x}$ can be written as in Theorem 1, ($\mu^1, \mu^{t-1} \in \mathcal{M}_{\mu^1}^{K'}$ and $\mu^{t-1} \succ_F \mu^t$), and using inequalities (11) and (12), we have that

$$\sum_{j \succeq_{\bar{f}} \bar{w}} \bar{x}_{\bar{f},j} < q_{\bar{f}} \text{ and } \sum_{i \succeq_{\bar{w}} \bar{f}} \bar{x}_{i,\bar{w}} < 1.$$

Then,

$$\left[q_{\bar{f}} - \sum_{j \succeq_{\bar{f}} \bar{w}} \bar{x}_{\bar{f},j} \right] \cdot \left[1 - \sum_{i \succeq_{\bar{w}} \bar{f}} \bar{x}_{i,\bar{w}} \right] > 0.$$

That is, condition (6) fails for the pair $(\bar{f}, \bar{w})$.

Case 2.2: $\sigma \in \Phi(\mu^1)$.

In this case, we have that $\mu^{t-1}[\sigma^\star] \in \mathcal{M}_{\mu^1} \setminus \mathcal{M}_{\mu^1}^{K'}$. Then by Lemma 4 item 2) and Lemma 5 item 2), there is a cycle $\sigma' \in \Phi\left(\mu^{t-1}[\sigma^\star]\right)$ such that

$$\mu^{t-1} \succ_F \mu^{t-1}[\sigma^\star] \succ_F \mu^{t-1}[\sigma^\star][\sigma'] \succ_F \mu^t.$$

Notice that, σ' may or may not belong to $\Phi(\mu^1)$. If $\sigma \notin \Phi(\mu^1)$, the arguments follows as in Case 2.1. If $\sigma' \in \Phi(\mu^1)$ we continue this process until, by

[2]Here, $\mu^{t-1}[\sigma^\star](f) \succ_f w$ denotes that worker w is less preferred for the firm f than all workers matched to firm f under the stable matching $\mu^{t-1}[\sigma^\star]$.

finiteness of the set $\Phi(\mu^1)$, there is $\tilde{\sigma} \notin \Phi(\mu^1)$, and the arguments follows as in Case 2.1.

Therefore, from cases 1 and 2, we have that there is a connected set $\mathcal{M}_{\mu^1}$ such that $\bar{x} \in Conv\left(\mathcal{M}_{\mu^1}\right)$.

($\Longleftarrow$) Let $\bar{x}$ be a convex combination of stable matchings from a connected set. That is, there are a stable matching μ, and a list of sets $K_1, \ldots, K_k \subseteq \Phi(\mu)$ with the corresponding cyclic matchings $\mu^1, \ldots, \mu^k$, such that $\bar{x} = \sum_{l=1}^{k} \alpha_l x^{\mu^l}$ with $0 \leq \alpha_l \leq 1$ and $\sum_{l=1}^{k} \alpha_l = 1$.

Since $\bar{x}$ is a convex combination of stable matchings from $\mathcal{M}_\mu$, we have that $\mu \succeq_F \mu^l$ for each $l = 1, \ldots, k$. Then, we have that $\bar{x}$ is a stable fractional matching for the matching market (F, W, P^μ). Moreover, since $S(P^\mu) \subseteq S(P)$, we have that $\bar{x}$ is also a stable fractional matching for the matching market $(F, W, P, \boldsymbol{q})$. By Lemma 8, we only need to prove that $\bar{x}$ is strongly stable in the reduced preference profile P^μ.

If $\alpha_1 = 1$ we have that $\bar{x} = x^{\mu^1}$. Since μ^1 is also a stable matching in the original preferences profile, then we have that $\bar{x}$ is strongly stable. Hence, we assume $0 < \alpha_l < 1$ for each $l = 1, \ldots, k$. Now, we prove that $\bar{x}$ fulfils condition (6) for each pair $(f, w) \in A(P^\mu)$.

Fix $f \in F$. Assume that firm f does not fill its quota. Theorem RHT assures that this firm is always assigned to the same set of workers in every stable matching. Then $\bar{x}_{f,j} = x^{\mu^l}_{f,j}$ for $l = 1, \ldots, k$ and for all j such that $(f, j) \in A(P^\mu)$. Since μ^l is a stable matching for $l = 1, \ldots, k$, it fulfils condition (6) for each $(f, j) \in A(P^\mu)$. Then, by Remark 1 we have that $\bar{x}$ also fulfil condition (6) for each $(f, j) \in A(P^\mu)$.

Assume now that f does fill its quota. Let $\mathcal{K} = \bigcup_{l=1}^{k} K_l$. Let $\mu(f) = \{w_1, \ldots, w_{q_f}\}$ and $w_i \succ_f w_{i+1}$. We analyse two cases separately.

Case 1: There is no $\sigma \in \mathcal{K}$ such that $f \in \sigma$.

By Definition 7,we have that $\mu[\mathcal{K}](f) = \mu(f)$. That is, for each $j \in W$, $\bar{x}_{f,j} = x^\mu_{f,j}$. Thus, if $w \preceq_f w_{q_f}$ we have that

$$\sum_{j \succeq^\mu_f w} \bar{x}_{f,j} = \sum_{j \succeq^\mu_f w} x^\mu_{f,j} = q_f.$$

If $w \succ_f w_{q_f}$, we have that

$$\sum_{j \succeq^\mu_f w} \bar{x}_{f,j} = \sum_{j \succeq^\mu_f w} x^\mu_{f,j} < q_f.$$

Then, $\sum_{j \succ^\mu_f w} x^\mu_{f,j} < q_f$. Moreover, by linear inequality (5) for the stable matching μ, we have that $\sum_{i \succeq^\mu_w f} x^\mu_{i,w} > 0$. Therefore, $\sum_{i \succeq^\mu_w f} x^\mu_{i,w} = 1$. Recall that μ is the firm oltimal stable matching in the reduced preference profile P^μ, then Lemma 7 states that $\bar{x} \succeq_W^\mu x^\mu$, i.e.,

$$\sum_{i \succeq_w^\mu f} \bar{x}_{i,w} \geq \sum_{i \succeq_w^\mu f} x^\mu_{i,w} = 1.$$

By the linear inequality (2), we have that $\sum_{i \succeq_w^\mu f} \bar{x}_{i,w} = 1$. Thus, for $(f, w) \in A(P^\mu)$, we have that

$$\left[q_f - \sum_{j \succeq_f^\mu w} \bar{x}_{f,j} \right] \cdot \left[1 - \sum_{i \succeq_w^\mu f} \bar{x}_{i,w} \right] = 0.$$

Case 2: There is $\sigma_f \in \mathcal{K}$ such that $f \in \sigma_f$.

By Lemma 5, there is a unique cycle $\sigma_f \in K_l$ such that $f \in \sigma_f$. But σ_f may be in more than one set K_l. We denote $L_f = \{l : \sigma_f \in K_l\}$. Therefore,

$$\bar{x}_{f,j} = \sum_{l=1}^{k} \alpha_l x^{\mu^l}_{f,j} = \sum_{l \in L_f} \alpha_l x^{\mu^l}_{f,j} + \sum_{l \notin L_f} \alpha_l x^{\mu^l}_{f,j}.$$

Since σ_f is unique, by Lemma 5 and Lemma 6, we have that $\mu[K_l](f) = \mu[\sigma_f](f)$ and $x^{\mu^l}_{f,j} = x^{\mu[\sigma_f]}_{f,j}$ for those $l \in L_f$. Also, $\mu[K_l](f) = \mu(f)$ and $x^{\mu^l}_{f,j} = x^\mu_{f,j}$ for those $l \notin L_f$. Hence,

$$\sum_{l \in L_f} \alpha_l x^{\mu^l}_{f,j} + \sum_{l \notin L_f} \alpha_l x^{\mu^l}_{f,j} = \sum_{l \in L_f} \alpha_l x^{\mu[\sigma_f]}_{f,j} + \sum_{l \notin L_f} \alpha_l x^{\mu}_{f,j} =$$

$$x^{\mu[\sigma_f]}_{f,j} \left(\sum_{l \in L_f} \alpha_l \right) + x^\mu_{f,j} \left(\sum_{l \notin L_f} \alpha_l \right). \tag{13}$$

Since $\sum_{l \in L_f} \alpha_l + \sum_{l \notin L_f} \alpha_l = 1$, then we define $\bar{\alpha} = \sum_{l \in L_f} \alpha_l$.

Then (13) is equal to $\bar{\alpha} x^{\mu[\sigma_f]}_{f,j} + (1 - \bar{\alpha})\, x^\mu_{f,j}$. That is, $\bar{x}_{f,j}$ is the convex combination of $x^{\mu[\sigma_f]}_{f,j}$ and $x^\mu_{f,j}$. Since $f \in \sigma_f$, then

$$|supp(x^{\mu[\sigma_f]}_{f,\cdot})| = |supp(x^\mu_{f,\cdot})| = q_f \text{ and } |supp(x^{\mu[\sigma_f]}_{f,\cdot}) \cap supp(x^\mu_{f,\cdot})| = q_f - 1.$$

Hence, there are two workers w_a and w_{q_f+1} such that $(f, w_a) \in supp(x^\mu) \setminus supp(x^{\mu[\sigma_f]})$ and $(f, w_{q_f+1}) \in supp(x^{\mu[\sigma_f]}) \setminus supp(x^\mu)$. Let $T = \{w_s : (f, w_s) \in supp(x^\mu) \cap supp(x^{\mu[\sigma_f]})\}$. Then,

$$\bar{x}_{f,j} = \begin{cases} 1 & \text{if } j \in T \\ 1-\bar{\alpha} & \text{if } j = w_a \\ \bar{\alpha} & \text{if } j = w_{q_f+1}. \end{cases} \tag{14}$$

Now, we prove that $\bar{x}$ fulfils condition (6) in P^μ for each w:

i) If $w \preceq_f w_{q_f+1}$, then

$$\sum_{j \succeq_f^\mu w} \bar{x}_{f,j} = \sum_{s \in T} \bar{x}_{f,w_s} + \bar{x}_{f,w_a} + \bar{x}_{f,w_{q_f+1}} = (q_f - 1) + (1-\bar{\alpha}) + \bar{\alpha} = q_f.$$

Then,

$$q_f - \sum_{j \succeq_f^\mu w} \bar{x}_{f,j} = 0,$$

hence,

$$\left[q_f - \sum_{j \succeq_f^\mu w} \bar{x}_{f,j} \right] \cdot \left[1 - \sum_{i \succeq_w^\mu f} \bar{x}_{i,w} \right] = 0.$$

ii) If $w \succ_f^\mu w_{q_f}$, then

$$\sum_{j \succeq_f^\mu w} \bar{x}_{f,j} < q_f.$$

Since

$$q_f > \sum_{j \succeq_f^\mu w} \bar{x}_{f,j} = (1-\bar{\alpha}) \sum_{j \succeq_f^\mu w} x_{f,j}^\mu + \bar{\alpha} \sum_{j \succeq_f^\mu w} x_{f,j}^{\mu[\sigma_f]},$$

then,

$$\sum_{j \succeq_f^\mu w} x_{f,j}^\mu < q_f \text{ and } \sum_{j \succeq_f^\mu w} x_{f,j}^{\mu[\sigma_f]} < q_f.$$

But μ and $\mu[\sigma]$ are stable matchings, and these stable matchings fulfil condition (6). So we have that

$$\sum_{i \succeq_w^\mu f} x_{i,w}^\mu = 1 \text{ and } \sum_{i \succeq_w^\mu f} x_{i,w}^{\mu[\sigma_f]} = 1,$$

in which case we can assure that

$$\sum_{i \succeq_w^\mu f} \bar{x}_{i,w} = (1-\bar{\alpha}) \sum_{i \succeq_w^\mu f} x_{i,w}^\mu + \bar{\alpha} \sum_{i \succeq_w^\mu f} x_{i,w}^{\mu[\sigma_f]} = 1.$$

That is,

$$\left[q_f - \sum_{j \succeq_f^\mu w} \bar{x}_{f,j} \right] \cdot \left[1 - \sum_{i \succeq_w^\mu f} \bar{x}_{i,w} \right] = 0.$$

iii) If $w = w_{q_f}$. Recall that $\mu(f) = \{w_1, \ldots, w_{q_f}\}$, then $\mu(w) = f$. Also, we have that , $\mu \succ_F^{\mu} \mu^l$ and $\mu^l \succeq_W^{\mu} \mu$ for all $l = 1, \ldots, k$. In particular, $\mu^l(w) \succeq_w^{\mu} \mu(w) = f$ for all $l = 1, \ldots, k$. This implies that

$$\sum_{i \succeq_w^{\mu} f} x_{i,w}^{\mu^l} = 1$$

for all $l = 1, \ldots, k$. Hence

$$\sum_{i \succeq_w^{\mu} f} \bar{x}_{i,w} = \sum_{i \succeq_w^{\mu} f} \sum_{l=1}^{t} \alpha_l x_{i,w}^{\mu^l} = \sum_{l=1}^{t} \alpha_l \sum_{i \succeq_w^{\mu} f} x_{i,w}^{\mu^l} = \sum_{l=1}^{t} \alpha_l = 1.$$

Then,

$$\left[q_f - \sum_{j \succeq_f^{\mu} w} \bar{x}_{f,j} \right] \cdot \left[1 - \sum_{i \succeq_w^{\mu} f} \bar{x}_{i,w} \right] = 0.$$

From cases 1 and 2, we have that for the pair (f, w),

$$\left[q_f - \sum_{j \succeq_f^{\mu} w} \bar{x}_{f,j} \right] \cdot \left[1 - \sum_{i \succeq_w^{\mu} f} \bar{x}_{i,w} \right] = 0.$$

Therefore, $\bar{x}$ is a strongly stable fractional matching in the reduced preference profile P^{μ}, and by Lemma 8, $\bar{x}$ is a strongly stable fractional matching in the original preference profile P. □

Once we have characterized all strongly stable fractional matchings for the many-to-one matching market $(F, W, P, \boldsymbol{q})$, we can characterize the set of all strongly stable fractional matchings. Recall that, $SSF(P)$ denotes the set of all strongly stable fractional matchings at the preference profile P.

Corollary 2 *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. Then,*

$$SSF(P) = \bigcup_{\mu \in S(P)} Conv\{\mathcal{M}_{\mu}\}.$$

The following corollary extends Corollary 21 from Roth *et al.* [16]. It gives an upper bound to the number of worker matched to each firm, and the number of firms matched to each worker.

Corollary 3 *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. Each strongly stable fractional matching fulfils the following two conditions:*

1. *Each worker has a positive probability with at most two distinct firms.*

2. *Each firm, all but possibly one position are assigned deterministically. For the one position that is assigned by a lottery, two workers have a positive probability of being employed.*

Notice that, from (14) in the proof of Theorem 2, the previous corollary follows straightforward. Another extension is due to Schlegel [18] for a school choice matching market with strict preferences (similar setting as ours). Our characterization gives an alternative proof for these two similar results, for the school choice set-up presented in Schlegel [18] is straightforward, and for the marriage market presented in Roth *et al.* [16], it's necessary only to set all quotas of all firms equal to one.

Conclusions.

In this paper we present a strong stability condition for a many-to-one matching market where firms' preference are q-responsive. Further, we prove that a strongly stable fractional matching can be represented as a convex combination of stable matchings that fulfil a decreasing order in the eyes of all firms. Although it was already known the "almost" integrability of a strongly stable fractional matchings, there may be more "almost" integral stable fractional matchings that are not strongly stable (the stable fractional matching x^1 in Example 1 illustrates this). Our characterization of strongly stable fractional matching allows us to describe precisely which are the agents matched. Also, we characterize the set of all strongly stable fractional matchings. We think that our results gives a complete description of the strongly stable fractional matchings.

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A Appendix

The reduction procedure:

Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. Let μ_F be the optimal stable matching for all firms, and μ_W be the optimal stable matching for all workers.

Step 1: Remove all w who are more preferred than the most preferred worker matched under $\mu_F(f)$ from f's list of acceptable workers. Remove all f who are more preferred than $\mu_W(w)$ from w's list of acceptable firms.

Therefore, the most preferred worker matched in $\mu_F(f)$ will be the first entry in f's reduced list, and $\mu_W(w)$ will be the first entry in w's reduced list.

Step 2: Remove all f who are less preferred than $\mu_F(w)$ from w's list of acceptable firms. Remove all w who are less preferred than the least worker matched under $\mu_W(f)$ from f's list of acceptable workers.

Thus, $\mu_F(w)$ will be the last entry in w's reduced list and the least preferred worker in $\mu_W(f)$ will be the last entry in f's reduced list.

Step 3: After steps 1 and 2, if f is not acceptable for w (i.e., if f is not on w's preference list as now modified), then remove w from f's list of acceptable workers, and similarly, remove from w's list of acceptable firms, any firm f to whom w is no longer acceptable.

Hence, f will be acceptable for w if and only if w is acceptable for f after Step 3.

For the matching market $(F, W, P^{\mu}, \boldsymbol{q})$, the stable matching μ is the $F-optimal$ stable matching, that is the stable matching that all firm prefer in the matching market $(F, W, P^{\mu}, \boldsymbol{q})$.

Lemmas and Proofs.

Proof of Lemma 2. Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. Let $\bar{x}$ be a strongly stable fractional matching. First, we will prove that $\mu_{\bar{x}}$ is a matching. Assume that all positive entries of $\bar{x}$ are equal to 1, then we have by Definition 4 that $x^{\mu_{\bar{x}}} = \bar{x}$. Since $\bar{x}$ is a strongly stable fractional matching, by Lemma 1 we have that $\mu_{\bar{x}}$ is a stable matching.

Assume now that not all positive entries of $\bar{x}$ are equal to 1. We will prove that $\mu_{\bar{x}}$ is a matching. Assume that is not a matching. That is, there is a worker w and two different firms f and f', such that $w \in \mu_{\bar{x}}(f)$ and $w \in \mu_{\bar{x}}(f')$. Since the preferences of the worker w are strict, without loss of generality, we can assume that $f \succ_w f'$. We will show that $\sum_{i \succeq_w f} \bar{x}_{i,w} = 1$, for this we analyse two cases:

Case 1: $|C_f^0(\bar{x})| \leq q_f$.

We have that if $\sum_{j \succeq_f w} \bar{x}_{f,j} < q_f$, since $\bar{x}$ is a stable fractional matching. Then condition (6) implies that

$$\sum_{i \succeq_w f} \bar{x}_{i,w} = 1.$$

Hence, $\sum_{i \prec_w f} \bar{x}_{i,w} = 0$, and $\bar{x}_{f',w} = 0$, which contradicts the assumption of $\bar{x}_{f',w} > 0$.

If $\sum_{j \succeq_f w} \bar{x}_{f,j} = q_f$, then $w \in C_f^{q_f}$ and $\bar{x}_{f,w} = 1$. This implies that

$$\sum_{i \succeq_w f} \bar{x}_{i,w} = 1.$$

Case 2: $|C_f^0(\bar{x})| > q_f$.

Notice that $C_f^{q_f}(\bar{x}) \subset C_f^0(\bar{x})$. Since $w \in \mu_{\bar{x}}(f)$, then $w \in C_f^{q_f}(\bar{x})$, and we have that

$$\sum_{j \succeq_f w} \bar{x}_{f,j} \leq \sum_{j \in C_f^{q_f}(\bar{x})} \bar{x}_{f,j} < \sum_{j \in C_f^0(\bar{x})} \bar{x}_{f,j} \leq q_f$$

Hence, $\sum_{j \succeq_f w} \bar{x}_{f,j} < q_f$. Since $\bar{x}$ is a strongly stable fractional matching, condition (6) implies that

$$\sum_{i \succeq_w f} \bar{x}_{i,w} = 1.$$

From cases 1 and 2, we have that $\sum_{i \succeq_w f} \bar{x}_{i,w} = 1$, then $\sum_{i \prec_w f} \bar{x}_{i,w} = 0$, and also $\bar{x}_{f',w} = 0$ since $f \succ_w f'$, which contradicts the assumption of $\bar{x}_{f',w} > 0$. Therefore $\mu_{\bar{x}}$ is a matching.

Now, we will prove that $\mu_{\bar{x}}$ is a stable matching. Let $w \in \mu_{\bar{x}}(f)$, then $w \in C_f^{q_f}(\bar{x}) \subseteq C_f^0(\bar{x})$. Hence $w \succ_f f$ and $f \succ_w w$. Then $\mu_{\bar{x}}$ in an individually rational matching.

Assume that there is a blocking pair $(\bar{f}, \bar{w})$ of $\mu_{\bar{x}}$. This means that we have the following three statements:

a) $\bar{w} \notin \mu_{\bar{x}}(\bar{f})$.

b) There is $w' \in \mu_{\bar{x}}(\bar{f})$ such that: either $\bar{w} \succ_{\bar{f}} w'$ if $|\mu_{\bar{x}}(\bar{f})| = q_{\bar{f}}$, or $\bar{w} \succ_{\bar{f}} \bar{f}$ if $|\mu_{\bar{x}}(\bar{f})| < q_{\bar{f}}$.

c) $\bar{f} \succ_{\bar{w}} \mu_{\bar{x}}(\bar{w})$.

Now, we will show that $\bar{x}_{\bar{f},\bar{w}} = 0$:

i) If $|C_f^{q_f}(\bar{x})| < q_f$, then $C_f^{q_f}(\bar{x}) = C_f^0(\bar{x})$. Hence, since $\mu_{\bar{x}}(\bar{f}) = C_f^0(\bar{x})$ and $\bar{w} \notin \mu_{\bar{x}}(\bar{f})$, we have that $\bar{x}_{\bar{f},\bar{w}} = 0$.

ii) If $|C_{\bar{f}}^{q_{\bar{f}}}(\bar{x})| = q_{\bar{f}}$. Let $w^\star \in C_{\bar{f}}^{q_{\bar{f}}}(\bar{x}) \setminus C_{\bar{f}}^{q_{\bar{f}}-1}(\bar{x})$. That is, $w^\star$ is the least preferred worker employed with firm $\bar{f}$ under the matching $\mu_{\bar{x}}$. Then, by item b), we have that

$$\sum_{j \succeq_{\bar{f}} \bar{w}} \bar{x}_{\bar{f},j} < \sum_{j \succeq_{\bar{f}} w^\star} \bar{x}_{\bar{f},j} \leq q_{\bar{f}}. \tag{15}$$

By items a) and b), we have that $\bar{w} \succ_{\bar{f}} w^\star$, and by Definition 4 we can assure that $\bar{x}_{\bar{f},\bar{w}} = 0$. If not, we have that $\bar{w} \in \mu_{\bar{x}}(\bar{f})$, a contradiction.

By (15) and the fact that $\bar{x}$ by hypothesis is strongly stable, we have that

$$1 - \sum_{i \succeq_{\bar{w}} \bar{f}} \bar{x}_{i,\bar{w}} = 0.$$

Then,

$$1 = \sum_{i \succeq_{\bar{w}} \bar{f}} \bar{x}_{i,\bar{w}} = \bar{x}_{\bar{f},\bar{w}} + \sum_{i \succ_{\bar{w}} \bar{f}} \bar{x}_{i,\bar{w}} = 0 + \sum_{i \succ_{\bar{w}} \bar{f}} \bar{x}_{i,\bar{w}}.$$

Hence, $\sum_{i \prec_{\bar{w}} \bar{f}} \bar{x}_{i,\bar{w}} = 0$, but this is a contradiction since from Definition 4, we have that $\bar{x}_{\mu_{\bar{x}}(\bar{w}),\bar{w}} > 0$ and by item c) we have $\bar{f} \succ_{\bar{w}} \mu_{\bar{x}}(\bar{w})$. Therefore, $\mu_{\bar{x}}$ is a stable matching. □

Proof of Lemma 3. Let $\bar{x}$ be a strongly stable fractional matching. Then, by Lemma (2) we have that $\mu_{\bar{x}}$ is a stable matching. By Definition 4 we have that $supp(x^{\mu_{\bar{x}}}) \subset supp(\bar{x})$, and by the fact that $\bar{x} \neq x^{\mu_{\bar{x}}}$ we have that

$$\bar{x} = \alpha x^{\mu_{\bar{x}}} + (1 - \alpha) y.$$

We need to prove that

$$y = \frac{\bar{x} - \alpha x^{\mu_{\bar{x}}}}{1 - \alpha}$$

is a strongly stable fractional matching. That is, y is a solution of CP and fulfils condition (6).

From $\bar{x} \neq x^{\mu_{\bar{x}}}$, we have that $\alpha > 0$. From definition of α, we have that $\alpha < 1$.
Assume that $\alpha = \bar{x}_{\bar{f},\bar{w}}$, with $\bar{w} \in C_{\bar{f}}^{q_{\bar{f}}}(\bar{x})$.

- **Inequality (1) of *CP*.** Following from the definition of y and the definition of α, we have that:

  If $|C_f^0(\bar{x})| \geq q_f$, then

  $$\sum_{j \in W} \bar{x}_{f,j} - \alpha \sum_{j \in W} x_{f,j}^{\mu_{\bar{x}}} = \sum_{i \in W} \bar{x}_{f,j} - \alpha q_f \leq q_f - \alpha q_f.$$

  Therefore,

  $$\sum_{j \in W} y_{f,j} = \frac{1}{1-\alpha} \left[\sum_{j \in W} \bar{x}_{f,j} - \alpha \sum_{j \in W} x_{f,j}^{\mu_{\bar{x}}} \right] \leq q_f.$$

  If $|C_f^0(\bar{x})| = r < q_f$, then $\sum_{j \in W} \bar{x}_{f,j} - \alpha \sum_{j \in W} x_{f,j}^{\mu_{\bar{x}}} = \sum_{i \in W} \bar{x}_{f,j} - \alpha r \leq r - \alpha r = r(1-\alpha) < q_f(1-\alpha)$. Then,

  $$\sum_{j \in W} y_{f,j} = \frac{1}{1-\alpha} \left[\sum_{j \in W} \bar{x}_{f,j} - \alpha \sum_{j \in W} x_{f,j}^{\mu_{\bar{x}}} \right] \leq q_f.$$

  That is, y satisfy linear inequality (1).

- **Inequality (2) of *CP*.** A similar argument that is used for inequality (1), proves that y satisfy linear inequality (2).

- **Inequality (3) of *CP*.** If $(f,w) \in supp(\bar{x})$, by definition of $\mu_{\bar{x}}$, we shall consider two cases:

  - If $(f,w) \notin supp(x^{\mu_{\bar{x}}})$, that is, $x_{f,w}^{\mu_{\bar{x}}} = 0$. Then, $y_{f,w} = \frac{\bar{x}_{f,w}}{1-\alpha} = 0$.
  - If $(f,w) \in supp(x^{\mu_{\bar{x}}})$, that is, $x_{f,w}^{\mu_{\bar{x}}} = 1$ and also $x_{\bar{f},\bar{w}}^{\mu_{\bar{x}}} = 1$.

  Then, $y_{f,w} = \frac{\bar{x}_{f,w} - \alpha x_{f,w}^{\mu_{\bar{x}}}}{1-\alpha} = \frac{\bar{x}_{f,w} - \alpha}{1-\alpha} \geq 0$. Then, for $(f,w) \in A$, $y_{f,w} \geq 0$, that is y satisfies linear inequality (3).

- **Inequality (4) of *CP*.** Inequality (4) it is easily satisfied form definition of y.

- **Condition (6).** By hypothesis we have that $\bar{x}$ is a strongly stable fractional matching, then $\bar{x}$ fulfils condition (6). Hence,

$$\left[q_f - \sum_{j\succeq_f w} \bar{x}_{f,j}\right] \cdot \left[1 - \sum_{i\succeq_w f} \bar{x}_{i,w}\right] = 0,$$

for each $(f,w) \in A(P)$. Since $\bar{x} = \alpha x^{\mu_{\bar{x}}} + (1-\alpha)y$, with $0 < \alpha < 1$, then for each $(f,w) \in A(P)$ we have that

$$\left[q_f - \sum_{j\succeq_f w} \left(\alpha x^{\mu_{\bar{x}}}_{f,j} + (1-\alpha)y_{f,j}\right)\right] \cdot \left[1 - \sum_{i\succeq_w f} \left(\alpha x^{\mu_{\bar{x}}}_{i,w} + (1-\alpha)y_{i,w}\right)\right] = 0$$

$$\left[q_f - \alpha \sum_{j\succeq_f w} x^{\mu_{\bar{x}}}_{f,j} - (1-\alpha) \sum_{j\succeq_f w} y_{f,j}\right] \cdot \left[1 - \alpha \sum_{i\succeq_w f} x^{\mu_{\bar{x}}}_{i,w} - (1-\alpha) \sum_{i\succeq_w f} y_{i,w}\right] = 0$$

$$\left[\alpha \left(q_f - \sum_{j\succeq_f w} x^{\mu_{\bar{x}}}_{f,j}\right) + (1-\alpha) \left(q_f - \sum_{j\succeq_f w} y_{f,j}\right)\right] \cdot$$
$$\left[\alpha \left(1 - \sum_{i\succeq_w f} x^{\mu_{\bar{x}}}_{i,w}\right) + (1-\alpha) \left(1 - \sum_{i\succeq_w f} y_{i,w}\right)\right] = 0. \quad (16)$$

Since $x^{\mu_{\bar{x}}}$ and y fulfil inequality (1) and (2), then

$$q_f - \sum_{j\succeq_f w} x^{\mu_{\bar{x}}}_{f,j} \geq 0, \;\; q_f - \sum_{j\succeq_f w} y_{f,j} \geq 0,$$
$$sum_{i\succeq_w f} x^{\mu_{\bar{x}}}_{f,j} \geq 0 \text{ and } 1 - \sum_{j\succeq_w f} y_{f,j} \geq 0. \quad (17)$$

Then, by (16) and (17), we have that either

$$q_f - \sum_{j\succeq_f w} x^{\mu_{\bar{x}}}_{f,j} = 0 \text{ and } q_f - \sum_{j\succeq_f w} y_{f,j} = 0$$

or

$$1 - \sum_{i\succeq_w f} x^{\mu_{\bar{x}}}_{i,w} = 0 \text{ and } 1 - \sum_{i\succeq_w f} y_{i,w} = 0.$$

Since, by Lemma 2 $\mu_{\bar{x}}$ is a stable matching, and by Remark 1 $x^{\mu_{\bar{x}}}$ fulfils condition (6) for each $(f,w) \in A(P)$. Therefore, y fulfils condition (6) for each $(f,w) \in A(P)$.

Since $supp(x^{\mu_{\bar{x}}}) \subseteq supp(\bar{x})$, we have that $supp(y) \subseteq supp(\bar{x})$. Moreover, since $y_{\bar{f},\bar{w}} = 0$, and $x^{\mu_{\bar{x}}}_{\bar{f},\bar{w}} = 1$, then $supp(y) \subset supp(\bar{x})$. □

Proof of Lemma 6. Let P^μ be a profile of reduced lists for (F, W, P), and let σ and σ' be two different cycles in P^μ.

1. By Lemma 5, we have that $\sigma \cap \sigma' = \emptyset$. Then, we can assume that $\sigma' = \{e_1, \ldots, e_r\}$ and $\sigma = \{e_{r+1}, \ldots, e_{r+r'}\}$. By Definition 7,

$$\mu[\sigma'](f) = \begin{cases} \mu(e_1) & \text{if } f = e_r \\ \mu(e_{k+1}) & \text{if } f = e_k,\ k = 1, \ldots, r-1 \\ \mu(f) & \text{if } f \notin \{e_1, \ldots, e_r\}. \end{cases}$$

   That is, the firms that do not belong to the cycle σ' do not change the set of workers in both stable matchings (μ and $\mu[\sigma']$). Then, by Lemma 5, the cycle σ is a subset of those firms that do not change. That is, σ is a cycle of $P^{\mu[\sigma']}$.

2. Let $K = \{\sigma, \sigma'\}$. Since K is a set of cycles (not an ordered set), by Definition 7 and item 1., we have that

$$\mu[K]\,(f) = \begin{cases} \mu[\sigma](f) & \text{if } f \in \sigma \\ \mu[\sigma'](f) & \text{if } f \in \sigma' \\ \mu(f) & \text{otherwise.} \end{cases}$$

   Then, $\mu[\sigma', \sigma] = \mu[\sigma, \sigma']$.

□

We say that a fractional matching x **weakly dominates** a fractional matching y with respect to the preference of the firm f, if for all workers w,

$$\sum_{j \succeq_f w} x_{f,j} \geq \sum_{j \succeq_f w} y_{f,j},$$

and it will be denoted by $x \succeq_f y$, using the same notation that is used for stable matchings.

Similarly x **strongly dominates** y, denoted by $x \succ_f y$, if the previous inequality holds strictly for at least one worker w. Weak and strong domination under a worker's preferences are defined analogously. We say that $x \succeq_F y$ when $x \succeq_f y$ for all $f \in F$. The relation $x \succeq_W y$ is defined analogously.

Lemma 7 *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. Let $\bar{x}$ be a strongly stable fractional matching. Then, $x^{\mu_F} \succeq_f \bar{x} \succeq_f x^{\mu_W}$ for all $f \in F$ and $x^{\mu_W} \succeq_w \bar{x} \succeq_w x^{\mu_F}$ for all $w \in W$.*

Proof. Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. Let $\bar{x}$ be a strongly stable fractional matching. By Theorem 1, there are stable matchings $\mu^1, \ldots, \mu^k$ and real numbers $\alpha_1, \ldots, \alpha_k$, such that $\bar{x} = \sum_{l=1}^{k} \alpha_l x^{\mu^l}$, with $0 < \alpha_l \leq 1$, $\sum_{l=1}^{k} \alpha_l = 1$ and

$\mu^1 \succ_F \mu^2 \succ_F \ldots \succ_F \mu^k$. Since $\mu_F \succeq_F \mu^l \succeq_F \mu_W$ for each $l = 1, \ldots, k$, then for $f \in F$ we have that

$$\sum_{j \succeq_f w} x^{\mu_F}_{f,j} = \sum_{j \succeq_f w} \left(\sum_{l=1}^{k} \alpha_l x^{\mu_F}_{f,j} \right) = \sum_{l=1}^{k} \alpha_l \left(\sum_{j \succeq_f w} x^{\mu_F}_{f,j} \right) \geq$$

$$\sum_{l=1}^{k} \alpha_l \left(\sum_{j \succeq_f w} x^{\mu^l}_{f,j} \right) = \sum_{j \succeq_f w} \left(\sum_{l=1}^{k} \alpha_l x^{\mu^l}_{f,j} \right) = \sum_{j \succeq_f w} \bar{x}_{f,j},$$

for all $w \in W$. Then $x^{\mu_F} \succeq_f \bar{x}$.

To prove that $\bar{x} \succeq_f x^{\mu_W}$,

$$\sum_{j \succeq_f w} \bar{x}_{f,j} = \sum_{j \succeq_f w} \left(\sum_{l=1}^{k} \alpha_l x^{\mu^l}_{f,j} \right) = \sum_{l=1}^{k} \alpha_l \left(\sum_{j \succeq_f w} x^{\mu^l}_{f,j} \right) \geq$$

$$\sum_{l=1}^{k} \alpha_l \left(\sum_{j \succeq_f w} x^{\mu_W}_{f,j} \right) = \sum_{j \succeq_f w} \left(\sum_{l=1}^{k} \alpha_l x^{\mu_W}_{f,j} \right) = \sum_{j \succeq_f w} x^{\mu_W}_{f,j},$$

for all $w \in W$. Then $\bar{x} \succeq_f x^{\mu_W}$.

A similar argument proves that $x^{\mu_W} \succeq_w \bar{x} \succeq_w x^{\mu_F}$. □

Lemma 8 *Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. Let $\mu \in S(P)$, and P^μ the reduced preference profile. Let $\bar{x}$ be a stable fractional matching for a many-to-one matching market $(F, W, P, \boldsymbol{q})$. If $\bar{x}$ is a strongly stable fractional matching for a many-to-one matching market $(F, W, P^\mu, \boldsymbol{q})$, then $\bar{x}$ is a strongly stable fractional matching for a many-to-one matching market $(F, W, P, \boldsymbol{q})$.*

Proof. Let $(F, W, P, \boldsymbol{q})$ be a many-to-one matching market. Let $\mu \in S(P)$, and P^μ be the reduced preference profile. Let $\bar{x}$ be a stable fractional matching for a many-to-one matching market $(F, W, P, \boldsymbol{q})$. Let $\bar{x}$ be a strongly stable fractional matching for a many-to-one matching market $(F, W, P^\mu, \boldsymbol{q})$, that is:

$$\left[q_f - \sum_{j \geq^\mu_f w} \bar{x}_{f,j} \right] \cdot \left[1 - \sum_{i \geq^\mu_w f} \bar{x}_{i,w} \right] = 0,$$

for all $(f, w) \in A(P^\mu)$.

We need to prove that, for all $(f, w) \in A(P)$, $\bar{x}$ fulfils

$$\left[q_f - \sum_{j \succeq_f w} \bar{x}_{f,j} \right] \cdot \left[1 - \sum_{i \succeq_w f} \bar{x}_{i,w} \right] = 0.$$

We consider the following two cases:

1. Let $(f,w) \in A(P^{\mu})$. That is, (f,w) was not eliminated in P^{μ}. So,

$$\sum_{j \succeq_f w} \bar{x}_{f,j} \geq \sum_{j \geq^{\mu}_f w} \bar{x}_{f,j}$$

holds, since for each firm f, there are more workers in the original preference list than in the reduced preference list.

Hence,

$$q_f - \sum_{j \succeq_f w} \bar{x}_{f,j} \leq q_f - \sum_{j \geq^{\mu}_f w} \bar{x}_{f,j}.$$

With a similar argument we have that

$$1 - \sum_{i \succeq_w f} \bar{x}_{i,w} \leq 1 - \sum_{i \geq^{\mu}_w f} \bar{x}_{i,w}.$$

By hypothesis, and linear inequalities (1) and (2) of PC,

$$0 = \left[q_f - \sum_{j \geq^{\mu}_f w} \bar{x}_{f,j} \right] \cdot \left[1 - \sum_{i \geq^{\mu}_w f} \bar{x}_{i,w} \right] \geq \left[q_f - \sum_{j \succeq_f w} \bar{x}_{f,j} \right] \cdot \left[1 - \sum_{i \succeq_w f} \bar{x}_{i,w} \right] \geq 0.$$

Then, for $(f,w) \in A(P^{\mu})$, we have that

$$\left[q_f - \sum_{j \succeq_f w} \bar{x}_{f,j} \right] \cdot \left[1 - \sum_{i \succeq_w f} \bar{x}_{i,w} \right] = 0.$$

2. Let $(f,w) \in A(P) - A(P^{\mu})$. Let $w_1 \in \mu(f)$ such that $w_1 \succeq_f w'$ for each $w' \in \mu(f)$. Notice that $w \neq w_1$. Then, we analyse two sub-cases:

   i) $w \succ_f w_1$.
   We have that $x^{\mu} \succeq_F \bar{x}$, then

$$\sum_{j \succ_f w} \bar{x}_{f,j} \leq \sum_{j \succ_f w} x^{\mu}_{f,j} = 0. \tag{18}$$

   Since, $\bar{x}$ is a stable fractional matching, $\bar{x}$ satisfy inequality (5) of SPC, i.e.

$$\sum_{j \succ_f w} \bar{x}_{f,j} + q_f \sum_{i \succ_w f} \bar{x}_{i,w} + q_f \bar{x}_{f,w} \geq q_f.$$

   Then, by condition (18)

$$q_f \sum_{i \succeq_w f} \bar{x}_{i,w} \geq q_f,$$

and for all $i \succeq_w f$, $\bar{x}_{i,w} = 1$. Hence,

$$\sum_{i \succeq_w f} \bar{x}_{i,w} \geq 1,$$

and by linear inequality (2), we have that

$$\sum_{i \succeq_w f} \bar{x}_{i,w} = 1,$$

then we have that

$$\left[q_f - \sum_{j \succeq_f w} \bar{x}_{f,j}\right] \cdot \left[1 - \sum_{i \succeq_w f} \bar{x}_{i,w}\right] = 0.$$

ii) $w_1 \succ_f w$. We analyse two sub-cases, if the firm f does or does not fill its quota.

a) If the firm f does not fill its quota, by Theorem RHT, the firm f is assigned to the same set of workers in every stable matching. Assume that $\mu(f) = \{w_1, \ldots, w_p\}$ with $p < q_f$. Recall that $w_1 \succ_f w$, then

$$0 < \sum_{j \succ_f w} \bar{x}_{f,j} < q_f.$$

Assume that $0 < \sum_{i \succeq_w f} \bar{x}_{i,w} < 1$, then $\sum_{i \prec_w f} \bar{x}_{i,w} > 0$. Since $\bar{x}$ is a strongly stable fractional matching for the reduced preference profile P^μ, then by Theorem 1 there are stable matchings $\mu^1, \ldots, \mu^k$ in $(F, W, P^\mu, \boldsymbol{q})$ and real numbers $\alpha_1, \ldots, \alpha_k$, such that $\bar{x} = \sum_{l=1}^k \alpha_l x^{\mu^l}$, with $0 < \alpha_l \leq 1$, $\sum_{l=1}^k \alpha_l = 1$ and $\mu^1 \succ_F \mu^2 \succ_F \ldots \succ_F \mu^k$. Since $\sum_{i \prec_w f} \bar{x}_{i,w} > 0$, then there is a stable matching μ^l for some $l = 1, \ldots, k$ such that $\sum_{i \prec_w f} x^{\mu^l}_{i,w} = 1$. Given that $(f, w) \in A(P)$, the firm f does not fill its quota, and $\sum_{i \prec_w f} x^{\mu^l}_{i,w} = 1$, then $\mu^l(w) \prec_w f$. Hence, (f, w) is a blocking pair for μ^l for some $l = 1, \ldots, k$, and this is a contradiction, then $\sum_{i \succeq_w f} \bar{x}_{i,w} = 1$. Therefore, $\bar{x}$ fulfils condition (6).

b) If the firm f fill its quota. Without loss of generality, we assume that $\mu(f) = \{w_1, \ldots, w_{q_f}\}$, $\mu_W(f) = \{w'_1, \ldots, w'_{q_f}\}$, $w_l \succ_f w_{l+1}$ and $w'_l \succ_f w'_{l+1}$ for $l = 1, \ldots, q_f - 1$. Notice that, $\mu(f) \cap \mu_W(f)$ is not necessarily empty.

We analyse the following 3 sub-cases:

b_1) $w_1 \succ_f w \succ_f w_{q_f}$.
Hence, $\sum_{j \succeq_f w} x^\mu_{f,j} < q_f$. Since each stable matching fulfils condition (6), then $\sum_{i \succeq_w f} x^\mu_{i,w} = 1$.

Since $\bar{x}$ is a stable fractional matching in P^{μ}, then $x^{\mu} \succeq_F \bar{x}$. By Lemma 7, we have that $\bar{x} \succeq_W x^{\mu}$. Thus,

$$1 = \sum_{i \succeq_w f} x^{\mu}_{i,w} \leq \sum_{i \succeq_w f} \bar{x}_{i,w} \leq 1.$$

Hence,

$$\sum_{i \succeq_w f} \bar{x}_{i,w} = 1,$$

which implies that

$$\left[q_f - \sum_{j \succeq_f w} \bar{x}_{f,j} \right] \cdot \left[1 - \sum_{i \succeq_w f} \bar{x}_{i,w} \right] = 0.$$

b2) $w_{q_f} \succ_f w \succ_f w'_{q_f}$.
Since $(f, w) \notin A(P^{\mu})$, then f was eliminated from the worker w's preference list P^{μ}. Then, for the worker w we have that $f \succ_w \mu_W(w)$ or $\mu(w) \succ_w f$. If $f \succ_w \mu_W$, then the pair (f, w) blocks the matching μ_W, then $\mu(w) \succ_w f$. Therefore, by Lemma (7) we have that $\bar{x} \succeq_W x^{\mu}$. Hence,

$$\sum_{i \succeq_w f} \bar{x}_{i,w} \geq \sum_{i \succeq_w f} x^{\mu}_{i,w} = x^{\mu}_{\mu(w),w} = 1.$$

Since $\bar{x}$ satisfies linear inequality (2), we have that $\sum_{i \succeq_w f} \bar{x}_{i,w} = 1$. Then,

$$\left[q_f - \sum_{j \succeq_f w} \bar{x}_{f,j} \right] \cdot \left[1 - \sum_{i \succeq_w f} \bar{x}_{i,w} \right] = 0.$$

b3) $w'_{q_f} \succ_f w$.
By Lemma (7) we have $\bar{x} \succeq_F x^{\mu_W}$. Hence,

$$\sum_{j \succeq_f w} \bar{x}_{f,j} \geq \sum_{j \succeq_f w} x^{\mu_W}_{f,j} = q_f.$$

Since $\bar{x}$ satisfies linear inequality (1), we have that $\sum_{j \succeq_f w} \bar{x}_{f,j} = q_f$. Then,

$$\left[q_f - \sum_{j \succeq_f w} \bar{x}_{f,j} \right] \cdot \left[1 - \sum_{i \succeq_w f} \bar{x}_{i,w} \right] = 0.$$

From cases 1 and 2, we conclude that $\bar{x}$ is a strongly stable fractional matching for the many-to-one matching market $(F, W, P, \boldsymbol{q})$. $\square$