Red Nacional de Investigadores en Economía

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DOCUMENTO DE TRABAJO 2020-20

Agosto de 2020

Citar como:
Lasso de la Vega, Casilda, Oscar Volij, y Federico Weinschelbaum (2020).
Theft in equilibrium. *Documentos de Trabajo RedNIE, 2020-20*

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July 8, 2020

Abstract

We incorporate theft in a partial equilibrium model. This model allows us to perform positive and normative analysis using traditional Marshallian tools, as well as to obtain results that do not hinge on specific parametric specifications. We analyze the model's implications under two scenarios, one in which all factors of production are subject to theft, and another in which only final goods are stealable.

JEL Classification numbers: D72, D74, H23, K42.

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1 Introduction

Early economic thinkers were well aware of the prevalence of crime, theft in particular, in society. For instance, Pareto (1971) unequivocally states that "the efforts of men are utilized in two different ways: they are directed to the production or transformation of economic goods, or else to the appropriation of goods produced by others." Likewise, J. S. Mill (1848) writes "it is lamentable to think how a great proportion of all efforts and talents in the world are employed in merely neutralizing one another" and claims that the role of government is "to reduce this wretched waste to the smallest amount". However, perhaps except for taxation, any appropriation activity is absent from the realm of the Walrasian model, which is the central model in economics. Only at the end of the 1960's, did economists begin to formally analyze the subject of crime, the seminal reference being Becker (1968). The first models focused on the decision-making process of a rational potential criminal, and although the aggregate behavior of economic agents is made mutually consistent through the adjustment of the relevant endogenous variables, they do not perfectly fit the standard Walrasian model, (see Ehrlich (1996) for an overview). There is a related strand of literature that deals with conflict, whose seminal ideas can be traced back to Haavelmo (1954) and Hirshleifer (1988). This literature adopts a game theoretic approach to conflict and appropriation and is related to the vast literature on contests.[1] An early general equilibrium model that incorporates appropriation is Grossman (1994).[2] The paper that best fits the Walrasian model is Dal Bó and Dal Bó (2011). It introduces appropriation activities in the celebrated 2x2 production model and analyzes, among other things, the effect of changes in the exogenous output prices and in the factor endowments on the level of crime. The authors also investigate the effect of tax, subsidies, and trade policies on crime.

Property theft is a pervasive phenomenon in all societies. In the US, to mention one,

[1]Notable examples include Skaperdas (1992) and Garfinkel (1990). For a very thorough overview, see Garfinkel and Skaperdas (2007).

[2]Other models that allow for appropriation activities are Burdett, Lagos, and Wright (2003, 2004), İmrohoroğlu, Merlo, and Rupert (2000), González (2007), Galiani, Cruz, and Torrens (2018) and Galiani, Jaitman, and Weinschelbaum (2020).

the FBI estimates that property crime in 2018 resulted in losses of \$16.4 billion. Although there is a vast theoretical literature that analyzes crime, there are very few papers that do so within a full Walrasian model. The purpose of this paper is to uncover the implications of this model once it is amended to allow the possibility of theft. In order to accomplish this, we introduce theft into the standard partial equilibrium model, which is perhaps the main tool in economics for designing well-founded theories and for obtaining testable implications. In the words of Friedman (1955), it is a powerful "engine for analyzing concrete problems." Furthermore, this model enables us to perform positive and normative analysis using traditional Marshallian tools and to obtain results that do not hinge on specific parametric specifications. In this paper, we allow agents to devote time to theft, the returns of which depend on the economy-wide crime level and on the wealth that is subject to theft. From the individual thief's point of view, he cannot affect the level of crime and the wealth subject to theft is a common resource. We also allow for the possibility of police protection, which can be either public or private. We study the existence and uniqueness of the competitive equilibrium, and analyze the nature of its inefficiency.

It turns out that the results depend on the wealth that is subject to theft. If the stealable wealth consists of the individuals' initial factor endowments, then a competitive equilibrium exists and is unique. As expected, the equilibium allocation is not efficient and a Pareto improvement can be achieved by means of an increase in output. Also, the provision of public police reduces the level of crime. If we allow for private police protection we obtain that although the competitive equilibrium is inefficient, conditional on the level of theft in the economy, the allocation of police protection is optimal.

When the wealth subject to theft is the gross domestic product, however, namely, when only produced goods can be stolen, most of the above results no longer hold. In particular, a competitive equilibrium may not exist, and that when it does, it may not be unique. Equilibrium allocations are generally inefficient but incentives to output production are not necessarily Pareto improving. Also, increases in police protection may lead to an increase in crime. Finally, unless the production technology is linear, equilibrium private police protection is no longer optimal conditional on the equilibrium level of theft.

We also investigate, within the scenario in which all factor endowments are subject to theft, the possibility of equilibria in which two or more regions coexist with different levels of crime and police protection, and in which the tax rates are determined by majority voting. In particular, we show that when tax rates are constrained to be proportional, regions vote to supply the optimal level of police protection and the economy is typically not segregated by income.

The paper is organized as follows. Section 2 introduces the basic definitions of an economy with theft. Section 3 develops the model in which the whole initial endowment is subject to theft. Section 4 considers the case where the stealable wealth consists of produced goods. Section 5 concludes.

2 The model

The primitives of the model are the following. There is a firm that transforms labor into a consumption good, which will be henceforth referred to as *peanuts*, according to the technology $\mathcal{T} = \{(-Z, Q) : 0 \leq c(Q) \leq Z\}$, where $c : \mathbb{R}_+ \to \mathbb{R}_+$ is the cost function, which is assumed to be convex. There is a continuum of agents $I = [0, 1]$. Each agent $i \in I$ is characterized by a quasilinear utility function $u_i(x_i, m_i) = \phi_i(x_i) + m_i$, an initial endowment of labor ω_i, and a share θ_i of the firm's profits. For simplicity, we assume that the consumption set is $\mathbb{R}_+ \times \mathbb{R}$, namely individuals can consume negative amounts of leisure. Further, we assume that ϕ_i is strictly increasing, concave, and that $\lim_{x\to\infty} \phi_i'(x) = 0$.

Individuals, apart from consuming peanuts and leisure, devote some time to theft and may obtain some police protection, which is measured in units of time. A *bundle* for individual i is thus a four-tuple $(x_i, m_i, y_i, t_i) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ whose components are the amounts of peanuts, leisure, time devoted to theft, and police protection.

For any real function f defined on $[0, 1]$, we will sometimes write $\int f$ for $\int_0^1 f_i \, di$. Aggregate (or per capita) values are denoted by capital letters. In particular, the per capita amount of resources in the economy is $\Omega = \int \omega$, the per capita consumption of leisure is $M = \int m$, the crime level is $Y = \int y$, and the average police protection is $T = \int t$. We assume that the agents in $[0,1]$ are the sole owners of the firm: $\int \theta = 1$.

If individual i devotes y_i units of time to theft he gets a share y_i/Y of the booty. Police protection may be public or private, being public when t_i is decided by the government and private when it is decided by agent i. When police protection is public, it is usually allocated uniformly across individuals. However, public police protection may very well be discriminatory. When public, police protection is financed by means of compulsory taxation. When private, it is purchased voluntarily by the consumers themselves.

As in Dal Bó and Dal Bó (2011), there is an appropriation technology described by $A : \mathbb{R}^2_+ \to [0,1]$. The value $A(Y,t_i)$ is the proportion of individual i's stealable income that gets stolen when the crime level is Y and enjoys t_i units of police protection. We call $A(Y,t_i)$ the *excise rate* associated with Y and t_i. We assume that $A(0,t_i) = 0$, that A is increasing and strictly concave in its first argument, decreasing and strictly convex in its second argument, and that $A_{12} < 0$, namely the marginal excise rate of crime is decreasing in police protection.[3] These assumptions imply that

$$A_1(Y,t_i) < \frac{A(Y,t_i)}{Y}$$

and that $\lim_{Y\to 0} A(Y,t_i)/Y = A_1(0,t_i)$. Namely, the marginal excise rate is lower than the average excise rate. We denote by $a(Y,t_i)$ the average excise rate, with the extension $a(0,t_i) = A_1(0,t_i)$. It is the proportion of wealth stolen per unit of time devoted to theft. It follows from our assumptions that $a(Y,t_i)$ is decreasing in both its arguments, and convex in its second argument. We summarize the data of the economy by $\mathcal{E} = \langle(\phi,\omega,\theta), c, A\rangle$.

We denote the set of bundles by $\mathcal{X}$. A *feasible allocation* consists of a production plan $(-Z,Q) \in \mathcal{T}$ and a function $(x,m,y,t) : [0,1] \to \mathcal{X}$ that assigns a bundle to each agent, such that

1. $\int x = Q$,

2. $\int m + Z + \int y + \int t = \Omega$.

[3] We denote by A_1 and A_2 the partial derivatives of A with respect to its first and second arguments. Also $A_{k\ell}$, for $k,\ell = 1,2$, stand for the corresponding second derivatives.

A feasible allocation is *efficient* if there is no alternative feasible allocation that can make all agents better off. Given our assumptions on the individuals' consumption set and preferences, an allocation is efficient if it maximizes, among the feasible allocations, the social welfare, namely, the average utility, $W(x,m) = \int_0^1 u_i(x_i, m_i)\, di$, of the individuals.

In the next two sections we introduce and analyze the competitive equilibrium for an economy with theft. They differ in how the wealth subject to theft is defined. In Section 3, the stealable wealth consists of the whole factor endowment. In Section 4, in contrast, only earned income can be stolen. That is, time devoted to leisure cannot be alienated. Specifically, given a price of peanuts p, the firm's profits are $\Pi = pQ - Z$, and individual i's share in these profits is $\pi_i = \theta_i \Pi$. When all wealth is subject to theft, individual i's stealable wealth is $(\omega_i + \pi_i)$ and the aggregate stealable wealth is $(\Omega + \Pi)$. When, alternatively, only earned income is subject to theft individual i's stealable wealth is $(\omega_i + \pi_i - m_i)$ and the aggregate stealable wealth is $(\Omega + \Pi - M)$.

3 All wealth can be stolen

In this section, we assume that all wealth can be stolen. We first restrict attention to the case where there is no police protection. To simplify notation, we will henceforth let $A(Y) = A(Y, 0)$, $a(Y) = a(Y, 0)$, etc.

3.1 Competitive equilibrium

When all wealth can be stolen, individual i's budget set is given by

$$\{(x_i, y_i, m_i) : px_i + m_i + y_i \leq (1 - A(Y))(\omega_i + \pi_i) + y_i a(Y)(\Omega + \Pi)\}$$

His disposable income consists of the fraction of his wealth that has not been stolen and the proceeds of his appropriation activities. Note that the consumer takes not only the price p of peanuts as given, but the crime level Y and the returns to crime $a(Y)(\Omega + \Pi)$ as well.

The concept of competitive equilibrium is the usual one.

Definition 1 A *competitive equilibrium* consists of a feasible allocation $\langle (x^*, y^*, m^*), (-Z^*, Q^*) \rangle$ and a price p, such that

1. $(-Z^*, Q^*) \in \mathcal{T}$ maximize profits given p.

2. For each $i \in [0,1]$, (x_i^*, y_i^*, m_i^*) maximizes i's utility given his budget.

Characterization of the equilibrium. For simplicity, and since we want to focus on theft, we now characterize the competitive allocations that assign interior consumption bundles. Assume that $\langle (x^*, y^*, m^*), (-Z^*, Q^*) \rangle$ and p constitute such a competitive equilibrium. Then $(-Z^*, Q^*)$ must satisfy the necessary (and sufficient) conditions for profit maximization:

$$p = c'(Q) \text{ and } Z = c(Q).$$

Also, for all $i \in [0,1]$, (x_i^*, y_i^*, m_i^*) must satisfy the first order conditions for utility maximization:

$$\begin{aligned} \phi_i'(x) &= p \qquad i \in [0,1] \\ 1 &\geq a(Y)(\Omega + \Pi) \text{ with equality if } Y > 0 \qquad (1) \\ px + y + m &= (1 - A(Y))(\omega + \pi) + ya(Y)(\Omega + \Pi) \end{aligned}$$

Finally, the allocation must be feasible:

$$\begin{aligned} \int x &= Q \\ M + Z + Y &= \Omega \end{aligned}$$

Condition (1) is a zero-profit condition for appropriation activities. It says that in equilibrium, if there is theft, individuals are indifferent between allocating an additional unit of time to leisure or to stealing. Observe that condition (1) implies that

$$Y^* = A(Y^*)(\Omega + \Pi^*).$$

In other words, in equilibrium the time spent stealing equals the value of the stolen goods. For that reason, it is justified to call Y^* the level of theft or of (property) crime.

It is routine to check that in order to find an equilibrium, it is enough to solve

$$p = c'(Q) \tag{2}$$

$$\phi_i'(x) = p \qquad i \in [0,1] \tag{3}$$

$$\int x = Q \tag{4}$$

$$1 \geq a(Y)(\Omega + \Pi) \text{ with equality if } Y > 0 \tag{5}$$

The equilibrium production plan and price are determined by conditions (2–4) and therefore, given our assumptions about preferences and technology, are unique. Furthermore, they are independent of the appropriation technology and activity. Consequently, so is the equilibrium level of profits Π^*. Given the equilibrium aggregate wealth, $(\Omega+\Pi^*)$, the equilibrium level of theft Y^* is characterized by equation (5). Therefore, given that $a(Y)$ is continuous, decreasing, and converges to 0 as Y goes to infinity, an application of the intermediate value theorem leads to the following.

Observation 1 A competitive equilibrium exists and is unique. If $a(0) \leq 1/(\Omega + \Pi^*)$, the equilibrium level of crime is 0. If $a(0) > 1/(\Omega+\Pi^*)$ the equilibrium level of crime is positive.

The equilibrium is *locally stable* in the sense that small perturbations of the level of theft unleash forces that return it to the equilibrium level. Indeed, if $Y < Y^*$, then since a is decreasing, the returns to theft are higher than 1, and thus induce an increase in theft. Similarly, if $Y > Y^*$, the returns to theft are lower than 1, and induce a decrease in theft.

When positive, the amount, Y^*, of criminal activity is pure waste because it does not produce anything; it only transfers resources from victims to thieves. Moreover, even from the viewpoint of the thieves there is too much criminal activity. If they wanted to increase $A(Y)(\Omega+\Pi^*)-Y$, namely the booty that is in excess of the criminal effort, they would choose a crime level lower than Y^*. To see this, note that since $a(Y)$ is decreasing in Y, by (5) we have that $a(Y)(\Omega + \Pi^*) > 1$ for all $0 < Y < Y^*$, which implies that $A(Y)(\Omega + \Pi^*) - Y > 0 = A(Y^*)(\Omega + \Pi^*) - Y^*$ for all $0 < Y < Y^*$. Namely all the thieves, and consequently everybody, could be made better off by simply reducing the level of crime.

The above discussion shows that there are feasible allocations that can make all individuals better off. However, these allocations may not be enforceable by a social planner. Indeed, whereas a social planner would be able to control the output level, he would not be able to simultaneously dictate the level of crime since the level of crime is determined by the thieves themselves. We will now show, however, that a social planner can enforce a feasible allocation that makes all agents better off.

If we look closely at equation (5) we see that any policy that reduces the stealable wealth, will also reduce the level of crime. One such policy would be to command the firm to produce a quantity Q that is larger than the equilibrium quantity Q^*.

Observation 2 Let $\langle (x(Q), m(Q), y(Q)), (-Z(Q), Q) \rangle$ be the equilibrium allocation when the firm is commanded to produce Q, and let $p(Q)$ be the equilibrium price and $W(Q) = \int (\phi(x(Q)) + m(Q))$ be the corresponding social welfare. Then $W'(Q^*) > 0$. Namely, starting from the equilibrium level of output, a slight increase in production increases social welfare.

Proof : See Appendix. □

The idea of the proof is as follows. Starting from the competitive equilibrium, a small increase in output leads to a decrease in price and profits. The additional output leads to an increase in the sum of utilities that is completely offset by the increase in cost since at the equilibrium marginal utilities are equal to marginal cost. Therefore, whether or not social welfare increases depends on whether or not the crime level goes down. By the zero-profit condition for appropriation activities, crime will go down if and only if wealth goes down, which it does due to the decrease in profits. We therefore conclude that welfare can be increased by increasing output.

3.2 Public police

Suppose now that the government levies a personalized tax $\hat{t}_i$ and that the total tax $T = \int \hat{t}$ is allocated to crime prevention. That is, $\hat{t}_i$ is the time that individual i contributes to the public police effort, and T is the per capita level of public police protection. Since public police is assumed to be enjoyed equally by all individuals, T is the actual time devoted to protecting individual i's wealth.

The definition of a competitive equilibrium is the same as before, except that now the budget of individual i consists of all the triples (x_i, m_i, y_i) that satisfy

$$px_i + m_i + y_i + \hat{t}_i \leq (1 - A(Y,T))(\omega_i + \pi_i) + y_i a(Y,T)(\Omega + \Pi)$$

The equilibrium allocation is still characterized by equations (2–5) with the proviso that $a(Y)$ is now replaced by $a(Y,T)$. The same argument as before shows that there is a unique equilibrium, which will exhibit positive levels of crime if $a(0,T) > 1/(\Omega + \Pi^*)$. In this case, the equilibrium level of redistributive activity is implicitly defined by

$$Y^*(T) = A(Y^*(T), T)(\Omega + \Pi^*)$$

By the implicit function theorem,

$$Y^{*\prime}(T) = \frac{(\Omega + \Pi^*)A_2(Y^*(T), T)}{1 - (\Omega + \Pi^*)A_1(Y^*(T), T)}. \tag{6}$$

Given that $A_2 < 0$ and that $(\Omega + \Pi^*)A_1(Y^*(T), T) < (\Omega + \Pi^*)A(Y^*(T), T)/Y^*(T) = 1$, we obtain the following.

Observation 3 When positive, the equilibrium level of theft $Y^*(T)$ is decreasing in T.

There is a vast literature that aims to identify a causal effect of police on crime. When police protection is measured by overall police manpower (as implicitly assumed in Observation 3) the evidence is mixed. However, when changes in police protection reflect changes in police deployments more convincing evidence has been found that crime responds to police.[4]

The *optimal tax level* is the one that induces an equilibrium with the maximum possible social welfare. In the present context, since neither the tax nor the level of theft affect

[4] For a non-exhaustive list of papers on this issue, see Marvell and Moody (1996), Levitt (1997, 2002), Evans and Owens (2007), Di Tella and Schargrodsky (2004), Klick and Tabarrok (2005), and Draca, Machin, and Witt (2011).

the output market, the optimal tax is the one that minimizes $Y^*(T) + T$. Since that for all $T > Y^*(0)$, we have that $Y^*(T)+T > Y^*(0)$, and since $Y^*(T)+T$ is continuous, it attains its minimum in $[0, Y^*(0)]$. Therefore, the optimal tax is well defined and satisfies $Y^{*\prime}(T) \geq -1$, with equality if $T > 0$. More specifically, it satisfies

$$\frac{(\Omega + \Pi^*)\, A_2(Y^*(T), T)}{1 - (\Omega + \Pi^*)\, A_1(Y^*(T), T)} \geq -1 \qquad \text{with equality if } T > 0. \tag{7}$$

This condition is also sufficient if $Y^*(T)$ is convex. Needless to say, the optimal level of public police does not necessarily lead to zero crime and clearly depends on $(\Omega + \Pi^*)$.

It would be interesting to compare the optimal level of public police with the one preferred by each individual. Individual i's preferred level of public police is the one that minimizes the wealth that is stolen from him plus the tax that he pays. Consequently, preferences over public police protection depend on how it is financed.

Assume that public police is financed by means of proportional income taxation, i.e., $\hat{t}_i = \tau(\omega_i + \pi_i^*)$ where $\tau = T/(\Omega + \Pi^*)$. Then, individual i's preferred level of police protection is the level T that minimizes $A(Y^*(T), T)(\omega_i + \pi_i^*) + (T/(\Omega + \Pi^*))(\omega_i + \pi_i^*)$. Multiplying by the constant $(\Omega + \Pi^*)/(\omega_i + \pi_i^*)$ and taking into account that $A(Y^*(T), T)(\Omega + \Pi^*) = Y^*(T)$, we obtain that this level is the one that minimizes $Y^*(T) + T$. That is, all agents prefer the socially optimal level of public police. To summarize,

Observation 4 When public police is financed by proportional income taxation, individuals unanimously prefer the socially optimal level of police protection.

The following example illustrates the concepts introduced so far.

Example 1 Assume that $\omega_i = 9(1+i)^2$ and that the appropriation technology is given by $A(Y, T) = \frac{Y}{(1+T)(1+Y)}$. Suppose that the production technology satisfies constant returns to scale and thus profits are 0 in equilibrium. Then,

$$(\Omega + \Pi^*) = \int 9(1+i)^2 = 21$$

The zero-profit condition for appropriation activities is given by

$$1 = \frac{21}{(1+T)(1+Y)}$$

and the equilibrium level of crime is

$$Y^*(T) = \max\{\frac{20-T}{1+T}, 0\}$$

The optimal level of police per capita is given by $\hat{T} = \sqrt{21} - 1 = 3.58258$, and the associated optimal level of crime is $\hat{Y} = \sqrt{21} - 1 = 3.58258$.

3.3 Private police

Suppose now that there is no public police but one can hire private police. Alternatively, one can spend some time protecting his own wealth. Given a level Y of crime, if individual i spends t_i on private police, the proportion of his wealth that gets stolen is $A(Y, t_i)$.

The definition of a competitive equilibrium is the same as before, except that now the budget of individual i consists of all the bundles (x_i, m_i, y_i, t_i) that satisfy

$$px_i + m_i + y_i + t_i \leq (1 - A(Y, t_i))(\omega_i + \pi_i) + y_i \int a(Y, t)(\omega + \pi).$$

Like before, the individual takes as given the price p, the level of crime Y, and the return to theft $\int a(Y, t)(\omega + \pi)$. The equilibrium price and quantity are still characterized by equations (2–4). The equilibrium allocation of private police, t^*, and the equilibrium level of theft, Y^*, are now characterized by the following conditions (since A is convex in t, the necessary conditions for utility maximization are also sufficient):

$$1 + (\omega_i + \pi_i)A_2(Y, t_i) \geq 0, \text{ with equality if } t_i > 0 \qquad i \in [0, 1] \tag{8}$$

$$\int a(Y, t_i)(\omega_i + \pi_i) \leq 1, \text{ with equality if } Y > 0 \tag{9}$$

Condition (8) implicitly defines individual i's demand $t_i(Y)$ for private police as a function of crime level. It can be checked that given our assumptions on A, when $t_i(Y) > 0$, we have that $t_i'(Y) > 0$. Namely, the higher the crime rate, the higher the preferred level of private police protection. Further, since $A(0, t_i) = 0$ for all $t_i \geq 0$, we have that $t_i(0) = 0$. Namely, when there is no crime, the individual does not demand police protection. All this, along with the fact that $a(Y, t_i(Y))$ is decreasing in Y and converges to 0 as Y goes to infinity, allows us to conclude the following.

Observation 5 An equilibrium with private police exists and is unique. If $a(0,0) \leq 1/(\Omega + \Pi^*)$, the equilibrium level of crime is 0. If $a(0,0) > 1/(\Omega + \Pi^*)$ the equilibrium level of crime is positive.

Like before, there is too much criminal activity even from the thieves' point of view. Indeed, since $\int a(Y, t_i(Y))(\omega_i + \pi_i^*)$ is decreasing in Y, we have that $\int a(Y, t_i(Y))(\omega_i + \pi_i^*) > \int a(Y^*, t_i(Y^*))(\omega_i + \pi_i^*) = 1$ for all $Y < Y^*$. Therefore, $\int A(Y, t_i(Y))(\omega_i + \pi_i^*) - Y > 0 = \int A(Y^*, t_i(Y^*))(\omega_i + \pi_i^*) - Y^*$ for all $Y < Y^*$. This means that $\int A(Y, t_i)(\omega_i + \pi_i^*) - Y$, namely the thieves' net benefit from appropriation activities could be increased by decreasing the level of criminal activity. That is, the decrease in the rewards to crime due to the reduction in criminal activity is more than compensated by the increase in these rewards resulting from the induced lower police protection plus the time saved by the thieves.

3.3.1 Optimal allocation of private police

We now investigate whether the competitive equilibrium allocates private police efficiently. The optimal allocation of private police minimizes total waste. Namely it solves

$$\min_{Y,t} Y + \int t$$
$$\text{s.t. } 1 \geq \int a(Y, t_i)(\omega_i + \pi_i^*) \quad \text{with equality if } Y > 0$$

It can be seen that since a is decreasing in its second argument, any solution to this problem satisfies the constraint with equality. We can solve this problem in two steps. First, we solve for the optimal allocation of police resources given an arbitrary level of crime, and then we solve for the optimal level of crime. Specifically, the *optimal allocation of police protection given a level of crime* Y is the solution to

$$V(Y) = \min_t \int t \tag{10}$$
$$\text{s.t. } \quad 1 = \int a(Y, t_i)(\omega_i + \pi_i^*)$$

and the *optimal level of crime* solves

$$\min_Y Y + V(Y).$$

Problem (10) is equivalent to

$$\begin{aligned} &\min_t \int t \\ \text{s.t. } &k_i' = a(Y,t_i)(\omega_i+\pi_i^*) \\ &k_0 = 0 \\ &k_1 = 1 \end{aligned}$$

The associated Hamiltonian is $H(t_i,k_i,\lambda_i,Y) = t_i + \lambda_i a(Y,t_i)(\omega_i+\pi_i^*)$, and the necessary conditions for a solution are

$$1+\lambda_i(\omega_i+\pi_i^*)a_2(Y,t_i) \geq 0 \quad \text{with equality if } t_i > 0 \tag{11}$$

$$\lambda_i' = 0 \tag{12}$$

$$k_i' = a(Y,t_i)(\omega_i+\pi_i^*) \tag{13}$$

$$k_0 = 0,\ k_1 = 1. \tag{14}$$

It follows from (11–12) that λ_i does not depend on i and that $\lambda_i \geq 0$. As a result, given that $a(Y,t_i)$ is convex in the second argument, the above conditions are also sufficient for the optimality of the distribution of police protection.

It also follows from (11) that for any level of crime Y, the optimal police function satisfies

$$(\omega_i+\pi_i^*)A_2(Y,t_i) = (\omega_j+\pi_j^*)A_2(Y,t_j) \qquad \text{for all } i,j \in [0,1] \text{ with } t_i,t_j > 0$$

namely, the property loss prevented by an additional unit of police protection is independent of whom this additional unit is allocated to. This means that it is never optimal to allocate police effort equally unless incomes are equal. In fact, richer people should be allocated higher police protection.

Inspecting equations (8–9), which characterize the competitive equilibrium, we see that the competitive private police allocation t_i^*, along with $\lambda_i = Y^*$ also satisfies conditions (11–14) when the level of crime is fixed to be at the competitive level Y^*. Furthermore, t_i^* is the only distribution of police protection that solves (10) for $Y = Y^*$. To see this, note that for each $\lambda > 0$, there is a unique distribution t that satisfies (11). This t is increasing in λ, which implies that, $\int a(Y^*, t_i(\lambda))(\omega_i + \pi_i^*)$ is decreasing in λ. This means that $\int a(Y^*, t_i(\lambda))(\omega_i + \pi_i^*) = 1$ has at most one solution. Summing up:

Observation 6 Given the competitive crime level Y^*, the only optimal allocation of police protection is the competitive one.

Private police exerts a positive externality; it reduces the returns to theft, which induces people to spend less time stealing from the whole population. This externality is not taken into account by the individual and, as a result, the competitive level of crime is not globally optimal. In fact, we can show the following.

Observation 7 At the competitive equilibrium, the level of crime is too high. Namely, the total waste could be reduced by increasing spending on police protection, thereby reducing crime.

Proof : It is enough to show that $-V'(Y^*) < 1$. Indeed, the value of $-V'(Y^*)$ is the additional spending on police required to reduce crime by one small unit. If at the equilibrium we had $-V'(Y^*) < 1$, there would be too much crime; reducing the time spent on crime by one unit costs less than one unit. The Lagrangian associated with problem (10) is

$$L = \int \big(t_i - \lambda_i(1 - a(Y, t_i)(\omega_i + \pi_i^*)\big).$$

By the envelope theorem

$$\begin{aligned} V'(Y^*) &= \int \lambda^* A_1(Y^*, t_i^*)(\omega_i + \pi_i^*) \\ &= \int Y^* A_1(Y^*, t_i^*)(\omega_i + \pi_i^*) \\ &= \int A_1(Y^*, t_i^*)(\omega_i + \pi_i^*) - \int a(Y^*, t_i^*)(\omega_i + \pi_i^*) \\ &\geq \int A_1(Y^*, t_i^*)(\omega_i + \pi_i^*) - 1 \\ &> -1 \end{aligned}$$

where we have used the fact that when the level of crime is the competitive one, $\lambda^* = Y^*$. □

If the excise rate $a(Y, t_i)$ is convex, then the function V is convex as well. In that case, the optimal level of crime $\hat{Y}$ is lower than the competitive level. In order to induce individuals to acquire the optimal level of police protection, the government could subsidize the cost of police by means of a quantity subsidy of $s = (\hat{\lambda} - \hat{Y})/\hat{\lambda}$, where $\hat{\lambda}$ is the value of the costate variable when the level of crime is the optimal one.[5] Interestingly, an income subsidy of $\sigma = \hat{\lambda} - 1$ (so that wealth becomes $\hat{\lambda}(\omega_i + \pi_i)$) does not achieve the optimal level of crime and private police protection because it also increases the stealable wealth with the resulting enhanced incentives to theft.

The following example illustrates the equilibrium and efficient allocation when there is private police.

Example 2 Consider the economy described in Example 1, where the appropriation technology is given by $A(Y, t_i) = \frac{Y}{(1+t_i)(1+Y)}$. We now calculate its competitive equilibrium. The utility maximizing level of private police, if positive, satisfies the first-order condition

$$1 - \frac{9(i+1)^2 Y}{(t_i+1)^2(Y+1)} = 0$$

which yields a demand of police protection given by

$$t_i(Y) = \max\{\frac{3(1+i)\sqrt{Y(Y+1)}}{Y+1} - 1, 0\}.$$

The condition of zero-profitability of crime is

$$1 = \frac{9\sqrt{Y(Y+1)}}{2Y(Y+1)}$$

which yields $Y^* = 4.02769$ and $t_i^* = 1.68513 + 2.68513\,i$.

We now calculate the optimal police allocation conditional on a positive level Y of crime. To avoid corner solutions, we assume that $Y \leq 25/2$. The optimal police allocation satisfies the necessary condition (11), which since $a(Y, t_i)$ is convex is also sufficient, and which in our case becomes

$$1 - \frac{9(1+i)^2\lambda}{(t_i+1)^2(Y+1)} = 0$$

[5] In this case, since the price of a unit of police protection is 1, a quantity subsidy and an ad-valorem subsidy amount to the same thing.

and yields

$$t_i = \frac{3(1+i)\sqrt{\lambda(Y+1)}}{Y+1} - 1.$$

We must also have that $\int a(Y, t_i)(\omega_i + \pi_i) = 1$, which gives

$$\lambda(Y) = \frac{81}{4(Y+1)}.$$

Substituting back into the formula of t_i we obtain that

$$\hat{t}_i(Y) = \frac{25 + 27i - 2Y}{2(Y+1)}.$$

Note that when the level of theft is the equilibrium one, namely when $Y = Y^*$, the optimal allocation of police protection is also the equilibrium one:

$$\hat{t}_i(Y^*) = 1.68513 + 2.68513\, i = t_i^*.$$

The optimal level of theft minimizes total waste, which is given by

$$Y + \int \hat{t}_i(Y) = \frac{77 + 4Y^2}{4(Y+1)}.$$

Consequently, the optimal level of crime and the associated private police allocation are

$$\widehat{Y} = 3.5, \qquad \hat{t}_i = 2 + 3i.$$

Since $\lambda(\hat{Y}) = \hat{\lambda} = 9/2$, we obtain that a subsidy on private police protection of $s = (\hat{\lambda} - \hat{Y})/\hat{\lambda} = 2/9$ will yield an equilibrium with the optimal level of crime.

3.4 Voting equilibrium

In this section, we introduce the notion of a voting equilibrium. Since individual preferences over levels of police protection depend on the tax regime, different tax regimes lead to different equilibria. A voting equilibrium, under a given tax regime, consists of a competitive equilibrium in which individuals are partitioned into groups, each one residing in a different region with its own crime level and its own level of public police protection. In such an equilibrium, nobody wants to leave his region and, furthermore, the level of police protection is preferred by a majority of residents to any other level, given the prevailing tax regime.

Here we restrict attention to the regime of proportional income taxation. Formally, given a partition of the individuals $\mathcal{P} = \{R_1, \ldots, R_K\}$ into K nonempty groups with associated per-capita levels of public police T_k for each group $k = 1, \ldots, K$, a feasible allocation consists of an assignment of bundles (x, y, m) and a production plan $(-Z, Q)$ such that

1. $\int x = Q$,
2. $\int m + Z + \int y + \sum_{k=1}^{K} \int_{R_k} T_k = \Omega$,
3. $\int_{R_k} \tau_k(\omega + \pi) = \int_{R_k} T_k$.

The interpretation is as follows. The individuals are partitioned into different groups and reside in different regions. Each region k sets a level T_k of public police which the residents finance by means of income taxation. The size of group k is $\mu_k = \int_{R_k} 1$. The wealth per capita in region k is $(\Omega_k + \Pi_k)$ where $\Omega_k = \int_{R_k} \omega/\mu_k$ and $\Pi_k = \int_{R_k} \pi/\mu_k$. The crime rate in region R_k is $Y_k = \int_{R_k} y/\mu_k$.

Given a price of peanuts, a level of crime Y_k, and police protection T_k, the budget of a resident i of region k is

$$\{(x_i, m_i, y_i) : px_i + m_i + y_i + \tau_k(\omega_i + \pi_i) \leq (1 - A(Y_{R_k}, T_k))(\omega_i + \pi_i) + y_i a(Y_k, T_k)(\Omega_k + \Pi_k)\}.$$

A *voting equilibrium with proportional taxation* consists of a partition of the individuals $\mathcal{P} = \{R_1, \ldots, R_K\}$ with associated level of public police $T_1, \ldots T_K$, a feasible allocation (x^*, y^*, m^*) and production plan $(-Z^*, Q^*)$, and a price p such that

1. (Z^*, Q^*) maximize profits given p.
2. For each $i \in R_k$, (x_i^*, y_i^*, m_i^*) maximizes his utility given his budget.
3. Residents of region R_k do not prefer to move to other regions.
4. Given proportional taxation, the level of public police protection T_k is preferred by a majority of residents in R_k to any other level.

Conditions (1–2) are the profit and utility maximizing conditions, which along with the peanut market clearing condition determine the equilibrium price, p, production plan, $(-Z^*, Q^*)$, and corresponding profits, Π^*, which, as noted earlier, are independent of the appropriation technology and activity. They also set the condition that the equilibrium level of theft in region k must satisfy:

$$1 \geq a(Y_k^*(T_k), T_k)(\Omega_k + \Pi_k^*), \qquad \text{with equality if } Y_k^*(T_k) > 0.$$

Condition 3 says that, given the tax rates and crime levels of the various regions, individuals do not want to emigrate from their own region.[6] This condition means that each individual resides in the region where his disposable income is highest. Alternatively, where the income excised by the thieves and by the government is lowest. Under the regime of proportional taxation, condition 3 reduces to

$$A(Y_k, T_k) + \tau_k = A(Y_{k'}, T_{k'}) + \tau_{k'} \qquad \text{for all } k, k'.$$

Finally, condition 4 says that the tax rate in each region is a Condorcet winner given the residents' preferences over tax rates. In the case of proportional taxation, this condition means that the level of public police in each region is the region's socially optimal one (see Observation 4).

It follows from the above discussion that there exist equilibrium partitions with no income segregation in which the economy-wide optimal tax rate is applied to every region. Formally,

Observation 8 Any partition $\{R_1, \ldots, R_K\}$ such that all the mean wealths are equal, namely, $(\Omega_1 + \Pi_1^*) = \cdots = (\Omega_K + \Pi_K^*) = (\Omega + \Pi^*)$, and such that each region's level of public police is the economy-wide optimal one, is an equilibrium partition with proportional taxation.

There may, however, be non-trivial equilibria with proportional taxation, as the following example illustrates.

[6] If the partition consists of a single group, this condition is superfluous.

Example 3 Assume that $\omega_i = 16\,i$ and that the appropriation technology is given by $A(Y,T) = \frac{Y}{(1+4T)(1+Y)}$. Suppose that the production technology satisfies constant returns to scale and thus profits are 0 in equilibrium. As a result, for any region R , the per capita wealth is

$$\Omega_R = \frac{1}{\mu(R)} \int_R 16\,i$$

The zero-profit condition for appropriation activities is

$$1 = \frac{\Omega_R}{(1+4T)(1+Y)}$$

and the equilibrium level of crime is

$$Y_R(T) = \frac{\Omega_R - 1 - 4T}{1+4T}$$

Using (7), the optimal level of police per capita is $T_R = \sqrt{\Omega_R}/2 - 1/4$, and the associated optimal level of crime is $Y_R = \sqrt{\Omega_R}/2 - 1$. Consider now a partition $R_1 = [1/32, 15/32)$, $R_2 = [0, 1/32) \cup [15/32, 1]$. The average wealths of each region are $\Omega_1 = 4$ and $\Omega_2 = 100/9$. Consequently, the corresponding optimal police and crime rates are

$$Y_1 = 0 \qquad T_1 = 3/4 \qquad Y_2 = 2/3 \qquad T_2 = 17/2$$

which results in the following excise rates:

$$A(Y_1, T_1) = 0 \qquad A(Y_2, T_2) = .06$$

Since income is taxed proportionally, we have that

$$\tau_1 = \frac{T_1}{\Omega_1} = .1875 \qquad \tau_2 = \frac{T_2}{\Omega_2} = .1275$$

Since $A(Y_1, T_1) + \tau_1 = A(Y_2, T_2) + \tau_2$, we conclude that $(\{R_1, R_2\}, \{\tau_1, \tau_2\})$ is an equilibrium partition.

4 Only earned income can be stolen

We now assume that leisure cannot be stolen. That is, individual i's stealable wealth is $(\omega_i + \pi_i - m_i)$ and the aggregate stealable wealth is $(\Omega + \Pi - M)$. We reintroduce the equilibrium concepts, first when there is no police protection, and later when there is public and private police protection.

4.1 Competitive equilibrium

According to our assumptions, given a price of peanuts p individual i's budget set is given by

$$\{(x_i, m_i, y_i) : px_i + y_i \leq (1 - A(Y))(\omega_i + \pi_i - m_i) + y_i a(Y)(\Omega + \Pi - M)\}$$

As before, the parameters that the individual takes as given are the price p, the crime rate Y, and the returns to theft, which now are $a(Y)(\Omega + \Pi - M)$. Note that the relative price of peanuts (in terms of leisure) faced by the consumers is $p/(1 - A(Y))$. This is so because for every unit of time that they devote to work, $A(Y)$ is stolen and therefore only $(1 - A(Y))$ can be used to purchase peanuts. Equivalently, a consumer who wants to bring home one unit of peanuts needs to buy $1/(1 - A(Y))$ units because a proportion $A(Y)$ of them will be stolen.

Since for each individual i his share of the booty is proportional to y_i, we can equivalently assume that neither leisure nor the time spent stealing can be stolen. Indeed, his budget can be equivalently written as

$$\{(x_i, m_i, y_i) : px_i \leq (1 - A(Y))(\omega_i + \pi_i - m_i - y_i) + y_i a(Y)(\Omega + \Pi - M - Y)\}.$$

That is, the wealth available to individual i for peanut consumption is the sum of two terms. One is the part of his legitimate income that has not been stolen and the other is the share of his neighbors' legitimate income that he stole. We will henceforth call $\omega_i - \pi_i - m_i - y_i$ agent *i's net stealable wealth* and $\Omega - \Pi - M - Y$ the *net stealable wealth*.

The definition of a competitive equilibrium is, *mutatis mutandis*, the one introduced in Section 3.1.

Characterization of the equilibrium. We now characterize the competitive allocations that assign interior consumption bundles. Assume that $\langle (x^*, m^*, y^*), (-Z^*, Q^*) \rangle$ and p constitute such a competitive equilibrium. Then $(-Z^*, Q^*)$ satisfies the necessary (and sufficient) conditions for profit maximization:

$$p = c'(Q) \text{ and } Z = c(Q).$$

Also, (x_i^*, m_i^*, y_i^*) satisfies the first-order conditions for utility maximization:

$$\begin{aligned}
\phi_i'(x_i) &= \frac{p}{1 - A(Y)} \qquad i \in [0,1] && (15)\\
1 &\geq a(Y)(\Omega + \Pi - M) \quad \text{with equality if } Y > 0 && (16)\\
px + y &= (1 - A(Y))(\omega + \pi - m) + ya(Y)(\Omega + \Pi - M) && (17)
\end{aligned}$$

Finally, the allocation must be feasible:[7]

$$\begin{aligned}
\int x &= Q && (18)\\
M + Z + Y &= \Omega && (19)
\end{aligned}$$

Observe that in equilibrium $A(Y^*) < 1$. This follows from condition (15).

Condition (16) is a zero-profit condition for appropriation activities. It says that in equilibrium either theft doesn't pay and nobody engages in crime, or individuals are indifferent between allocating an additional unit of time to leisure or to stealing. Specifically, if an individual spends one additional unit of time on theft, he gives up one unit of leisure. In other words, the opportunity cost of theft is 1. On the other hand, the benefit of that same unit of time devoted to theft is its share in the stolen wealth, $a(Y)(\Omega + \Pi - M)$. If this share is less than one, nobody wants to engage in theft. Only when this share is one, will an individual devote part of his time to theft. Note that the aggregate wealth subject to theft considered in condition (16) is the gross stealable wealth, namely the one that includes the time spent stealing. That is, in their calculation of the marginal benefit of theft, the thieves include not only the income legitimately earned but also the one acquired by stealing. We will discuss this point later.

Integrating (17) and using (18) we see that in equilibrium

$$pQ^* + Y^* = (\Omega + \Pi^* - M^*). \tag{20}$$

[7] In fact, by Walras's law, condition 19 is redundant.

Namely, the gross stealable wealth equals the sum of the GDP and the value of the stolen goods. We also have that

$$Y^* = A(Y^*)(\Omega + \Pi^* - M^*). \tag{21}$$

This equality is trivially satisfied if $Y^* = 0$, and if $Y^* > 0$, it follows from (16). But using equations (20–21) we obtain that the value of the stolen goods is

$$Y^* = \frac{A(Y^*)}{1 - A(Y^*)} pQ^*. \tag{22}$$

Using (20) and (22) we can see that at the equilibrium level of crime, the gross stealable wealth can be written as the sum of two components:

$$\Omega + \Pi^* - M^* \;=\; pQ^* + \frac{A(Y^*)}{1 - A(Y^*)} pQ^*.$$

One component is the value of the peanuts actually produced. The other is the value of the stolen peanuts, where stolen peanuts are counted as many times as they are stolen. Indeed,

$$\begin{aligned} \Omega + \Pi^* - M^* &= pQ^* + \frac{A(Y^*)}{1 - A(Y^*)} pQ^* \\ &= pQ^* + A(Y^*)pQ^* + A(Y^*)^2 pQ^* + \cdots \end{aligned}$$

Namely, the gross stealable wealth considered in condition (16) consists of both the income legitimately earned and illegitimately earned. Referring to equation (22), we see that as in the previous section, the equilibrium time devoted to theft equals the value of the stolen goods. For this reason, Y^* is aptly referred to as the level of crime.

4.2 Existence

It is routine to check that in order to find an interior equilibrium, it is enough to solve

$$p = c'(Q) \tag{23}$$

$$\phi_i'(x_i) = \frac{p}{1 - A(Y)} \qquad i \in [0, 1] \tag{24}$$

$$\frac{a(Y)}{1 - A(Y)} p \int x \leq 1 \quad \text{with equality if } Y > 0 \tag{25}$$

$$\int x = Q \tag{26}$$

Once this system is solved, the remaining variables are obtained by mere substitution.

Let $x_i^d : \mathbb{R}_+ \to \mathbb{R}_+$ be the demand function of individual i as a function of the effective relative price of peanuts $p^d = p/(1 - A(Y))$. That is, $x_i^d(p^d)$ solves $\phi_i'(x_i) = p^d$. Also let $X^d = \int x_i^d$ be the aggregate demand as a function of the effective relative price of peanuts. Similarly, let $Q^s : \mathbb{R}_+ \to \mathbb{R}_+$ be the supply function of peanuts. Namely, $Q^s(p)$ solves $c'(Q) = p$. Then, the equilibrium conditions can be written as

$$X^d\left(\frac{p}{1-A(Y)}\right) = Q^s(p) \tag{27}$$

$$\frac{a(Y)}{1-A(Y)}pQ^s(p) \leq 1 \text{ with equality if } Y > 0 \tag{28}$$

Figure 1 depicts the equilibrium in the peanut market. As can be seen, theft has a similar effect to that of an ad valorem tax of $A(Y^*)/(1-A(Y^*))$. It introduces a wedge between the effective price paid by the consumers and the one received by the firm. The difference is the value of the peanuts being stolen when one ends up acquiring one unit of peanuts. However, since the value of the stolen peanuts equals the value of the time spent on appropriation activities, this value ultimately dissipates.

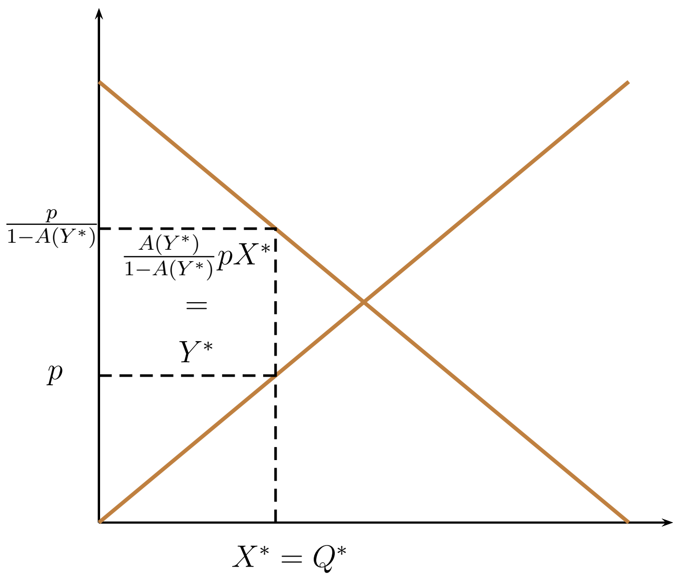

Figure 1: The peanut market.

Equation (27) implicitly defines p as a function of Y. Let $p(Y)$ denote this function.

Also let $X(Y) = X^d(\frac{p(Y)}{1-A(Y)})$ denote the associated aggregate demand. It can be checked that $p(Y)$ is non-increasing and that $X(Y)$ is strictly decreasing. As a result, $p(Y)X(Y)$ is strictly decreasing. Making use of $p(Y)$ and $X(Y)$, the equilibrium conditions are reduced to:

$$\frac{a(Y)}{1-A(Y)}p(Y)X(Y) \leq 1 \quad \text{with equality if } Y > 0. \tag{29}$$

That is, the returns to theft as a function of the crime level when the peanut market is in equilibrium should be 1, unless the crime level is 0, in which case it should not exceed 1.

As opposed to the case of Section 3, an equilibrium is not guaranteed to exist as the following example illustrates.

Example 4 Consider an economy where consumers have a common utility function given by $\phi_i(x) = 1-1/x$, the cost function is $c(Q) = 4Q$, and the appropriation technology is given by $A(Y) = 1 - e^{-Y}$. Given the linear technology, the equilibrium price must be $p(Y) = 4$, and therefore we have that $X(Y) = \sqrt{1-A(Y)}/2$. Hence, in equilibrium we have that $p(Y)X(Y) = 2\sqrt{1-A(Y)}$. It can be checked that

$$\frac{a(Y)}{1-A(Y)} = \begin{cases} 1 & Y = 0 \\ \frac{e^Y-1}{Y} & Y > 0 \end{cases}$$

As a result, the returns to crime as a function of Y is

$$\frac{a(Y)}{1-A(Y)}p(Y)X(Y) = \begin{cases} 2 & Y = 0 \\ 2\frac{e^{Y/2}-e^{-Y/2}}{Y} & Y > 0 \end{cases}$$

which is greater than 1 for all $Y \geq 0$. We conclude that this economy has no equilibrium.

In this model, an equilibrium is *locally stable* if the returns to theft $\frac{a(Y)}{1-A(Y)}p(Y)X(Y)$ are decreasing at the equilibrium level of theft. When an equilibrium does exist, it may be neither unique nor stable. The following example illustrates this point.

Example 5 Consider an economy where consumers have a common utility function defined on $[0, 10]$ given by $\phi_i(x) = x(10 - x/2)$, the cost function is $c(Q) = Q/20$, and the appropriation technology is given by $A(Y) = 1 - e^{-Y}$. Given the linear technology, the equilibrium price must be $p(Y) = 1/20$. Therefore,

$$\begin{aligned} X(Y) &= \max\{10 - \frac{p(Y)}{1-A(Y)}, 0\} \\ &= \max\{10 - \frac{1}{20e^{-Y}}, 0\}. \end{aligned}$$

and, hence, in equilibrium we have that

$$p(Y)X(Y) = \max\{\frac{1}{2} - \frac{1}{400e^{-Y}}, 0\}.$$

It can be checked that

$$\frac{a(Y)}{1-A(Y)} = \begin{cases} 1 & Y = 0 \\ \frac{e^Y-1}{Y} & Y > 0 \end{cases}$$

As a result, the returns to crime as a function of Y is

$$\frac{a(Y)}{1-A(Y)}p(Y)X(Y) = \begin{cases} 199/400 & Y = 0 \\ -\frac{\left(e^Y-200\right)\left(e^Y-1\right)}{400Y} & 0 < Y \leq \ln(200) \\ 0 & Y > \ln(200) \end{cases}$$

which intersects 1 at $Y = 1.28$ and $Y = 5.24$. We conclude that this economy has three equilibria. One, with no theft, one with $Y = 1.28$, and one with $Y = 5.24$. Only the first and third equilibra are locally stable.

Observation 9 A sufficient condition for the existence of an equilibrium is that the returns to theft, $\frac{a(Y)}{1-A(Y)}p(Y)X(Y)$, be less than 1 for some Y. This condition holds if $\lim_{Y\to\infty} \frac{a(Y)}{1-A(Y)} = 0$. For example, if A is bounded away from 1, an equilibrium exists. If, furthermore, $\frac{a(Y)}{1-A(Y)}$ is non-increasing, the equilibrium is unique.

Proof : If $\frac{a(Y)}{1-A(Y)}p(Y)X(Y) < 1$ for all Y, then $Y^* = 0$ satisfies the equilibrium condition (29). Otherwise, if there is some Y, for which this inequality does not hold, then by an application of the intermediate value theorem there must be a Y^* for which condition (29) holds with equality. The rest of the proof follows from the fact that $p(Y)X(Y)$ is decreasing in Y.

□

4.3 Inefficiency of the equilibrium

If the equilibrium level of theft is 0, then it coincides with the equilibrium of a standard economy in which theft is not allowed and therefore it is efficient. Since we are interested in equilibria with positive theft, in this section we assume that the equilibrium level of theft is positive. Furthermore, we assume that there is a unique equilibrium and therefore

$$\frac{a(Y)}{1-A(Y)}p(Y)X(Y) > 1 \qquad \text{for all } Y < Y^*. \tag{30}$$

In this case, it is clear that the market equilibrium is inefficient. There are two reasons for this inefficiency. First, the equilibrium conditions (23–24) imply that the marginal utility of peanuts is higher than its marginal cost, and hence in equilibrium there is underproduction and underconsumption of peanuts. Second, as in the previous section, the amount Y^* of criminal activity is pure waste; it merely transfers resources from victims to thieves. But even worse, for the thieves as well there is too much criminal activity. Note that when the crime level is Y, and taking into account (21) and (22), the resulting booty is the value of the stolen goods:

$$A(Y)\,(\Omega+\Pi-M) \;=\; \frac{A(Y)}{1-A(Y)}p(Y)X(Y).$$

If the thieves, as a union, wanted to maximize $\frac{A(Y)}{1-A(Y)}p(Y)X(Y)-Y$, conditional on consumers choosing their consumption bundles (x_i, m_i) optimally, namely the booty in excess of their criminal effort, they would choose a criminal level that is lower than the equilibrium one. This follows from (30), which implies that for all $0 < Y < Y^*$, we have that $\frac{A(Y)}{1-A(Y)}p(Y)X(Y)-Y > 0 = \frac{A(Y^*)}{1-A(Y^*)}p(Y^*)X(Y^*)-Y^*$, where the equality follows from the definition of equilibrium. Thus, any level $0 < Y < Y^*$ attains a better result for the thieves, meaning that everybody could be made better off by simply reducing the level of crime.

We have seen that there are feasible allocations that can make all individuals better off. However, these allocations may not be enforceable by a social planner, because he would not be able to implement arbitrary combinations of output and crime levels. The above discussion, however, suggests that any tool, such as a quantity subsidy, that induces a slight increase in the production and consumption of peanuts would be welfare improving. We will

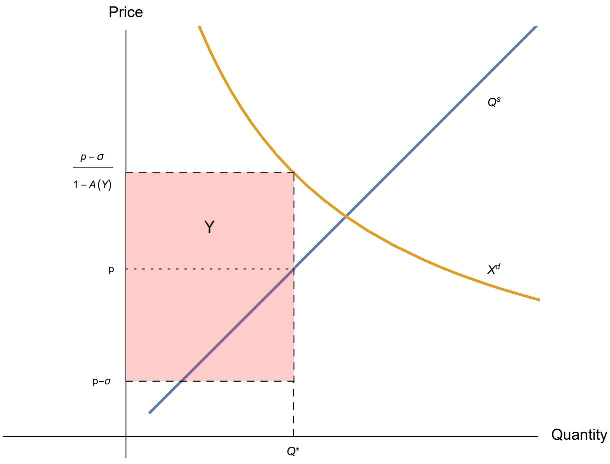

Figure 2: The equilibrium level of crime with a quantity subsidy.

now see that this is not always the case. While sometimes, a subsidy on peanuts induces an increase in social welfare, there are instances when a tax on peanuts is the appropriate policy. Furthermore, it is possible that neither a tax nor a subsidy can lead to a welfare improvement.

When a government imposes a quantity subsidy σ on peanuts, the relevant equilibrium conditions become

$$X^d(\frac{p-\sigma}{1-A(Y)}) = Q^s(p) \tag{31}$$

$$\frac{a(Y)}{1-A(Y)}(p-\sigma)Q^s(p) = 1 \tag{32}$$

Figure 2 depicts such an equilibrium. The resulting level of crime is represented by the shaded area.

A quantity subsidy will improve social welfare if its social marginal benefit is higher than its social marginal cost. The social marginal benefit is the marginal utility of peanuts. The social marginal cost of the subsidy has two components. One is the marginal production cost and the other is the additional crime level, Y', induced by the subsidy. Since, at the equilibrium, the marginal utility of peanuts is $\frac{p}{1-A(Y)}$, and the marginal cost of peanuts is p, a quantity subsidy is welfare-improving if and only if

$$\frac{A(Y)}{1-A(Y)}pQ' > Y'.$$

At the competitive equilibrium, however, this condition does not necessarily hold. Specifically, while a small increase in output leads to an increase in individuals' utility that is higher than the additional production cost, it also leads to a change in the level of crime that may offset the associated increase in social surplus. The following example illustrates this point.

Example 6 Consider an economy where $\phi_i(x) = x(6 - x/d)$, $c(Q) = Q^2/2$, and $A(Y) = Y/(1+Y)$, where $d \in [1, 4]$. The peanut supply function is, therefore, given by $Q(p) = p$. If there is a quantity subsidy σ on peanuts, aggregate demand is

$$X^d\left(\frac{p-\sigma}{1-A(Y)}\right) = 3d - \frac{p-\sigma}{2(1-A(Y))}$$

It can be checked that the price, quantity, and crime level that satisfy equilibrium conditions (31–32) are

$$\begin{aligned} p(\sigma) = Q(\sigma) &= \frac{1}{2}\left(\sigma + \sqrt{\sigma^2+4}\right) \\ Y(\sigma) &= \frac{d\left(-\sqrt{\sigma^2+4}+\sigma+12\right) - 2\left(\sqrt{\sigma^2+4}+\sigma\right)}{d\left(\sqrt{\sigma^2+4}-\sigma\right)}. \end{aligned}$$

In particular, when there is no subsidy, the equilibrium price, quantity and crime levels are

$$p^* = 1, \ X^* = Q^* = 1, \ Y^* = 5 - \frac{2}{d}.$$

Social welfare is

$$\frac{(d+2)\sqrt{\sigma^2+4} + (2-3d)\sigma}{2d\left(\sqrt{\sigma^2+4}-\sigma\right)}.$$

It can be checked that when $d = 2$, social welfare does not depend on σ. As a result, no subsidy can improve social welfare. When $d < 2$ a small positive subsidy improves welfare, and when $d > 2$, a small quantity tax improves welfare.

4.4 Public police

Under the regime of public police, a level of protection $t_i = T$ is allocated uniformly across individuals and is financed by a personalized compulsory contribution $\hat{t}_i$ such that $\int \hat{t} = T$.

We assume that, like leisure, individuals' tax payments are not subject to theft. Therefore, individual i's budget set is now

$$\left\{(x_i, m_i, y_i) : px_i + y_i \leq (1 - A(Y,T))(\omega_i + \pi_i - \hat{t}_i - m_i) + y_i a(Y,T)(\Omega + \Pi - T - M)\right\}$$

The parameters that the individual takes as given are the price p, his contribution to police protection $\hat{t}_i$, the crime rate Y, and the returns to theft $a(Y,T)(\Omega + \Pi - M - T)$.

A competitive allocation consists of a feasible allocation $\langle (x^*, m^*, y^*, T), (-Z^*, Q^*) \rangle$ and a price p, such that

1. Production plan $(-Z^*, Q^*)$ maximizes profits given p.

2. For each $i \in [0,1]$, (x_i^*, m_i^*, y_i^*) maximize utility given his budget set.

We can see that a competitive equilibrium with public police T and tax schedule $\hat{t}$ is equivalent to a competitive equilibrium with no police in the economy $\left\langle \left(\phi, \omega - \hat{t}, \theta\right), c, A(\cdot, T) \right\rangle$. Furthermore, given T, and except for equilibrium leisure m^*, the tax schedule $\hat{t}$ does not affect the equilibrium outcome. As a result, the equilibrium is still characterized, *mutatis mutandis*, by conditions (23–26), or, equivalently, by conditions (27–28). In particular, the equilibrium level of theft is implicitly defined by

$$X^d\left(\frac{p}{1 - A(Y,T)}\right) = Q^s(p) \tag{33}$$

$$\frac{a(Y,T)}{1 - A(Y,T)} p Q^s(p) \leq 1 \text{ with equality if } Y > 0 \tag{34}$$

As opposed to the model in Section 3, the equilibrium level of crime is not necessarily decreasing in the police level. The reason is that, other things being equal, more police reduces crime, which induces individuals to work and produce more, which itself increases crime. Since, in equilibrium, the value of the stolen goods equals the level of crime, it may well be that an increase in public police protection leads to an increase in the value of the appropriated goods.

Although equilibrium theft is not necessarily decreasing in police protection, at the optimal level of police protection it does decrease. As mentioned before, the optimal level of public police is the one that induces an equilibrium that maximizes social welfare. An increase in public police protection will increase social welfare if its marginal social benefit is greater than its marginal social cost. Clearly, this is possible only if an increase in police protection reduces crime. In this case, the marginal social benefit is composed of the decrease in crime and of the marginal utility of the consumption induced by the lower crime level. The marginal social cost, on the other hand, is composed of the marginal cost of output and the marginal cost of police protection. Taking into account that in equilibrium the marginal utility is $\frac{p}{1-A(Y^*,T)}$ and the marginal cost is p, we conclude that the optimal level of public police satisfies

$$\frac{A(Y^*,T)p}{1-A(Y^*,T)}\frac{\partial Q^*}{\partial T} = \frac{\partial Y^*}{\partial T} + 1.$$

The next example illustrates the above points.

Example 7 Assume that $\phi_i(x) = x\,(10(1+i) - x/2)$ and that the appropriation technology is given by $A(Y,T) = \frac{Y}{(1+T)(1+Y)}$. Suppose that the cost of peanuts in terms of work hours is given by $c(q) = q^2/2$. Then, individual i's demand, as a function of the excise rate and the peanut price is

$$x_i^d(\frac{p}{1-A}) = 10(1-i) - \frac{p}{(1-A)}$$

Consequently, the aggregate demand is

$$X^d(\frac{p}{1-A}) = \int x = 15 - \frac{p}{1-A}$$

The peanut supply function is given by $Q(p) = p$. Therefore, the equilibrium conditions (33–34) are

$$\begin{aligned} 15 - \frac{p(1+T)(1+Y)}{1+T(1+Y)} &= p \\ \frac{p^2}{1+T(1+Y)} &= 1 \end{aligned}$$

This system of equations implicitly define the equilibrium level of crime and price as functions of police protection. It can be checked that the equilibrium level of crime is not decreasing in the police level. Specifically, for low levels of police protection, the crime rate is increasing in T. (See Figure 3). The optimal level of police per capita is $T^* = 6.70109$, and the associated level of crime is $Y^* = 6.27845$.

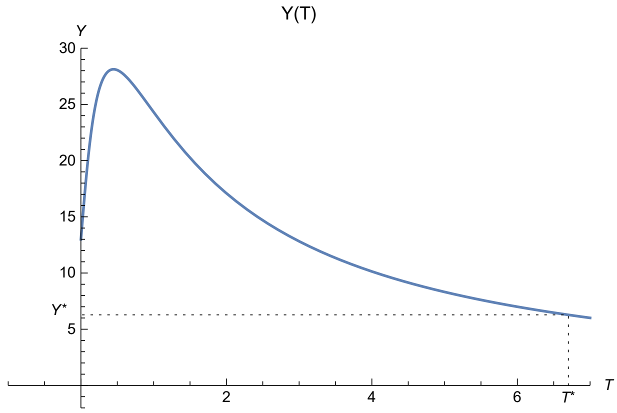

Figure 3: The equilibrium level of crime as a function of police protection.

4.5 Private police

Suppose now that there is no public police but one can hire private police protection. The definition of a competitive equilibrium is the same as before, except that now the budget of individual i consists of all the bundles (x_i, m_i, y_i, t_i) that satisfy

$$px_i + y_i \leq (1 - A(Y, t_i))(\omega_i + \pi_i - t_i - m_i) + y_i \int a(Y, t)(\omega + \pi - t - m),$$

where the data that the agent considers as given are the price p, his share of the profits π_i, and the returns to crime $\int a(Y, t)(\omega + \pi - t - m)$. As a result, the equilibrium allocation is now characterized by the following conditions (assuming an interior solution):

$$c'(Q) = p \tag{35}$$

$$(1 - A(Y, t_i)) + (\omega_i + \pi_i - t_i - m_i)A_2(Y, t_i) = 0 \qquad i \in [0, 1] \tag{36}$$

$$(1 - A(Y, t_i))\phi_i'(x_i) = p \qquad i \in [0, 1] \tag{37}$$

$$(1 - A(Y, t))(\omega + \pi - t - m) = px \tag{38}$$

$$\int a(Y, t)(\omega + \pi - t - m) = 1 \tag{39}$$

$$\int x = Q \tag{40}$$

Condition (36) is the condition that the choice of private police must satisfy if it is to be utility-maximizing.

4.5.1 Optimal allocation of private police

Let $\mathcal{E} = \langle(\phi, \omega, \theta), c, A\rangle$ be an economy and let $\langle(x^*, m^*, y^*, t^*), (-Z^*, Q^*)\rangle$ be a competitive allocation. The corresponding crime level and tax collection are $Y^* = \int y^*$ and $T^* = \int t^*$. For the reasons discussed in the previous sections, this equilibrium is not efficient. However, one may ask whether, as was the case in the model of Section 3, the allocation of police protection is efficient, *given the equilibrium level of crime*. It turns out that this is not the case. As the following example illustrates, the government can impose a different allocation of police protection whose resulting equilibrium attains a higher level of social welfare.

Example 8 Consider an economy with two types of consumers. For $i \in [0, 1/2]$, the utility of peanuts is given by $\phi_i(x) = x(10 - x/2)$, and for $i \in (1/2, 1]$, it is given by $\phi_i(x) = x(50 - x/2)$. The cost function is $c(Q) = Q^4/42348$ and the appropriation technology is given by $A(Y, t_i) = \frac{Y}{(1+Y)(1+t_i)}$. It can be checked that the competitive equilibrium consists of $p^* = 2$, $Q^* = 27.66$, $Y^* = 6.35$ along with an allocation of peanuts given by $x_i^* = 7.52$ for $i \leq 1/2$ and $x_i^* = 47.82$ for $i > 1/2$, and an allocation of private police protection given by $t_i^* = 3.47$ for $i \leq 1/2$ and $t_i^* = 8.95$ for $i > 1/2$. However, a social planner can impose public but discriminatory police protection given by $\hat{t}_i = 3.38$ if $i \leq 1/2$ and $\hat{t}_i = 9.03$ if $i > 1/2$ and the resulting competitive equilibrium (the one that solves the system of equation (37–40) would yield $\hat{p} = 2$, $\hat{Q} = 27.66$, $\hat{Y} = Y^*$, along with the peanut allocation $\hat{x}_i = 7.51$ if $i \leq 1/2$ and $\hat{x}_i = 47.81$ if $i > 1/2$. As we can see, in both equilibria the level of crime is the same, but it can be checked that the equilibrium with discriminatory public police protection attains a higher level of social welfare.

The above example shows that Observation 6 cannot be extended to the case in which only produced output can be stolen. However, it can be checked that if we modify the example by changing the cost function to be $c(Q) = 2Q$, we obtain the same equilibrium which this time turns out to be conditional efficient. This is no accident. As we will show next, in this section's scenario, if the production technology is linear, the competitive allocation of police protection is efficient conditional on crime being at the equilibrium level.

To see this, assume that the production technology is linear and let $p = c'(Q)$ be the corresponding constant marginal cost. Conditional on the level of crime being the competitive one, Y^*, the optimal allocation of private police maximizes social welfare, given that the other variables are determined in equilibrium. Formally, it solves

$$\begin{aligned} & \max_{t,x,m} \int (m_i + \phi_i(x_i)) \\ \text{s.t.} \quad & (37) - (40) \end{aligned}$$

We will show that the competitive allocation, $\langle (x^*, m^*, y^*, t^*), (-Z^*, Q^*) \rangle$, which satisfies condition (37), solves the following problem, which is obtained from the previous one by eliminating this restriction.

$$\begin{aligned} & \max_{t,x,m} \int (m_i + \phi_i(x_i)) \\ \text{s.t.} \quad & 1 = \int a(Y^*, t)(\omega - t - m) \\ & px = (1 - A(Y^*, t))(\omega - t - m) \\ & \int x = Q \end{aligned}$$

The above problem's constraints readily imply that $\Omega = M + pQ + T + Y^*$. As a result, this problem can be rewritten as

$$\begin{aligned} & \max_{t,x} \int (\phi_i(x_i) + \omega_i - px_i - t_i) \\ \text{s.t.} \quad & k' = \frac{a(Y^*, t)}{1 - A(Y^*, t)} px \\ & k_0 = 0, \;\; k_1 = 1 \end{aligned} \tag{41}$$

The associated Hamiltonian is

$$H = \phi_i(x_i) + \omega_i - t_i - px_i - \lambda\big(a(Y^*, t)\frac{px}{1 - A(Y^*, t)}\big)$$

and the necessary conditions for a solution are

$$\phi_i'(x_i) - p\left(1 + \lambda \frac{a(Y^*, t)}{(1 - A(Y^*, t))}\right) = 0 \tag{42}$$

$$-1 - \lambda \frac{A_2(Y^*, t)px}{Y^* (1 - A(Y^*, t))^2} = 0 \tag{43}$$

$$\lambda' = 0 \tag{44}$$

$$\int a(Y^*, t)\frac{px}{(1 - A(Y^*, t))} = 1 \tag{45}$$

Now let $\langle (x^*, m^*, y^*, t^*), (-Z^*, Q^*) \rangle$ be a competitive equilibrium allocation along with $\lambda = Y^*$. Since $p = (1 - A(Y^*, t_i^*))\phi_i'(x_i^*)$ we have that condition (42) is satisfied. Also, since $(1 - A(Y^*, t_i^*)) + \frac{px_i^*}{1-A(Y^*,t_i^*)} A_2(Y^*, t_i^*) = 0$, we have that

$$\frac{px_i^*}{(1 - A(Y^*, t_i^*))^2} A_2(Y^*, t_i^*) = -1$$

and condition (43) is satisfied as well. Since Y^* is constant, $\lambda' = 0$ and condition (44) is also satisfied. Finally, by definition, condition (45) is also satisfied by the competitive allocation.

We conclude that the competitive equilibrium satisfies the necessary conditions for an optimal allocation of police, conditional on the level of crime. We cannot conclude, however, that the allocation is optimal, because the necessary conditions may not be sufficient. But even if the allocation of police protection is constrained efficient, the competitive level of crime is not globally optimal. Indeed, an argument analogous to the one used to prove Observation 7, shows that social welfare can be increased by increasing spending on police protection and reducing the level of crime. The reason is the same as that of the scenario in Section 3. Namely, private police exerts a positive externality; it reduces the returns to theft, which induces people to spend less time stealing from the whole population. This externality is not taken into account by the individual.

5 Concluding remarks

We have introduced theft into the standard partial equilibrium model of an economy. We considered two models that differ in the kind of goods that are subject to theft. In the first model, we allow thieves to steal from the initial endowments of factors of production. In the second one, only produced goods can be stolen. In particular, time devoted to leisure, theft, and property protection is not subject to appropriation. The two models generate different conclusions. While in the first an equilibrium exists and is unique, in the second there may be non-existence and multiplicity of equilibria. In both models, theft generates the obvious inefficiency associated with the fact that time spent stealing is itself a waste of resources. In the second model, there is an additional source of inefficiency due to the fact that theft acts as a quantity tax that introduces a wedge between the consumers' marginal utility and the firms' marginal cost of production. The two models also differ in their policy recommendations. While in the first one, a policy that increases output is beneficial in the sense that it reduces crime, this is not necessarily so in the second one. Also, whereas in the first model, public police protection reduces crime, in the second one it may very well increase it. The allocation of protection granted by public police is typically inefficient since equal protection is awarded to all individuals, independent of their stealable wealth. Although allowing for private police does not induce an optimal level of crime, it has the potential to distribute police efficiently (conditional on the level of crime). It turns out that in the first model, the competitive equilibrium does allocate police protection efficiently (conditional on the level of crime). In the second model, however, private police is typically inefficient, unless the technology of peanut production is linear. Finally, within the first model, we have investigated the notion of a voting equilibrium under proportional taxation. Under this tax regime, communities are not necessarily classified by income brackets, but in all of them, the sum of the excise and tax rates are the same. Furthermore, the equilibrium crime tax rates in each community are the optimal ones.

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A Appendix

Proof of Observation 2. By definition, $(p(Q), x(Q))$ solves

$$\begin{aligned} \phi'(x(Q)) &= p(Q) \\ \int x(Q) &= Q. \end{aligned}$$

It is routine to show that these conditions imply that $p' < 0$. Letting $\Pi = p(Q)Q - c(Q)$, the level of crime $Y(Q) = \int y(Q)$ is the solution to the equation

$$1 = a(Y)(\Omega + \Pi) \tag{46}$$

Since the equilibrium allocation satisfies $\int m = \Omega - Z - Y$, and $Z = c(Q)$, we have that

$$W(x(Q), m(Q)) = \int \phi(x(Q)) + \Omega - c\,(Q) - Y(Q).$$

As a result,

$$\frac{\partial W}{\partial Q} = \int \phi'(x)x' - c' - Y'$$

Since in equilibrium, when $Q = Q^*$, we have that $\phi'(x) = c'$, and $\int x' = 1$ we conclude that

$$\frac{\partial W}{\partial Q}(x(Q^*), m(Q^*)) = -Y'(Q^*),$$

That is, the increase in welfare is exactly the decrease in time devoted to theft. In order to calculate this value, note that $Y(Q)$ solves equation (46). Therefore, since $a(Y)$ is a decreasing function, Y is decreasing in Q if and only if $\Pi' < 0$. By Hotelling's lemma, $\Pi' = Qp'$. Therefore, since $p' < 0$, we have that $\Pi' < 0$ and we can conclude that

$$\frac{\partial W}{\partial Q}(x(Q^*), m(Q^*)) = -Y'(Q^*) > 0.$$

□