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Sensitivity Analysis in Unconditional Quantile Effects

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Abstract

This paper proposes a framework to analyze the effects of counterfactual policies on the unconditional quantiles of an outcome variable. For a given counterfactual policy, we obtain identified sets for the effect of both marginal and global changes in the proportion of treated individuals. To conduct a sensitivity analysis, we introduce the quantile breakdown frontier, a curve that quantifies the maximum amount of selection bias consistent with a given conclusion of interest across different quantiles. We obtain a $\sqrt{n}$-consistent estimator of the curve, and propose a bootstrap-based inference procedure. To illustrate our method, we perform a sensitivity analysis on the effect of unionizing low income workers on the quantiles of the distribution of (log) wages.

Keywords: unconditional quantile effects, partial identification, sensitivity analysis, directional differentiability.

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1 Introduction

In this paper we propose a sensitivity analysis on the effect of counterfactual policies that change the proportion of treated individuals. Consider a situation where a policy maker is interested in treating non-treated individuals. The key identification challenge is that we do not the observe the counterfactual outcome of individuals who switch groups, that is, the *newly* treated individuals. In some cases, however, it is still possible to recover the distribution of the unobserved counterfactual outcome. For example, suppose that treatment status is randomly assigned, and a policy maker increases the proportion of treated individuals by randomly selecting non-treated individuals.[1] Although we do not observe the counterfactual outcome of the *newly* treated individuals, we know it is drawn from the same distribution as the *already* treated individuals. Hence, we identified the counterfactual distribution of *newly* treated individuals.

When treatment status is *not* randomly assigned in the first place, the identification strategy previously described breaks down. The reason is that due to the selection bias in the original treatment status, a random selection of individuals from the control group will be drawn from a different distribution. Thus, in the presence of selection bias, identification of the counterfactual distribution requires that the policy maker has enough information to device a policy such that the (unobservable) distribution of the *newly* treated "matches" the distribution of the *already* treated individuals. This is usually unfeasible. Even if the policy maker has this information, such as when treatment status is randomly assigned, they might not be interested in a policy that merely selects the *newly* treated individuals at random.

The previous discussion highlights that identification of counterfactual distributions results in either very stringent information requirements, or in policies that might not be interesting. In both cases, the distribution of the *newly* treated individuals is restricted. From the point of view of the policy maker, this can rule out many interesting policies. To see this, consider the following example. A policy maker might like to know if an increase in the unionization rate reduces inequality. If unionized workers are relatively high-skilled, and a policy expands unionization with low-skilled workers, then the distribution of wages conditional on being in the union, is likely to change.

In order to analyze a richer set of counterfactual policies, we drop the restrictions on the distribution of the *newly* treated individuals and provide partial identification results for two effects. The first one is a global effect that compares the quantiles of the observed outcome, to those of the counterfactual outcome, where the proportion of treated individuals has been increased by δ. The second one is a marginal effect where we let δ go to zero, and analyze its limiting effect on the unconditional quantiles of the outcome.

Another important contribution of this paper is to propose a framework for a sensitivity analysis on certain conclusions of interest. To do this, we quantify the departure from point identification by the vertical distance between the distributions of the *newly* treated individuals

[1] We assume full compliance in both randomizations.

and the *already* treated individuals. We introduce a curve called the *quantile breakdown frontier*, which quantifies the maximum departure from point identification such that a given set of conclusions holds across different quantiles. Next, we bound the global effects curve using this maximum departure derived from the quantile breakdown frontier. In this way, we obtain an identified region for the global effect curve consistent with the desired conclusions.

The departure from point identification is due to the selection bias induced by the counterfactual policy. We call this the *policy selection bias*. The usual selection bias states that treated and non-treated individuals are different in a sense, and that is what explains the selection in the first place. Instead, the policy selection bias is the difference between the distributions of the *newly* treated individuals and the *already* treated individuals. Returning to the unionization example, the policy selection bias arises because the union wages of *newly* unionized workers may not be drawn from distribution of the *already* unionized workers. We do not know the distribution of union wages of newly unionized workers, hence we can only partially identify the global and marginal effects.

The policy selection bias can be non-negligible even if the original selection into treatment is randomly assigned. The reason is that, for the policy selection bias, what matters is who the *newly* treated individuals are. Conversely, if there is selection bias initially, but the distribution of the *newly* treated "matches" the distribution of the *already* treated individuals, then there will be no policy selection bias. Thus, the policy selection bias depends on the particular counterfactual policy being analyzed, not whether there is selection bias in the original selection mechanism.

Estimation of both the quantile breakdown frontier and the bounds on the global effect are based on empirical distribution functions and empirical quantiles, and are $\sqrt{n}$-consistent. Inference is more challenging, though. The reason is that the bounds derived from the quantile breakdown frontier are not a fully Hadamard differentiable function of the underlying distributions; there are a few kinks where differentiability fails. However, directional differentiability holds, and we can still exploit the functional Delta method to obtain asymptotic distributions. These limiting laws are not Gaussian. So, as shown in Fang and Santos (2019), the standard or "naive" bootstrap is not valid. Instead, we resort to the numerical bootstrap of Hong and Li (2018, 2020) to construct pointwise confidence intervals and uniform confidence bands.

We apply these methods to the study of unions and inequality, which has long been of interest to labor economics. A recent comprehensive review of this extensive literature is provided by Farber et al. (2020). Using the data in Firpo, Fortin and Lemieux (2009), our empirical application considers the effect of expanding unionization on the quantiles of the distribution of (log) wages. Our approach allows us to tackle the question from a different perspective. Using the tools developed in this paper, we can quantify the amount of policy selection bias that is consistent with a policy that increases the unionization rate by unionizing low earnings workers. By looking at the global effect in the 20^{th} and 80^{th} quantiles of the distribution of wages we investigate the amount of policy selection bias consistent with unions reducing overall inequality. To this end, we look at two conclusions: whether the 20^{th} quantile increases by more than 10%, and whether the 80$^{\text{th}}$ quantile increases less than 10%. We find that this is consistent with moderate values of policy selection bias.

Related Literature There is an extensive literature devoted to the analysis of counterfactual distributions. A good reference is Firpo, Fortin and Lemieux (2011). In this paper, we focus on counterfactual distributions that arise as a result of a counterfactual policy that changes the proportion of treated individuals. The Policy Relevant Treatment Effect (PRTE) of Heckman and Vytlacil (2001, 2005), and the Marginal PRTE (MPRTE) of Carneiro, Heckman and Vytlacil (2010, 2011) are examples of the aforementioned global and marginal effects. The difference is that they analyze the unconditional mean of the outcome. Identification relies on the a separable threshold model for the selection equation, and the availability of a continuous instrumental variable. In this setting, the proportion of treated individuals is changed by manipulating the instrumental variable. Our analysis does not make any assumptions on the selection equation. We do not require an instrumental variable either.

The marginal effect on the unconditional quantiles of an outcome was first studied by Firpo, Fortin and Lemieux (2009). The identification arguments of Firpo, Fortin and Lemieux (2009) are based on a distributional invariance assumption: the distribution of the outcome for the original treatment group (under the original policy regime) is the same as that for the new treatment group (under the new policy regime), and this also holds for the control groups under the two policy regimes.[2] For the case of an endogenous binary covariate, where distributional invariance might not hold, Martinez-Iriarte and Sun (2020) achieve identification by generalizing the Marginal Treatment Effect framework. Kasy (2016) also analyzes counterfactual policies which assign a binary treatment, but focuses on a welfare ranking. Kaplan (2019) takes a closer look at the conditional independence assumption in the case of counterfactual assignments, and concludes that it must hold not only for the original assignment, but also for the counterfactual assignment/policy. We analyze the conditions Kaplan (2019) in more detail in Example B.4 in Appendix B.

Rothe (2012) provides a general treatment for functionals of the unconditional distribution of the outcome. What we call a global effect, Rothe (2012) refers to as a *Fixed Partial Policy Effect*, and what we call a marginal effect, Rothe (2012) refers to as a *Marginal Partial Distributional Policy*. However, Rothe (2012) imposes different identifying assumptions, namely a form of conditional exogeneity, which also yield a partial identified set. We do not impose such assumptions in order to broaden the types of policies we can analyze.

It is important to highlight that we do not estimate a quantile treatment effect. The quantile treatment effect is the difference between the τ-quantile under treatment and the τ-quantile under control, and depends on the distribution of the covariates. In a recent contribution, Hsu, Lai and Lieli (2020) investigate the changes in this effect when the distribution of the covariates is manipulated. Aside from treatment status, we do not manipulate the distribution of covariates.

Our sensitivity analysis is based on the breakdown analysis of Kline and Santos (2013) and

[2]See the proof to Corollary 3 of the working paper version Firpo, Fortin and Lemieux (2007).

Masten and Poirier (2020). Kline and Santos (2013) perform a sensitivity analysis in a different context: departures from a missing (data) at random assumption. In a manner similar to us, this departure is measured as the Kolmogorov-Smirnov distance between the distribution of observed outcomes and the (unobserved) distribution of missing outcomes. Our quantile breakdown frontier builds on the breakdown frontier introduced by Masten and Poirier (2020). However, the quantile breakdown frontier takes advantage of the unique features of the policy selection bias: for each quantile the breakdown frontier of Masten and Poirier (2020) is a rectangle. This allows us to plot the higher-dimensional quantile breakdown frontier in a plane.

Notation All the CDFs are denoted by F with a subscript indicating the random variable. So, the CDF of Y is $F_Y(y)$. Conditional CDFs are denoted similarly. For example, the CDF of Y conditional on $D = 1$ and $X = x$ is denoted by $F_{Y|D=1,X=x}(y)$. The τ-quantile of Y is denoted by $F_Y^{-1}(\tau)$. Weak convergence is denoted by $\rightsquigarrow$.

Organization The paper is organized as follows: Section 2 introduces our framework and shows how to construct the identified regions; Section 3 introduces the quantile breakdown frontier and explains the sensitivity analysis procedure; Section 4 discusses estimation and inference; Section 5 contains the empirical application; and Section 6 concludes. We relegate all proofs to Appendix A.

2 Counterfactual Policies and Unconditional Effects

We will work with the potential outcomes framework. For some unknown functions h_0 and h_1

$$Y(0) = h_0(X, U_0),$$
$$Y(1) = h_1(X, U_1),$$

where X are observed covariates and U_0 and U_1 consist of unobservables. We do not impose any restriction on the dimension of the unobservables. The observed outcome is thus

$$\begin{aligned} Y &= D \cdot h_1(X, U_1) + (1 - D) \cdot h_0(X, U_0). \\ &:= h(D, X, U), \end{aligned}$$

for a general nonseparable function h, where D is a binary random variable taking values 0 and 1, and $U := (U_0, U_1)'$. The variable D can be interpreted as the treatment status, and $p := \Pr(D = 1)$ is the proportion of treated individuals.

In the rest of the paper, we maintain a continuity assumption about the outcome Y. This is not essential to our results, but allows us to reduce the notational burden.

Assumption 1 (Continuity)**.** *The observed outcome Y is continuous, with positive density in its support $\mathcal{Y}$.*

A counterfactual policy is an alternative assignment of individuals to treatment. It is given by

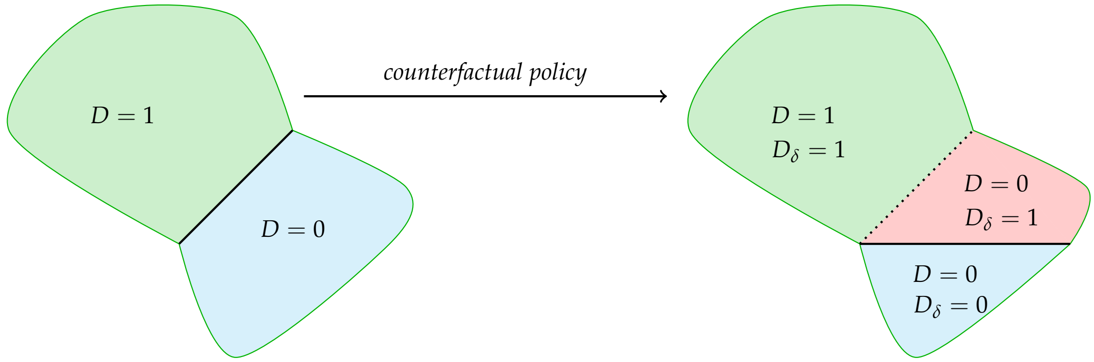


Figure 1: *A counterfactual policy where $D_\delta - D \geq 0$.*

a binary random variable D_δ, such that $\Pr(D_\delta = 1) = p + \delta$ for a fixed $\delta \in (-p, 1-p)$. It is called counterfactual because it may assign $D_\delta = 1$ to an individual whose $D = 0$. As δ varies over $(-p, 1-p)$, we obtain a collection of counterfactual policies which is denoted by $\mathcal{D}$. Somewhat casually, we also call the collection $\mathcal{D}$ a sequence of policies. When a particular counterfactual policy D_δ belongs to $\mathcal{D}$ we write $D_\delta \in \mathcal{D}$. The counterfactual outcome we would observe for a given $D_\delta \in \mathcal{D}$ is

$$Y_{D_\delta} = h(D_\delta, X, U),$$

where we implicitly assumes that the potential outcomes are not affected by the manipulation of D.

Strictly speaking, the counterfactual outcome Y_{D_δ} is not well defined until we define $\mathcal{D}$, the collection of counterfactual policies. We will restrict ourselves to policies that shift a portion of individuals in the control group to the treatment group. We refer to such individuals as *newly treated*. This means that for every individual, $D_\delta - D \geq 0$. This is shown in Figure 1.

Assumption 2 (Counterfactual Policies)**.** *The sequence of policies $\mathcal{D}$ satisfies*

1. $\Pr(D_\delta = 1) = p + \delta$ *for* $\delta \in [0, 1-p)$ *and* $D_\delta \in \mathcal{D}$*;*
2. *Monotonicity:* $D_\delta - D \geq 0$*;*
3. *The counterfactual outcomes* Y_{D_δ} *are continuous with positive density on their support* $\mathcal{Y}$*.*

The monotonicity assumption $D_\delta - D \geq 0$ is mainly for expositional simplicity. We can do without this assumption, but we need to make some minor changes to our approach. However, there is also a practical purpose. In a context where D is union status, and $D = 1$ denotes unionized individuals, Assumption 2 requires that we increase the unionization rate by unionizing previously nonunionized workers. It would probably be hard to simultaneously unionize and deunionize different workers.

Another way to look at the monotonicity assumption is by inspecting the joint distribution of D and D_δ it induces.

	$D_\delta = 0$	$D_\delta = 1$
$D = 0$	$1 - p - \delta$	δ
$D = 1$	0	p

In other words, Assumption 2 rules out the presence of *newly untreated* individuals. Also, in the limit, when $\delta = 0$, we return to the original distribution of individuals. We will evaluate the effect of a counterfactual policy with two parameters: the global and the marginal effects. Let $F_Y^{-1}(\tau)$ and $F_{Y_{D_\delta}}^{-1}(\tau)$ denote the τ-quantiles of Y and Y_{D_δ} respectively.

Definition 1 (Global and Marginal Effects)**.** *For a given sequence of policies $\mathcal{D}$, the unconditional global effect at the τ-quantile is*

$$G_{\tau,D_\delta} := F_{Y_{D_\delta}}^{-1}(\tau) - F_Y^{-1}(\tau),$$

and the unconditional marginal effect at the τ-quantile is

$$M_{\tau,\mathcal{D}} := \lim_{\delta \to 0} \frac{F_{Y_{D_\delta}}^{-1}(\tau) - F_Y^{-1}(\tau)}{\delta}$$

whenever this limit exists.

The global effect G_{τ,D_δ} is the comparison of quantiles of the counterfactual distribution vs. the observed distribution. For example, it could tell us what could happen to the median under a particular policy D_δ. The marginal effect $M_{\tau,\mathcal{D}}$ can be interpreted as an ordinary derivative: for small δ, it provides an approximation to the direction of the change in a given τ-quantile.

The next task is to define *who* are the *newly treated* individuals, that is, how does D_δ determine who receives treatment among the individuals whose $D = 0$? In this paper we will focus on two types of policies: a policy that simply chooses individuals whose $D = 0$ at random and assigns them to $D_\delta = 1$, and a policy that chooses individuals based on a user-specified criterion. We will refer to these two types of policies as *randomized policy* and *non-randomized policy* respectively. A randomized policy might be more in line with a concept of fairness: everyone in the untreated group gets an equal chance of being treated. A non-randomized policy, in contrast, might be more in line with a situation where the policy maker has a particular loss function they are trying to optimize.

Example 1 (Randomized policy)**.** *A randomized policy satisfies: for any $\delta \in [0, 1-p)$*

$$D_\delta = \begin{cases} 1 & \text{if } D = 1 \\ 0 \text{ or } 1 & \text{if } D = 0 \end{cases}$$

and the newly treated *are selected at random. Using the conditional independence notation of Dawid*

(1979),[3] *we write $D_\delta \perp Y(1), Y(0) \| D = 0$:*

$$Pr(D_\delta = 1 | D = 0) = \Pr(D_\delta = 1 | D = 0, Y(1), Y(0)) = \frac{\delta}{1 - p}. \tag{1}$$

Example 2 (Non-randomized policy)**.** *An example of a non-randomized policy is the following: for any $\delta \in [0, 1 - p)$*

$$D_\delta = \begin{cases} 1 & \text{if } D = 1 \\ 1 & \text{if } D = 0 \text{ and } Z \leq F^{-1}_{Z|D=0}\left(\frac{\delta}{1-p}\right) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

for some observable random variable Z. In this case, the individuals in the group $\{D = 0\}$ whose Z is less than the $\frac{\delta}{1-p}$-quantile of this group are shifted to $D_\delta = 1$. This rule guarantees that, in expectation, a proportion δ of individuals is shifted.

The following theorem characterizes the counterfactual distribution associated with an arbitrary policy.

Theorem 1 (Counterfactual Distribution)**.** *For a sequence of policies $\mathcal{D}$ that satisfies Assumptions 2, the counterfactual distribution for a given $D_\delta \in \mathcal{D}$ is*

$$F_{Y_{D_\delta}}(y) = F_a(y) + \delta\left[F_{Y(1)|D=0,D_\delta=1}(y) - F_{Y(1)|D=1,D_\delta=1}(y)\right], \tag{3}$$

where

$$\begin{aligned} F_a(y) :&= (1 - p - \delta)F_{Y(0)|D=0,D_\delta=0}(y) + (p + \delta)F_{Y(1)|D=1,D_\delta=1}(y) \\ &= (1 - p - \delta)F_{Y|D=0,D_\delta=0}(y) + (p + \delta)F_{Y|D=1,D_\delta=1}(y). \end{aligned} \tag{4}$$

The distribution F_a is called an apparent counterfactual distribution because it is obtained by imputing $F_{Y(1)|D=1,D_\delta=1}$ to the *newly* treated subpopulation.[4] The true distribution, which may not be identifiable, is $F_{Y(1)|D=0,D_\delta=1}$, so the second term corrects this. In a sense, F_a proceeds as if $F_{Y(1)|D=0,D_\delta=1}$ were equal to $F_{Y(1)|D=1,D_\delta=1}$, something which is unlikely to be true. This can be seen in Figure 1. The apparent distribution ignores the red shaded area, and combines the green and the blue areas. The second term in (3) is the difference between the red and green areas.

The apparent distribution F_a is identified because the policy maker knows the composition of the subpopulations $\{D = 0, D_\delta = 0\}$, the *never* treated, and $\{D = 1, D_\delta = 1\}$, the *already* treated. For both of these subpopulations we observe the "correct" potential outcome. More specifically,

[3] Dawid (1979) writes $X \perp Y \| Z$ to denote that X and Y are independent conditional on $Z = z$ for any z. Here, we require independence to hold conditionally only on $D = 0$.

[4] The correct notation for F_a is F_{a,D_δ}, that is, it should include the policy D_δ. However, to keep the notation simple, we omit this. The reader should bear in mind that for two different sequences of policies the apparent distributions might differ.

for the *never* treated subpopulation, we observe $Y(0)$, and for the *already* treated subpopulation, we observe $Y(1)$. The distributions of Y, which is equal to $Y(0)$ and $Y(1)$, respectively, for the two subpopulations, are identified. As a result, F_a is identified. It is worth pointing out that, under Assumption 2, $F_{Y(1)|D=1,D_\delta=1} = F_{Y(1)|D=1}$, and $F_{Y(0)|D=0,D_\delta=0} = F_{Y(0)|D_\delta=0}$. We maintain the "long" notation in order to emphasize the role of D_δ. Also, while $F_{Y(0)|D=0,D_\delta=0}$ is identified, it may not equal $F_{Y|D=0}$ unless D_δ is randomized.

The only unidentified term in (3) is $F_{Y(1)|D=0,D_\delta=1}$. This is the potential outcome $Y(1)$ for the *newly* treated individuals. However, for this subgroup, we only observe $Y(0)$. As we mentioned before, a consequence of Assumption 2 is that we lose point identification of the counterfactual distribution.

Remark 1 (Firpo, Fortin and Lemieux (2009))**.** *The marginal effect $M_{\tau,\mathcal{D}}$ was originally studied by Firpo, Fortin and Lemieux (2009). Instead of Assumption 2, Firpo, Fortin and Lemieux (2009) assume a form of distributional invariance: $F_{Y_{D_\delta}|D_\delta=d} = F_{Y|D=d}$ and obtain point identification. See the proof to Corollary 3 of the working paper version Firpo, Fortin and Lemieux (2007). When both D and D_δ are independent of U and X, then distributional invariance will be satisfied. In this particular case, a policy maker can randomize D_δ so that for a given δ, a fraction $p + \delta$ of individuals is randomly assigned to treatment. However, if we allow for D to be endogenous, and if, as is usually the case, the structural form of endogeneity is unknown, then it may be impossible for the policy maker to design a sequence $\mathcal{D}$, such that for every $D_\delta \in \mathcal{D}$, $F_{Y_{D_\delta}|D_\delta=d}$ "matches" $F_{Y|D=d}$. From the point of view of the policy maker, this is a significant restriction on the types of counterfactual policies they can consider.*

Remark 2 (Policy Relevant Treatment Effect)**.** *Heckman and Vytlacil (2001, 2005) and Carneiro, Heckman and Vytlacil (2010, 2011) investigate the effect on the unconditional mean of the outcome. Using our notation, the Policy Relevant Treatment Effect (PRTE) of Heckman and Vytlacil (2001, 2005) is*

$$PRTE_{D_\delta} = \frac{E(Y_{D_\delta}) - E(Y)}{\delta}$$

and taking the limit $\delta \to 0$ yields the Marginal PRTE (MPRTE) of Carneiro, Heckman and Vytlacil (2010, 2011):

$$\mathrm{MPRTE}_{\mathcal{D}} = \lim_{\delta \to 0} PRTE_\delta.$$

Martinez-Iriarte and Sun (2020) show how to generalize the MPRTE to cover the case of Firpo, Fortin and Lemieux (2009) as well.

Remark 3 (Rothe (2012))**.** *Rothe (2012) also studies the global and marginal effects but under a different identifying assumption, namely a form of conditional exogeneity. This assumption also yields an identified set. Let the outcome be $Y = h(D, X, U)$. For uniformly distributed random variables $\tilde{U}_1$ and $\tilde{U}_2$, the outcome can be represented as $Y = h(Q_D(\tilde{U}_1), Q_X(\tilde{U}_2), U)$ where Q_D and Q_X are the quantile functions. Then Q_D is changed to another quantile function Q_D^*, generating a counterfactual distribution, which is identified when $\tilde{U}_1 \perp U \| X$ and D is continuous. When D is discrete, $\tilde{U}_1$ is not uniquely determined, so that a range of possible counterfactual distributions is possible resulting in partial identification.*

2.1 Global Effect: Bounds and Identification Region

The difference between F_{Y_δ} and F_a is called the *policy selection bias*, that we denote $\text{psb}(y)$. Since different policies can induce different newly treated individuals, they can induce different counterfactual distributions. Thus the selection bias is policy dependent, and hence the name "policy selection bias."

$$\begin{aligned}\text{psb}(y) :&= \underbrace{F_{Y(1)|D=0,D_\delta=1}(y)}_{\textit{newly}\text{ treated}} - \underbrace{F_{Y(1)|D=1,D_\delta=1}(y)}_{\textit{already}\text{ treated}} \\ &= F_{Y(1)|D=0,D_\delta=1}(y) - F_{Y(1)|D=1}(y),\end{aligned}$$

where the second line follows from Assumption 2: the subpopulations $\{D = 1, D_\delta = 1\}$ and $\{D = 1\}$ are identical. Under a non-randomized policy, the random variable D_δ is usually a function of D and other observables as in (2). So, even if $D \perp Y(0), Y(1)$, that would lead us to

$$\text{psb}(y) = F_{Y(1)|D=0,D_\delta=1}(y) - F_{Y(1)|D=0}(y),$$

which is not zero unless $D_\delta \perp Y(1)|D = 0$. For example, if we want to choose individuals whose observed outcome is below a certain threshold, then most likely D_δ will be correlated with $Y(1)$ conditional on $D = 0$. Indeed, the fact that $\text{psb}(y)$ is unlikely to be zero seems to be an inevitable feature of the problem of analyzing non-randomized policies.

As a measure of departure from point identification, we will bound the the policy selection bias both from below and from above. For simplicity, we denote by $\mathcal{Y}$ the common support of $Y(1)|D = 0, D_\delta = 1$ and $Y(1)|D = 1, D_\delta = 1$.

Assumption 3 (L-U Bounds)**.** *For any $D_\delta \in \mathcal{D}$, there exists a pair of real numbers $L \in [-1, 0]$ and $U \in [0, 1]$ such that for every $y \in \mathcal{Y}$*

$$L \leq F_{Y(1)|D=0,D_\delta=1}(y) - F_{Y(1)|D=1,D_\delta=1}(y) \leq U.$$

The distribution of *newly* treated individuals is $F_{Y(1)|D=0,D_\delta=1}(y)$, while $F_{Y(1)|D=1,D_\delta=1}(y)$ is the distribution of the *already* treated individuals. A more precise way to define L is as the infimum over the differences $F_{Y(1)|D=0,D_\delta=1}(y) - F_{Y(1)|D=1,D_\delta=1}(y)$, while U is the supremum over such differences.

If $Y(1)$ is higher for *already* treated individuals than for *newly* treated individuals uniformly over $\mathcal{Y}$, then $F_{Y(1)|D=0,D_\delta=1}(y) \geq F_{Y(1)|D=1,D_\delta=1}(y)$, and we can set $L = 0$ and $U \leq 1$. This is a situation in which $F_{Y(1)|D=1,D_\delta=1}(y)$ first-order stochastically dominates $F_{Y(1)|D=0,D_\delta=1}(y)$. In the more general case, where the two distributions cross each other, then $L \in [-1, 0]$ and $U \in [0, 1]$, and we do not necessarily need to have $U = -L$. Finally, setting $L = -1$ and $U = 1$ corresponds to a trivial bounds situation.

Assumption 3 implies via (3) that the discrepancy between the counterfactual distribution

$F_{Y_{D_\delta}}$ and the apparent distribution F_a is further shrunk by a factor of δ:

$$\delta L \leq F_{Y_{D_\delta}}(y) - F_a(y) \leq \delta U. \tag{5}$$

We are now ready to state the main result of this section.

Theorem 2 (Global Effect Bounds)**.** *For a given sequence of policies that satisfies Assumptions 1, 2 and 3, the global effect is bounded by*

$$G_{\tau,D_\delta} \in \left[F_a^{-1}(\tau - \delta U) - F_Y^{-1}(\tau), F_a^{-1}(\tau - \delta L) - F_Y^{-1}(\tau)\right] \tag{6}$$

for any $\tau \in (\delta U, 1 + \delta L)$*.*

The proof is best given by a picture. Figure 2 shows $F_a(y)$ in solid blue, along with the uniform bounds for $F_{Y_{D_\delta}}(y)$ for given values of L and U. For a fixed τ, $F^{-1}_{Y_{D_\delta}}(\tau)$ must lie between the points ℓ and u. The point ℓ satisfies $F_a(\ell) + \delta U = \tau$, from which we obtain $\ell = F_a^{-1}(\tau - \delta U)$. A similar reasoning applied to u yields $u = F_a^{-1}(\tau - \delta L)$. Finally, the bound for the global effect is obtained by subtracting $F_Y^{-1}(\tau)$ from both ℓ and u.

The identified region in (6) is obtained by correcting the evaluation point of the quantile of the apparent distribution: instead of τ, we evaluate the quantile of the apparent distribution at $\tau - \delta U$ and $\tau - \delta L$. The farther away are U and L from zero, where point identification holds, the bigger is the region where the counterfactual distribution can lie. This is reflected in the widening of the identified region.

An important quantity that we will use later on is the *apparent global effect*. This is the estimand that neglects the policy selection bias by setting $L = U = 0$. It is given by

$$G^a_{\tau,D_\delta} := F_a^{-1}(\tau) - F_Y^{-1}(\tau), \tag{7}$$

where the superscript "a" conveys the fact that it captures an apparent effect. Indeed, G^a_{τ,D_δ} proceeds as if the counterfactual distribution $F_{Y_{D_\delta}}$ equals the apparent distribution F_a.

The bounds are "monotone" in L and U as we move away from point identification, that is, when $L = U = 0$. Indeed, as we move in the L direction towards -1, the upper bound $F_a^{-1}(\tau - \delta L) - F_Y^{-1}(\tau)$ increases. As move away from point identification in the U direction towards 1, the lower bound $F_a^{-1}(\tau - \delta U) - F_Y^{-1}(\tau)$ decreases. It is important to recall that L is non-positive, and U is non-negative, so that $F_a^{-1}(\tau - \delta U) \leq F_a^{-1}(\tau - \delta L)$.

Remark 4 (Range of τ)**.** *The requirement* $\tau \in (\delta U, 1 + \delta L)$ *comes from* $\tau - \delta U \geq 0$ *and* $\tau - \delta L \leq 1$*. However, later on, when we fix* τ *and* δ*, we want* U *and* L *to not be restricted,* i.e.*, they both can achieve* 1 *and* -1 *respectively. In order for this to happen, we need* $\delta < \tau < 1 - \delta$*. In the empirical application we work with* $\delta = 0.1$*, so there will not be a significant restriction on the quantiles we can analyze.*

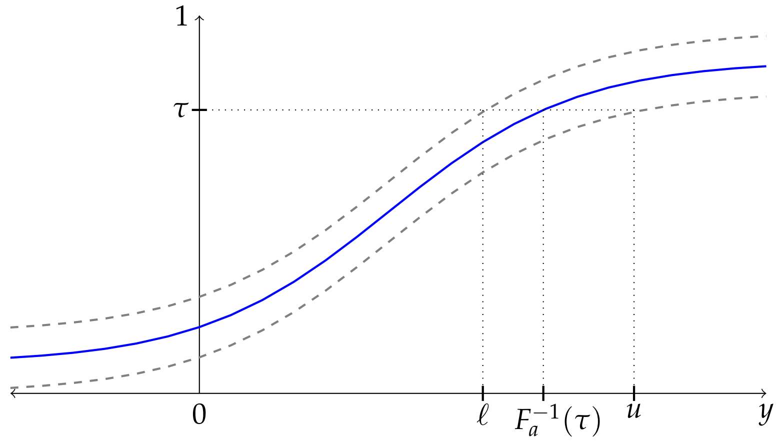


Figure 2: *The solid blue line is $F_a(y)$, while the grey dashed lines are the uniform bounds given by (5). The points ℓ and u are $F_a^{-1}(\tau - \delta U)$ and $F_a^{-1}(\tau - \delta L)$ respectively.*

Remark 5 (Trivial Bounds)**.** *In principle, the global effect need not be restricted. The trivial bounds, $U = 1$ and $L = -1$, provide a bounded region which contains the global effect:*

$$F_a^{-1}(\tau - \delta) - F_Y^{-1}(\tau) \leq G_{\tau,D_\delta} \leq F_a^{-1}(\tau + \delta) - F_Y^{-1}(\tau),$$

so, in the language of Manski (1989, 1990), the trivial bounds are always informative. However, the identified set derived from the trivial bounds always contains 0 for all quantiles. The intuition is that for a given δ, and the common support assumption, we know the counterfactual distributions for a proportion $1 - \delta$ of individuals. Thus, we are able to bound the quantiles. See Apprendix A for a proof.

Remark 6 (c-dependence)**.** *A common way to relax $D \perp Y(1)$ is a version*[5] *of the c-dependence approach of Masten and Poirier (2018) which posits a $c \in [0, 1]$ such that*

$$\sup_{y \in supp(Y(1))} |\Pr(D = 1|Y(1) = y) - \Pr(D = 1)| \leq c. \tag{8}$$

When $c = 0$, then $D \perp Y(1)$. If $c > 0$, then some sort of dependence is allowed between D and $Y(1)$. Alas, this approach would only help us in the case of randomized policies. For non-randomized policies, to achieve point identification, we need an extra conditional independence assumption, namely $D_\delta \perp Y(1)|D = 0$. We could, in addition to the c-dependence condition in (8), impose

$$\sup_{y \in supp(Y(1))} |\Pr(D_\delta = 1|Y(1) = y, D = 0) - \Pr(D_\delta = 1|D = 0)| \leq c^*.$$

for some $c^* \in [0, 1]$. However, the drawback is that the relationship between c and c^* is not at all clear. More importantly, their interpretation is not straightforward either.

[5]We do not follow exactly the definition of c-dependence of Masten and Poirier (2018) which includes covariates.

2.2 Marginal Effect: Bounds and Identification Region

Before we proceed, we will settle the question of existence of the marginal effect. Theorem 3 provides sufficient conditions.

Theorem 3 (Existence of Marginal Effect)**.** *Consider a sequence of policies $\mathcal{D}$ such that*

1. $F_{Y_{D_0}}(y) = F_Y(y)$ *for any* $y \in \mathcal{Y}$*;*
2. *The map* $\delta \mapsto F_{Y_{D_\delta}}(y)$ *is differentiable at* $\delta = 0$ *uniformly in* $y \in \mathcal{Y}$*, with derivative* $\dot{F}_{Y,\mathcal{D}}(y)$*, that is*
$$\lim_{\delta \downarrow 0} \sup_{y \in \mathcal{Y}} \left| \frac{F_{Y_{D_\delta}}(y) - F_Y(y)}{\delta} - \dot{F}_{Y,\mathcal{D}}(y) \right| = 0;$$
3. *The map* $y \mapsto \dot{F}_{Y,\mathcal{D}}(y)$ *is continuous at* $F_Y^{-1}(\tau)$*.*

Then, $M_{\tau,\mathcal{D}}$ *exists and is given by*

$$M_{\tau,\mathcal{D}} = -\frac{\dot{F}_{Y,\mathcal{D}}(F_Y^{-1}(\tau))}{f_Y(F_Y^{-1}(\tau))}.$$

The conditions and the proof of this Theorem come from viewing the marginal effect as a Hadamard derivative. The first condition, $F_{Y_{D_0}}(y) = F_Y(y)$, is particular to this setting, though. A primitive condition for $F_{Y_{D_0}}(y) = F_Y(y)$ is Assumption 2, because it implies the expansion of Theorem 1. Setting $\delta = 0$ in (3) yields $F_{Y_{D_0}}(y) = F_Y(y)$. It states that for $D_0 \in \mathcal{D}$, the limiting counterfactual distribution F_{D_0} matches the observed distribution F_Y. This might not necessarily be the case; see Example B.2 in Appendix B. Indeed, we could define a marginal effect with respect to $F_{Y_{D_0}}$ instead which would avoid the "discontinuity" at $\delta = 0$. However, this would be of limited interest.

The second condition, that of uniform differentiability of the map $\delta \mapsto F_{Y_{D_\delta}}(y)$, is more abstract. To understand what it entails, consider the following rearrangement[6] of equation (3):

$$\frac{F_{Y_{D_\delta}}(y) - F_Y(y)}{\delta} = F_{Y(1)|D=0,D_\delta=1}(y) - F_{Y(0)|D=0,D_\delta=1}(y).$$

The right hand side is the difference in potential outcomes for the *newly treated*. We require this change to be continuous in δ: small departures from 0 to $\delta > 0$ should not induce large (uniform) changes in the counterfactual distribution $F_{Y_{D_\delta}}$. This is automatically satisfied when the sequence of policies are randomized. The next example shows this.

[6] We can write $F_Y(y) = (1-p-\delta)F_{Y(0)|D=0,D_\delta=0}(y) + \delta F_{Y(0)|D=0,D_\delta=1}(y) + pF_{Y(1)|D=1,D_\delta=1}(y)$, and $F_Y(y) = (1-p-\delta)F_{Y(0)|D=0,D_\delta=0}(y) + \delta F_{Y(0)|D=0,D_\delta=1}(y) + pF_{Y(1)|D=1,D_\delta=1}(y)$. Subtracting $F_Y(y)$ to $F_{Y_{D_\delta}}(y)$ we get

$$F_{Y_{D_\delta}}(y) - F_Y(y) = \delta \left(F_{Y(1)|D=0,D_\delta=1}(y) - F_{Y(0)|D=0,D_\delta=1}(y) \right).$$

Example 3 (Marginal Effect of Randomized Policy)**.** *For the case of a randomized policy that satisfies Assumption 2, by (1) we can simplify the counterfactual distribution in (3) to*

$$F_{Y_{D_\delta}}(y) = F_Y(y) + \delta\left[F_{Y(1)|D=0}(y) - F_{Y(0)|D=0}(y)\right],$$

which implies that $\dot{F}_{Y,\mathcal{D}}(y) = F_{Y(1)|D=0}(y) - F_{Y(0)|D=0}(y)$. We obtain that $\dot{F}_{Y,\mathcal{D}}$ is independent of $\mathcal{D}$.

When $\dot{F}_{Y,\mathcal{D}}$ is independent of $\mathcal{D}$, two different randomized policies $\mathcal{D}$ and $\mathcal{D}'$ will deliver the same marginal effect. But at the same time, at the population level, there can only be one randomized policy. This result reflects precisely that. On the other hand, for non-randomized policies, the marginal effect can easily be sequence dependent. Appendix B contains more examples from the literature that show that the uniform differentiability of $\delta \mapsto F_{Y_{D_\delta}}(y)$ might be a non-trivial requirement.

For a randomized policy, $\dot{F}_{Y,\mathcal{D}}$ is well-defined, though not necessarily identified since it involves $F_{Y(1)|D=0}$. For this reason, when analyzing the marginal effect, we will focus on randomized policies. Thus, we will write M_τ instead of $M_{\tau,\mathcal{D}}$.

The bounds for the marginal effect will be obtained as the limiting bounds, as δ goes to 0, for the global effect under a randomized policy. That is, the bounds for M_τ will be given by

$$\lim_{\delta\to 0}\frac{F_a^{-1}(\tau - \delta U) - F_Y^{-1}(\tau)}{\delta},$$

and

$$\lim_{\delta\to 0}\frac{F_a^{-1}(\tau - \delta L) - F_Y^{-1}(\tau)}{\delta},$$

provided these limits exist.

These limits can be seen as derivatives with respect to δ of $\delta \mapsto F_a^{-1}(\tau - \delta L)$ at $\delta = 0$. There is a minor complication which makes the computation a bit more involved. The reason is that δ plays a dual role in the map $\delta \mapsto F_a^{-1}(\tau - \delta L)$: first, it enters in the argument of $F_a^{-1}(\tau - \delta L)$; second it is used in the construction of the apparent distribution $F_a := (1 - p - \delta)F_{Y|D=0} + (p + \delta)F_{Y|D=1}$ (see (4)). We resort to the chain rule and treat each case separately. The first case can be solved as an ordinary derivative of the inverse of a function, while the second case takes advantage of the Hadamard differentiability of the function $\delta \mapsto F_a$, which maps a scalar into the space of right-continuous functions with left limits, composed with the function $F_a \mapsto F_a^{-1}(\tau)$ which maps an increasing right-continuous function with left limits into the real numbers. The details can be found in Appendix A. Heuristically, we have

$$\lim_{\delta\to 0}\frac{F_a^{-1}(\tau - \delta U) - F_Y^{-1}(\tau)}{\delta} = \lim_{\delta\to 0}\frac{F_Y^{-1}(\tau - \delta U) - F_Y^{-1}(\tau)}{\delta} + \lim_{\delta\to 0}\frac{F_a^{-1}(\tau) - F_Y^{-1}(\tau)}{\delta},$$

where the first term can be dealt with the inverse function theorem, and the second term with a

Hadamard derivative to account how the function F_a^{-1} moves when we move δ.

Theorem 4 (Marginal Effect Bounds)**.** *For a sequence of randomized policies that satisfies Assumptions 1, 2 and 3 the marginal effect is bounded by*

$$-\frac{U}{f_Y(F_Y^{-1}(\tau))} \leq M_\tau - M_\tau^a \leq -\frac{L}{f_Y(F_Y^{-1}(\tau))} \tag{9}$$

for any $\tau \in (0,1)$, where

$$M_\tau^a := -\frac{F_{Y|D=1}(F_Y^{-1}(\tau)) - F_{Y|D=0}(F_Y^{-1}(\tau))}{f_Y(F_Y^{-1}(\tau))} \tag{10}$$

is the apparent effect.

The apparent effect in (10) is the estimand of Firpo, Fortin and Lemieux (2009). Hence, Theorem 4 states that the usual estimand should be enlarged by $-\frac{U}{f_Y(F_Y^{-1}(\tau))}$ and $-\frac{L}{f_Y(F_Y^{-1}(\tau))}$ in order to contain M_τ. Recall that L is non-positive, and U is non-negative. As opposed to the bounds on the global effect, the result in Theorem 4 holds for any $\tau \in (0,1)$. However, there is not much to gain from this because as τ approaches 0 or 1, the density $f_Y(F_Y^{-1}(\tau))$ is likely to approach zero and the bounds will diverge to $+\infty$ or $-\infty$.

Remark 7 (Trivial Bounds)**.** *Setting $L = -1$ and $U = 1$ corresponds to a trivial bounds case. It is a matter of simple algebra to show that 0 will always be in the identified set in this case. For example, if $M_\tau^a \geq 0$, then $0 \in [M_\tau^a - 1/f_Y(F_Y^{-1}(\tau)), M_\tau^a + 1/f_Y(F_Y^{-1}(\tau))]$. As in the case with the global effect, the boundedness of the outcome is not needed for the trivial bounds to be informative.*

3 Quantile Breakdown Frontier

In our framework, the amount of policy selection bias is controlled by L and U. Figure 3 shows this in the $L \times U$ plane. When $L = U = 0$, there is no policy selection bias, and hence we achieve point identification. Any other value of $L \in [-1,0)$ and $U \in (0,1]$ admits some policy selection bias, and consequently the effects are only partially identified. A special case of this are the trivial bounds: when $L = -1$ and $U = 1$. We refer to any combination (L,U) distinct from $(0,0)$ as a *departure from point identification*.

The following language convention is important. Because L is always non-positive, we say that we have *more* policy selection bias (due to L) in the point $(L,U) = (-1,u)$ than in the point $(L,U) = (-0.5,u)$, even though L is bigger in the latter, -0.5, than in the former, -1. Thus, we quantify the selection as how far we move (L,U) from $(0,0)$, rather than by the value of L or U.

The quantile breakdown frontier is a curve that quantifies the amount of policy selection bias compatible with a given conclusion of interest across quantiles. Suppose we are interested in a certain policy D_δ, and we would like to know if its global effect on the median of Y is positive.

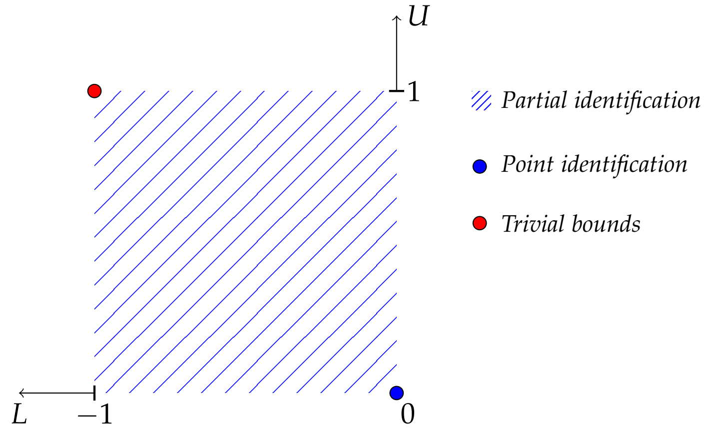


Figure 3: *Only* $L = U = 0$ *delivers point identification.*

That is, we want to know whether $G_{.5,D_\delta} > 0$ or not. If we were certain that there is no policy selection bias, we would just estimate the apparent effect $G^a_{.5,D_\delta}$ using (7):

$$G^a_{.5,D_\delta} = F_a^{-1}(.5) - F_Y^{-1}(.5).$$

However, it is very likely that the apparent effect $G^a_{.5,D_\delta}$ is biased for the true global effect $G_{.5,D_\delta}$. Sensitivity analysis, in a sense, asks the reverse question: how much policy selection bias is compatible with $G_{.5,D_\delta} > 0$? The quantile breakdown frontier answers this question by indicating the amount of departure from point identification such that the conclusion holds.

In order to answer the question posed by sensitivity analysis, we recall Theorem 2 which states that there are L and U such that

$$F_a^{-1}(.5 - \delta U) - F_Y^{-1}(.5) \leq G_{.5,D_\delta} \leq F_a^{-1}(.5 - \delta L) - F_Y^{-1}(.5). \tag{11}$$

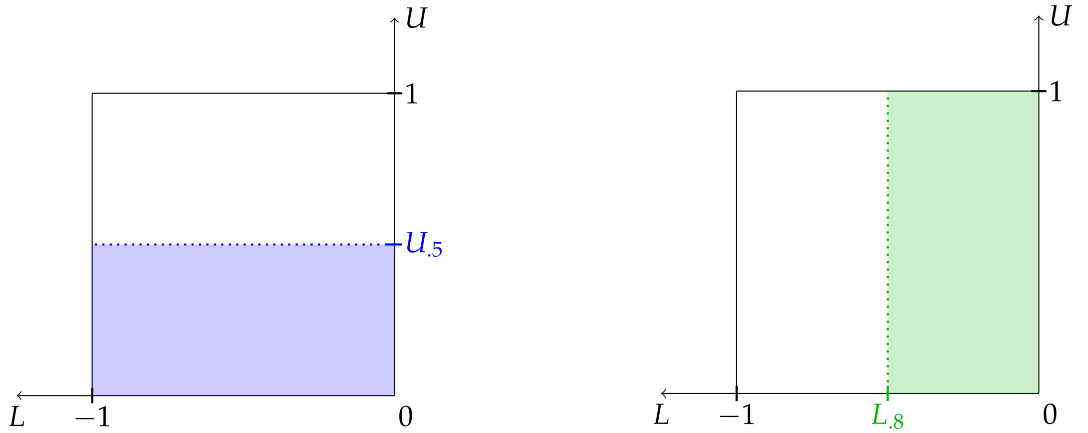
Hence, for $G_{.5,D_\delta} > 0$ to hold, we need that the lower bound in (11) be greater than zero. That is, we need all the values of U such that

$$0 < F_a^{-1}(.5 - \delta U) - F_Y^{-1}(.5) \leq G_{.5,D_\delta} \tag{12}$$

First, we note that $F_a^{-1}(.5 - \delta U)$ is decreasing in U. Suppose $G_{.5,D_\delta}$>0. We start with $U = 0$ and then move it towards 1. On the other hand, L is left unrestricted. So, all the values of U such that (12) holds and any value of $L \in [-1, 0]$ are compatible with $G_{.5,D_\delta} > 0$. In particular, let $U_{.5}$ be the value of U such that the lower bound in (12) is equal to zero: $0 = F_a^{-1}(.5 - \delta U_{.5}) - F_Y^{-1}(.5)$. Thus, the combination of L and U compatible with $G_{.5,D_\delta} > 0$ are

$$\{(L, U) : -1 \leq L \leq 0 \text{ and } 0 \leq U < U_{.5}\} \tag{13}$$

A value of U greater than $U_{.5}$ induces too much bias and fails to guarantee that $G_{.5,D_\delta} > 0$.



(a) *Compatible values of L and U for $G_{.5,D_\delta} > 0$.*

(b) *Compatible values of L and U for $G_{.8,D_\delta} < 0$.*

Figure 4: *Compatible values for $G_{.5,D_\delta} > 0$ and $G_{.8,D_\delta} < 0$.*

Figure 4a shows the compatible values of L and U in the $L \times U$ plane.

Now suppose we are also interested in the 80th quantile. However, it may be the case that there is no value of U such that

$$0 < F_a^{-1}(.8 - \delta U) - F_Y^{-1}(.8).$$

holds. That is, for any value of U,

$$F_a^{-1}(.8 - \delta U) - F_Y^{-1}(.8) \leq 0,$$

or equivalently $G^a_{.8,D_\delta} \leq 0$, thus, no combination of L and U can *guarantee* that $G_{.8,D_\delta} > 0$ holds. Therefore, we look at the reverse conclusion $G_{.8,D_\delta} < 0$, and find all the values of L such that

$$G_{.8,D_\delta} \leq F_a^{-1}(.8 - \delta L) - F_Y^{-1}(.8) < 0.$$

We denote by $L_{.8}$ the value of L that solves: $F_a^{-1}(.8 - \delta L) - F_Y^{-1}(.8) = 0$. The values of L and U such that $G_{.8,D_\delta} < 0$ are

$$\{(L, U) : L_{.8} < L \leq 0 \text{ and } 0 \leq U \leq 1\}, \tag{14}$$

and are shown in Figure 4b.

The extension of this procedure to more than two quantiles gives rise to the quantile breakdown frontier. For a collection of conclusions indexed by $\tau \in (\delta, 1 - \delta)$,[7] for example $G_{\tau,D_\delta} > g_\tau$, the quantile breakdown frontier shows the combinations of L and U compatible with each conclusion.

Figure 5 contains an hypothetical quantile breakdown frontier constructed for all $\tau \in (\delta, 1 -$

[7]See Remark 4 for an explanation of this restriction.

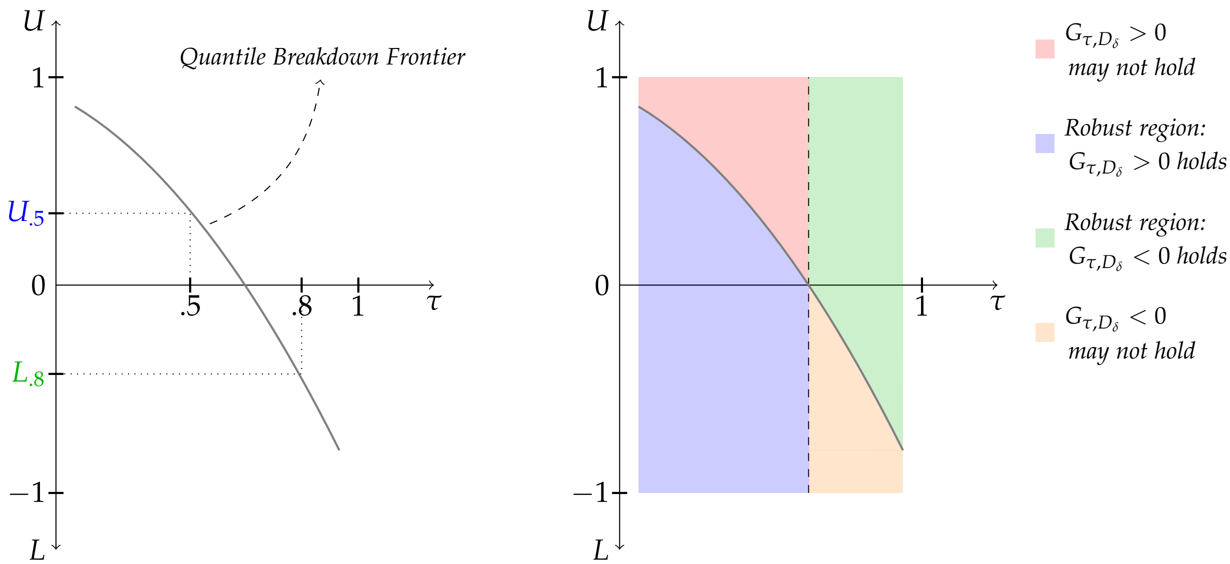
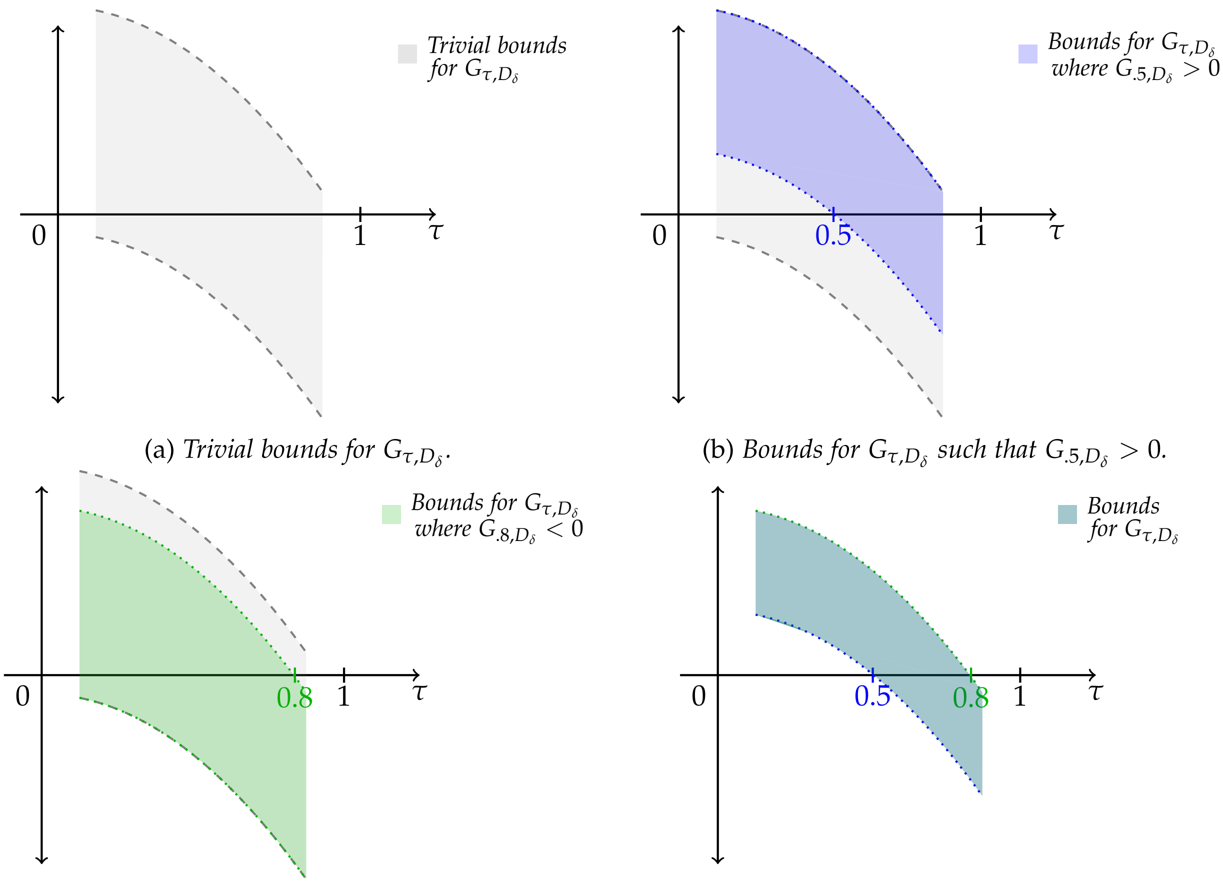


Figure 5: *Quantile Breakdown Frontier.*

δ). On the left side, at $\tau = .5$, we can see that below the curve we have the region described in (13) under which $G_{.5,D_\delta} > 0$ holds. At $\tau = .8$, we have that *above* the curve we have the region described in (14) where $G_{.8,D_\delta} < 0$ holds. The right hand side shows this for all the quantiles in $(\delta, 1-\delta)$. Values of U in the red area include possible negative values of the global effect. The green area is the counterpart of the blue area: a robust region for $G_{\tau,D_\delta} < 0$. Finally, the orange area is the counterpart of the red area: values of L such that the global effect might be positive.

Consider again the left panel in Figure 5. We can use the values $L_{.8}$ and $U_{.5}$ to construct bounds for the global effect curve: $\tau \mapsto G_{\tau,D_\delta}$. These bounds have the property that at $\tau = 0.5$, the identified region for the global effect is positive, while at $\tau = 0.8$, the identified region for the global effect is negative. Moreover, the identified region for the global effect derived from $L_{.8}$ and $U_{.5}$ will provide statements about the global effect at other quantiles as well. This can be seen in Figure 6. The solid line in Panel (a) shows the trivial bounds. These are obtained by setting $L = -1$ and $U = 1$ in (6). Note how the identified region of the trivial bounds contains 0 for all the quantiles, in line with Remark 5. Panel (b) shows the restriction on U such that $G_{.5,D_\delta} > 0$: the lower bounds is tightened and crosses 0 at $\tau = 0.5$. Similarly, panel (c) tightens the upper bound consistent with restricting L to be $L_{.8}$ in order for $G_{.8,D_\delta} < 0$ to hold. Note how the upper bound now crosses 0 at $\tau = 0.8$. Panel (d) gives simultaneous bounds for the global effect such that $G_{.5,D_\delta} > 0$ and $G_{.8,D_\delta} < 0$. The interpretation of the grey shaded area in Panel (d) of Figure 6 is the following: the global effect curve has to lie in the gray area in order for the conclusions $G_{.5,D_\delta} > 0$ and $G_{.8,D_\delta} < 0$ to hold.

One of the building blocks for the construction of the quantile breakdown frontier is the breakdown frontier of Masten and Poirier (2020). Figure 4 shows two examples a breakdown frontier. The quantile breakdown frontier takes advantage of the fact that the frontiers in Figures 4 are straight lines. This simplicity allows us to plot the higher dimensional quantile breakdown



(a) *Trivial bounds for* G_{τ,D_δ}.

(b) *Bounds for* G_{τ,D_δ} *such that* $G_{.5,D_\delta} > 0$.

(c) *Bounds for* G_{τ,D_δ} *such that* $G_{.8,D_\delta} < 0$.

(d) *Bounds for* G_{τ,D_δ} *with* $G_{.5,D_\delta} > 0$ *and* $G_{.8,D_\delta} < 0$.

Figure 6: *Bounds on the global effect.*

frontier in a plane as in Figure 5.

In the rest of this section we will derive analytical expressions for L_τ, U_τ, the quantile breakdown frontier, and the bounds on the global effect. We will also derive the quantile breakdown frontier for the sign of the marginal effect.

3.1 Global Effect

Suppose that, for a given τ and D_δ, we are interested in the global effect. By Theorem 2, there are L and U such that

$$F_a^{-1}(\tau - \delta U) - F_Y^{-1}(\tau) \le G_{\tau,D_\delta} \le F_a^{-1}(\tau - \delta L) - F_Y^{-1}(\tau).$$

In order not to impose restrictions of L and U, we will focus on $\tau \in (\delta, 1-\delta)$ (See Remark 4). We further recall that the bounds are "centered" around $G^a_{\tau,D_\delta} := F_a^{-1}(\tau) - F_Y^{-1}(\tau)$. For a given τ we are interested in the values of L and U such that either $G_{\tau,D_\delta} > g_\tau$ or $G_{\tau,D_\delta} < g_\tau$ holds. In order to build the breakdown frontier we must look at the location of G^a_{τ,D_δ} with respect to g_τ.

Figure 7 illustrates the case of $G^a_{\tau,D_\delta} > g_\tau$. The blue part of the axis shows the possible values of G_{τ,D_δ}. The dashed lines show three different combination of L and U. The two blue dashed lines allow us to conclude that the effect is greater than g_τ. The red dashed line include values lower than g_τ, and hence it is excluded. Since only the lower bounds concern us, this means that we do not want U to get too close to 1. Thus there is a maximum departure from point identification due to U that ensures that $G_{\tau,D_\delta} > g_\tau$ holds in the case where $G^a_{\tau,D_\delta} > g_\tau$.

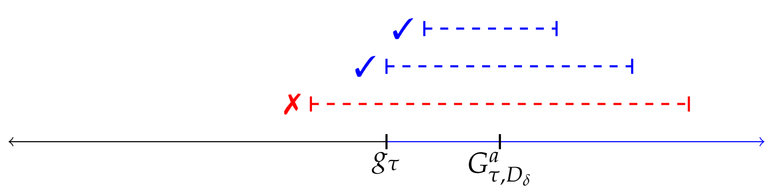


Figure 7: *The red segment includes values of $G_{\tau,D_\delta} < g_\tau$.*

This maximum U is denoted by U_τ, and it solves (see middle dashed line in Figure 7)

$$F_a^{-1}(\tau - \delta U_\tau) - F_Y^{-1}(\tau) = g_\tau.$$

which implies that

$$U_\tau = \min\left\{\max\left\{0, \frac{\tau - F_a(F_Y^{-1}(\tau) + g_\tau)}{\delta}\right\}, 1\right\}. \tag{15}$$

Figure 8 shows the other possibility, which is $G^a_{\tau,D_\delta} < g_\tau$. In this case we can analyze conclusions of the form $G_{\tau,D_\delta} < g_\tau$. As we move L towards -1, the right end of the identified regions

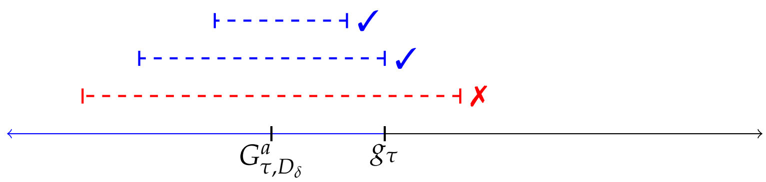


Figure 8: *The red segment includes values of $G_{\tau,D_\delta} > g_\tau$*

approaches g_τ. In the red segment, L is too close to -1, so the identified region contains values contrary to the conclusion. So, it is excluded.

In this case, we have

$$L_\tau = \max\left\{\min\left\{0, \frac{\tau - F_a(F_Y^{-1}(\tau) + g_\tau)}{\delta}\right\}, -1\right\}. \tag{16}$$

The common ingredient for U_τ in (15) and L_τ in (16) is

$$\theta(\tau) = \frac{\tau - F_a(F_Y^{-1}(\tau) + g_\tau)}{\delta}. \tag{17}$$

The map $\tau \mapsto \theta(\tau)$ is the quantile breakdown frontier. Alternatively, for a given τ, the quantile breakdown frontier is value of the policy selection bias such that the global effect $G_{\tau,D_\delta} = g_\tau$. If this value is positive, it is taken to be U, if it is negative, it is taken to be L.

Continuity of the quantile breakdown frontier is important for inference purposes. Inspection of the formulas in (17) shows that continuity of F_a, F_Y^{-1}, and of the map $\tau \mapsto g_\tau$ is enough. Continuity of F_a and F_Y^{-1} is true by assumption, but continuity of $\tau \mapsto g_\tau$ is up to the user. In our empirical application we will choose a g_τ which is constant-across-τ. An arbitrary collection of $\{g_\tau : \tau \in (\delta, 1-\delta)\}$ might be problematic.

Summarizing, we follow the following steps. First, we need to fix the set of quantiles τ in which we are interested and compute the quantile breakdown frontier for a given collection of g_τ. Then, we have to check the sign of the quantile breakdown frontier at these τ's. If the quantile breakdown frontier is positive, we can derive the values of U such that positive conclusions hold: $G_{\tau,D_\delta} > g_\tau$. If the quantile frontier is negative, we can derive the values of L such that the negative conclusions hold: $G_{\tau,D_\delta} < g_\tau$.

3.2 Bounds derived from the QBF

Often times, researchers are interested in quantile contrasts: for example Farber et al. (2020) examine 10^{th} vs. 90^{th} of a marginal increase in unionization. In this case, following Masten and Poirier (2020) we can visualize the result in a joint breakdown frontier/robust region. This is shown in Figure 9. The intersection contains the values of L and U compatible with both conclusions of interest.

We can use the quantile breakdown frontier to derive bounds on the global effect for every

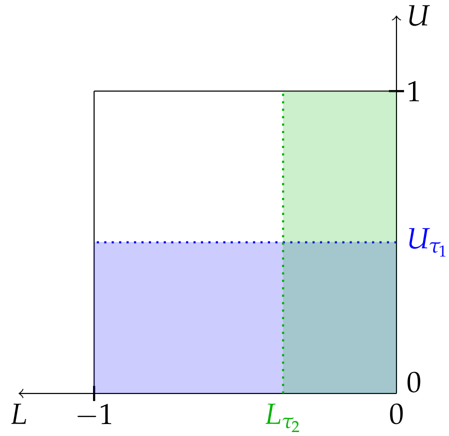


Figure 9: *Joint Breakdown Frontier.*

$\tau \in (\delta, 1-\delta)$:

$$\tau \mapsto G_{\tau, D_\delta}.$$

To do so, we find τ_1 and τ_2 such that we can analyze $G_{\tau_1} > 0$ and $G_{\tau_2} < 0$. That is, the quantile breakdown is positive at τ_1 and negative at τ_2.[8] Following Theorem 2, we can use U_{τ_1} to construct a lower bound for the global effect, and L_{τ_2} to construct an upper bound for the global effect. These bounds are given by

$$\tau \mapsto B(U_{\tau_1}; \tau) := F_a^{-1}(\tau - \delta U_{\tau_1}) - F_Y^{-1}(\tau), \tag{18}$$

and

$$\tau \mapsto B(L_{\tau_2}; \tau) := F_a^{-1}(\tau - \delta L_{\tau_2}) - F_Y^{-1}(\tau). \tag{19}$$

and are shown in Figure 6.

3.3 Marginal Effect

For the marginal effect the situation is a bit more delicate, and some care must be exercised with the density in the denominator. By Theorem 4, the identified region for M_τ is

$$\left[M_\tau^a - \frac{U}{f_Y(F_Y^{-1}(\tau))}, M_\tau^a - \frac{L}{f_Y(F_Y^{-1}(\tau))} \right]$$

[8] The empirical quantile breakdown frontier might be negative or positive everywhere. In that case this analysis would not apply. However, in our empirical analysis, the quantile breakdown is positive in a region, and negative in another region.

where recall that

$$M_\tau^a := -\frac{F_{Y|D=1}(F_Y^{-1}(\tau)) - F_{Y|D=0}(F_Y^{-1}(\tau))}{f_Y(F_Y^{-1}(\tau))}.$$

Consider a single quantile τ. For the conclusion $M_\tau > g_\tau$ to hold, then, as before, the restriction on U, denoted by U_τ, solves

$$-\frac{F_{Y|D=1}(F_Y^{-1}(\tau)) - F_{Y|D=0}(F_Y^{-1}(\tau))}{f_Y(F_Y^{-1}(\tau))} - \frac{U_\tau}{f_Y(F_Y^{-1}(\tau))} = g_\tau$$

which implies

$$U_\tau = \min\left\{1, \max\left\{0, F_{Y|D=0}(F_Y^{-1}(\tau)) - F_{Y|D=1}(F_Y^{-1}(\tau)) - g_\tau f_Y(F_Y^{-1}(\tau))\right\}\right\}.$$

For the opposite conclusion, $M_\tau < g_\tau$, similar calculations, this time on the upper bound, yield

$$L_\tau = \max\left\{\min\left\{0, F_{Y|D=0}(F_Y^{-1}(\tau)) - F_{Y|D=1}(F_Y^{-1}(\tau)) - g_\tau f_Y(F_Y^{-1}(\tau))\right\}, -1\right\}.$$

When it comes to estimation, the quantile breakdown frontier contains a non-parametric ingredient, namely the density f_Y evaluated at a quantity that must estimated: $F_Y^{-1}(\tau)$. This can be avoided if we set $g_\tau = 0$ for every τ. In such a case, we are interested in the sign of the marginal effect. This is natural conclusion to be interested in since the marginal effect has the interpretation of a derivative. When $g_\tau = 0$, these expressions simplify to

$$U_\tau = \min\left\{\max\left\{0, F_{Y|D=0}(F_Y^{-1}(\tau)) - F_{Y|D=1}(F_Y^{-1}(\tau))\right\}, 1\right\},$$

and

$$L_\tau = \max\left\{\min\left\{0, F_{Y|D=0}(F_Y^{-1}(\tau)) - F_{Y|D=1}(F_Y^{-1}(\tau))\right\}, -1\right\}.$$

The quantile breakdown frontier for the sign of marginal effect is then given by

$$\tau \mapsto \theta(\tau) = F_{Y|D=0}(F_Y^{-1}(\tau)) - F_{Y|D=1}(F_Y^{-1}(\tau)). \tag{20}$$

Remark 8. *Coincidentally, in this case where $g = 0$ for every τ, the quantile breakdown frontiers for the global and the marginal effects coincide. This reflects the fact that the apparent marginal effect and the apparent global effects have the same sign. Of course, the true effects might differ in sign. To see this, we note that the apparent distribution can be written as*[9] $F_a(y) = F(y) + \delta\left[F_{Y|D=1,D_\delta=1}(y) - F_{Y|D=0,D_\delta=1}(y)\right]$. *So, that for $g = 0$, and plugging the previous expression for $F_a(y)$ in (17), we obtain (20).*

[9] See the proof for the statement of Remark 5.

4 Estimation and Inference

There are two main results in the sensitivity analysis we propose. The first one is the quantile breakdown frontier $\tau \mapsto \theta(\tau)$. The second important result is the case when we use the estimated values of U_{τ_1} and L_{τ_2} to construct bounds for the effect across all quantiles in the manner of Figure 6. We will provide asymptotic results both pointwise, for a given τ, and uniform, when the objects are seen as a random function.

We work in the space $\ell^\infty(\delta, 1-\delta)$ of bounded real-valued functions defined on $(\delta, 1-\delta)$. As usual, we endow this space with the supremum norm: $\|x\|_\infty := \sup_{t\in(\delta,1-\delta)} |x(t)|$. The reason we restrict the space to be $\ell^\infty(\delta, 1-\delta)$ and not $\ell^\infty(0,1)$ is due to the fact that for a given δ, we cannot reach quantiles below δ or above $1-\delta$. See Remark 4 above.

In order to simplify notation, and ensure the continuity of the quantile breakdown frontier, we are going to focus on the case where the threshold g_τ is constant across τ.

Assumption 4 (Constant Threshold)**.** *For some scalar g, the threshold g_τ satisfies $g_\tau = g$ for any $\tau \in (\delta, 1-\delta)$.*

This assumption can be relaxed at the expense of more complicated notation. However, we still require smoothness in the map $\tau \mapsto g_\tau$. For the case of the quantile breakdown for the sign of the marginal effect, we will set $g = 0$.

4.1 Quantile Breakdown Frontier: Global Effect

Under Assumption 4, the quantile breakdown frontier is

$$\theta(\tau) := \frac{\tau - F_a(F_Y^{-1}(\tau) + g)}{\delta}.$$

The empirical apparent distribution is $\hat{F}_a(y) = (1-\hat{p}-\delta)\hat{F}_{Y|D=0,D_\delta=0}(y) + (\hat{p}+\delta)\hat{F}_{Y|D=1,D_\delta=1}(y)$, where $\hat{p} := n^{-1}\sum_{i=1}^n D_i$, and

$$\hat{F}_{Y|D=0,D_\delta=0}(y) := \frac{\sum_{i=1}^n \mathbb{1}\{Y_i \le y\}(1-D_i)(1-D_{\delta,i})}{\sum_{i=1}^n (1-D_i)(1-D_{\delta,i})},$$
$$\hat{F}_{Y|D=1,D_\delta=1}(y) := \frac{\sum_{i=1}^n \mathbb{1}\{Y_i \le y\} D_i D_{\delta,i}}{\sum_{i=1}^n D_i D_{\delta,i}}.$$

The empirical quantiles $\hat{F}_Y^{-1}$, are computed using the generalized inverse: $\hat{F}_Y^{-1}(\tau) := \inf\{y : \hat{F}_Y(y) \ge \tau\}$. Here, $\hat{F}_Y(y)$ is the empirical CDF: $\hat{F}_Y(y) := n^{-1}\sum_{i=1}^n \mathbb{1}\{Y_i \le y\}$. For given τ, g and δ, the estimated counterpart of $\theta(\tau)$ is then

$$\hat{\theta}(\tau) := \frac{\tau - \hat{F}_a(\hat{F}_Y^{-1}(\tau) + g)}{\delta}. \tag{21}$$

We can view the map $\tau \mapsto \hat{\theta}(\tau)$ as a random element of $\ell^\infty(\delta, 1-\delta)$. In that case, we denote it simply by $\hat{\theta}$. We want to investigate the weak convergence of $\sqrt{n}(\hat{\theta}-\theta)$ in $\ell^\infty(\delta, 1-\delta)$:

$$\sqrt{n}(\hat{\theta}-\theta) = -\frac{1}{\delta}\sqrt{n}\left(\hat{F}_a \circ (\hat{F}_Y^{-1}+g) - F_a \circ (F_Y^{-1}+g)\right).$$

This is similar to a quantile-quantile transformation (see Exercise 4 in Chapter 3.9 in van der Vaart and Wellner (1996)). We base our proof of the asymptotic distribution of $\sqrt{n}(\hat{\theta}-\theta)$ on the proof of Lemma A.1 in Beare and Shi (2019).[10] The main assumption is

Assumption 5 (Functional CLT)**.** *The following multivariate functional central limit theorem holds*

$$\sqrt{n}\begin{pmatrix} \hat{F}_Y - F_Y \\ \hat{F}_{Y|D=0,D_\delta=0} - F_{Y|D=0,D_\delta=0} \\ \hat{F}_{Y|D=1,D_\delta=1} - F_{Y|D=1,D_\delta=1} \\ \hat{p}-p \end{pmatrix} \rightsquigarrow \begin{pmatrix} \mathbb{G}_Y \\ \mathbb{G}_{0,0} \\ \mathbb{G}_{1,1} \\ \mathbb{Z}_p \end{pmatrix},$$

where $\mathbb{G}_Y$, $\mathbb{G}_{0,0}$, and $\mathbb{G}_{1,1}$ are Brownian bridges in $\ell^\infty(\mathcal{Y})$, and $\mathbb{Z}_p$ is a (real-valued) normal random variable.

The following assumption is needed to establish the Hadamard differentiable of different functions used in the construction of θ.

Assumption 6 (Conditions for Hadamard Differentiability)**.**

1. *For some $\varepsilon > 0$, F_Y is continuously differentiable in $[F_Y^{-1}(\delta)-\varepsilon, F_Y^{-1}(1-\delta)+\varepsilon] \subset \mathcal{Y}$ with strictly positive derivative f_Y.*

2. *The distribution functions $F_{Y|D=0,D_\delta=0}(y)$ and $F_{Y|D=1,D_\delta=1}(y)$ are differentiable, with uniformly continuous and bounded derivatives on their support $\mathcal{Y}$.*

The first item in Assumption 6 concerns the support $\mathcal{Y}$ and the smoothness of F_Y. It is used to guarantee the Hadamard differentiability of the quantile process $\tau \mapsto F_Y^{-1}(\tau)$ for $\tau \in (\delta, 1-\delta)$. The second item ensures that the apparent distribution $F_a(y)$ has a uniformly continuous and bounded derivative. This derivative is denoted by $f_a(y)$. It is needed to establish the Hadamard differentiability of the composition map $(F_a, F_Y^{-1}) \mapsto F_a \circ (F_Y^{-1}+g)$.[11]

Theorem 5 (Asymptotic Distribution of QBF)**.** *Under Assumptions 4, 5, and 6*

$$\sqrt{n}(\hat{F}_a - F_a) \rightsquigarrow \mathbb{G}_a := (1-\delta)\mathbb{G}_{0,0} + \delta\mathbb{G}_{1,1} + (F_{Y|D=1,D_\delta=1} - F_{Y|D=0,D_\delta=0})\mathbb{Z}_p,$$

where $\mathbb{G}_a$ is a Gaussian tight element of $\ell^\infty(\mathcal{Y})$, and

$$\sqrt{n}(\hat{\theta}-\theta) \rightsquigarrow \mathbb{G}_\theta := -\frac{1}{\delta}\mathbb{G}_a \circ (F_Y^{-1}+g) + \frac{1}{\delta} f_a \circ (F_Y^{-1}+g)\frac{\mathbb{G}_Y \circ F_Y^{-1}}{f_Y \circ F_Y^{-1}},$$

[10] Beare and Shi (2019) also offer some interesting historical context for the result.
[11] Section 3.9 in van der Vaart and Wellner (1996) studies the Hadamard differentiability of composition maps.

where $\mathbb{G}_\theta$ is Gaussian tight element of $\ell^\infty(\delta, 1-\delta)$.

The second convergence result of Theorem 5 is uniform in $\tau \in (\delta, 1-\delta)$. If we are interested in a particular quantile τ, we can evaluate $\sqrt{n}(\hat{\theta} - \theta)$ at τ to obtain

$$\sqrt{n}(\hat{\theta}(\tau) - \theta(\tau)) \rightsquigarrow \mathbb{G}_\theta(\tau) = -\frac{1}{\delta}\mathbb{G}_a \circ (F_Y^{-1}(\tau) + g) + \frac{1}{\delta} f_a \circ (F_Y^{-1}(\tau) + g)\frac{\mathbb{G}_Y \circ F_Y^{-1}(\tau)}{f_Y \circ F_Y^{-1}(\tau)}.$$

Instead of providing a closed form expression and a consistent estimator for the variance of $\mathbb{G}_\theta(\tau)$, we note that, by Theorem 23.9 in van der Vaart (1998), the empirical bootstrap is valid. Confidence intervals for $\theta(\tau)$ can be constructed in the following way:

Algorithm 1 (Bootstrap for $\theta(\tau)$)**.**

1. *Given the data $\{Y_i, D_i, D_{\delta,i}\}_{i=1}^n$ and a value $\tau \in (\delta, 1-\delta)$, compute $\hat{\theta}(\tau)$ as in (21).*
2. *Obtain B bootstrap samples of size n from $\{Y_i, D_i, D_{\delta,i}\}_{i=1}^n$, and compute $\sqrt{n}(\hat{\theta}^b(\tau) - \hat{\theta}(\tau))$, where $\hat{\theta}^b(\tau)$ is computed as in (21) for $b = 1, \ldots, B$.*
3. *Obtain the $(100 \times \alpha/2)\%$ and $(100 \times (1-\alpha/2))\%$ percentiles of $\left\{\sqrt{n}(\hat{\theta}^b(\tau) - \hat{\theta}(\tau))\right\}_{b=1}^B$. These are denoted $\xi_{\alpha/2,\theta(\tau)}$ and $\xi_{1-\alpha/2,\theta(\tau)}$.*

The $1-\alpha$ confidence intervals are then computed as

$$\mathcal{CI}(\theta(\tau), \alpha) = \left[\hat{\theta}(\tau) - \frac{\xi_{1-\alpha/2,\theta(\tau)}}{\sqrt{n}}, \hat{\theta}(\tau) - \frac{\xi_{\alpha/2,\theta(\tau)}}{\sqrt{n}}\right].$$

It is also possible to construct uniform confidence bands for $\tau \in (\delta, 1-\delta)$. In this case, we look for the smallest scalar c such that, under the bootstrap probability measure,

$$\Pr{}^*\left(\sup_{\tau \in (\delta, 1-\delta)} \left|\sqrt{n}(\hat{\theta}(\tau)^* - \hat{\theta}(\tau))\right| \leq c \,\middle|\, \{Y_i, D_i, D_{\delta,i}\}_{i=1}^n\right) \geq 1-\alpha.$$

The unknown scalar c can be obtained by the simulation procedure outlined below.

Algorithm 2 (Bootstrap for θ)**.**

1. *Given the data $\{Y_i, D_i, D_{\delta,i}\}_{i=1}^n$ and a grid of values $\{\tau_k\}_{k=1}^K \subset (\delta, 1-\delta)$, compute $\hat{\theta}(\tau_k)$ as in (21) for each $k = 1, \ldots, K$.*
2. *Obtain B bootstrap samples of size n from $\{Y_i, D_i, D_{\delta,i}\}_{i=1}^n$, and compute $\max_{k=1,\ldots,K} \left|\sqrt{n}(\hat{\theta}^b(\tau_k) - \hat{\theta}(\tau_k))\right|$, where $\hat{\theta}^b(\tau_k)$ is computed as in (21) for $b = 1, \ldots, B$ and each $k = 1, \ldots, K$.*
3. *Obtain the $(100 \times (1-\alpha))\%$ percentile of $\left\{\max_{k=1,\ldots,K} \left|\sqrt{n}(\hat{\theta}^b(\tau_k) - \hat{\theta}(\tau_k))\right|\right\}_{b=1}^B$. This is denoted $\xi_{1-\alpha,\theta}$.*

The $1-\alpha$ confidence bands are then computed as

$$\mathcal{CB}(\theta(\tau),\alpha) = \left[\hat{\theta}(\tau) - \frac{\xi_{1-\alpha,\theta}}{\sqrt{n}}, \hat{\theta}(\tau) + \frac{\xi_{1-\alpha,\theta}}{\sqrt{n}}\right].$$

4.2 Bounds on the Global Effect

An important case is when we are interested in two conclusions $G_{\tau_1} > g$ and $G_{\tau_2} < g$ for $\tau_1 \neq \tau_2$, both in $(\delta, 1-\delta)$. This is the case in Figure 6. For the case of the global effect, by Theorem 2, the bounds are $\tau \mapsto F_a^{-1}(\tau - \delta U_{\tau_1}) - F_Y^{-1}(\tau)$, and $\tau \mapsto F_a^{-1}(\tau - \delta L_{\tau_2}) - F_Y^{-1}(\tau)$, for fixed values of U_{τ_1} and L_{τ_2}. The goal is to make inference on these bounds when U_{τ_1}, L_{τ_2}, F_a^{-1} and F_Y^{-1} are estimated.[12]

Define $B(U_{\tau_1};\tau) := F_a^{-1}(\tau - \delta U_{\tau_1}) - F_Y^{-1}(\tau)$ and $B(L_{\tau_2};\tau) := F_a^{-1}(\tau - \delta L_{\tau_2}) - F_Y^{-1}(\tau)$. The estimated counterparts are

$$\begin{pmatrix} \hat{B}(\hat{U}_{\tau_1};\tau) \\ \hat{B}(\hat{L}_{\tau_2};\tau) \end{pmatrix} = \begin{pmatrix} \hat{F}_a^{-1}(\tau - \delta \hat{U}_{\tau_1}) - \hat{F}_Y^{-1}(\tau) \\ \hat{F}_a^{-1}(\tau - \delta \hat{L}_{\tau_2}) - \hat{F}_Y^{-1}(\tau) \end{pmatrix},$$

where by (15), and (16) we have

$$\begin{pmatrix} \hat{U}_{\tau_1} \\ \hat{L}_{\tau_2} \end{pmatrix} = \begin{pmatrix} \min\{\max\{0, \hat{\theta}(\tau_1)\}, 1\} \\ \max\{\min\{0, \hat{\theta}(\tau_2)\}, -1\} \end{pmatrix}.$$

To find the distributions of $\hat{U}_{\tau_1}$ and $\hat{L}_{\tau_2}$, we define the map $\phi : \ell^\infty(\delta, 1-\delta) \mapsto [-1, 0] \times [0, 1]$ given by

$$\phi(H) = \begin{pmatrix} \min\{\max\{0, H(\tau_1)\}, 1\} \\ \max\{\min\{0, H(\tau_2)\}, -1\} \end{pmatrix}. \tag{22}$$

Though continuous, the composition of max and min (and vice versa) is not smooth. However, a form of differentiability, namely Hadamard directional differentiability, is still preserved. More importantly, the Delta method is still valid under this weaker differentiability notion. See Shapiro (1990), Dümbgen (1993), and, more recently and with applications to econometric theory, Fang and Santos (2019).

Theorem 6. *Under the Assumptions of Theorem 5,*

$$\sqrt{n}\begin{pmatrix} \hat{U}_{\tau_1} - U_{\tau_1} \\ \hat{L}_{\tau_2} - L_{\tau_2} \end{pmatrix} \rightsquigarrow \phi'_\theta(\mathbb{G}_\theta),$$

[12] It is assumed that both U_{τ_1} and L_{τ_2} exist, in the sense that there is a robust region for the conclusions $G_{\tau_1} > g$ and $G_{\tau_2} < g$.

where

$$\phi'_\theta(\mathbb{G}_\theta) = \begin{pmatrix} \mathbb{G}_\theta(\tau_1)\mathbb{1}_{\{0<\theta(\tau_1)<1\}} + \max(0, \mathbb{G}_\theta(\tau_1))\mathbb{1}_{\{\theta(\tau_1)=0\}} + \min(0, \mathbb{G}_\theta(\tau_1))\mathbb{1}_{\{\theta(\tau_1)=1\}} \\ \mathbb{G}_\theta(\tau_2)\mathbb{1}_{\{-1<\theta(\tau_2)<0\}} + \min(0, \mathbb{G}_\theta(\tau_2))\mathbb{1}_{\{\theta(\tau_2)=0\}} + \max(0, \mathbb{G}_\theta(\tau_2))\mathbb{1}_{\{\theta(\tau_2)=-1\}} \end{pmatrix}. \quad (23)$$

It is important to point out that the distribution of $\phi'_\theta(\mathbb{G}_\theta)$ is *not* Gaussian. This is not only due to the presence of the min and max functions, but also because when $\theta(\tau_1) \notin [0,1]$ the first coordinate is degenerate in 0. The same comment applies to the second coordinate, which is degenerate when $\theta(\tau_2) \notin [-1,0]$. See Example 2.1 in Fang and Santos (2019) for a similar situation.

We need the following assumption in order to establish the Hadamard differentiability of the quantile process $\tau \mapsto F_a^{-1}(\tau)$ for $\tau \in (\delta, 1-\delta)$. Recall that the support of F_a is assumed to be $\mathcal{Y}$.

Assumption 7. *For some $\varepsilon > 0$, F_a is continuously differentiable in $[F_a^{-1}(\delta) - \varepsilon, F_a^{-1}(1-\delta) + \varepsilon] \subset \mathcal{Y}$ with strictly positive derivative f_a.*

When the bounds are viewed as a map in $\ell^\infty(\delta, 1-\delta) \times \ell^\infty(\delta, 1-\delta)$, we use "$\cdot$" to keep track of where the argument of the function should be, and we write

$$\sqrt{n}\begin{pmatrix} \hat{B}(\hat{U}_{\tau_1}; \cdot) - B(U_{\tau_1}; \cdot) \\ \hat{B}(\hat{L}_{\tau_2}; \cdot) - B(L_{\tau_2}; \cdot) \end{pmatrix} \quad (24)$$

Theorem 7. *Under Assumptions 4, 5, 6, and 7*

$$\sqrt{n}\begin{pmatrix} \hat{B}(\hat{U}_{\tau_1}; \cdot) - B(U_{\tau_1}; \cdot) \\ \hat{B}(\hat{L}_{\tau_2}; \cdot) - B(L_{\tau_2}; \cdot) \end{pmatrix} \rightsquigarrow \begin{pmatrix} \mathbb{G}_{U_{\tau_1}} \\ \mathbb{G}_{L_{\tau_2}} \end{pmatrix}$$

a tight process in $\ell^\infty(\delta, 1-\delta) \times \ell^\infty(\delta, 1-\delta)$ given by

$$\begin{pmatrix} \mathbb{G}_{U_{\tau_1}} \\ \mathbb{G}_{L_{\tau_2}} \end{pmatrix} := \begin{pmatrix} -\frac{\mathbb{G}_a \circ F_a^{-1}(\,\cdot\, - \delta U_{\tau_1})}{f_a \circ F_a^{-1}(\,\cdot\, - \delta U_{\tau_1})} - \frac{\delta \phi'_\theta(\mathbb{G}_\theta)_2}{f_a \circ F_a^{-1}(\,\cdot\, - \delta U_{\tau_1})} - \frac{\mathbb{G}_Y \circ F_Y^{-1}(\cdot)}{f_Y \circ F_Y^{-1}(\cdot)} \\ -\frac{\mathbb{G}_a \circ F_a^{-1}(\,\cdot\, - \delta L_{\tau_2})}{f_a \circ F_a^{-1}(\,\cdot\, - \delta L_{\tau_2})} - \frac{\delta \phi'_\theta(\mathbb{G}_\theta)_1}{f_a \circ F_a^{-1}(\,\cdot\, - \delta L_{\tau_2})} - \frac{\mathbb{G}_Y \circ F_Y^{-1}(\cdot)}{f_Y \circ F_Y^{-1}(\cdot)} \end{pmatrix}, \quad (25)$$

where the map $\phi'_\theta(\mathbb{G}_\theta)$ is given in (23), and $\phi'_\theta(\mathbb{G}_\theta)_1$ and $\phi'_\theta(\mathbb{G}_\theta)_2$ are the first and second coordinates respectively.

The limiting process in (25) is *not* Gaussian because of the presence of $\phi'_\theta(\mathbb{G}_\theta)$ given in Theorem 5. Hence, by Corollary 3.1 in Fang and Santos (2019), the standard bootstrap will fail. This means that if we attempt to construct confidence intervals in the usual way by resampling $\hat{B}(\hat{U}_{\tau_1}; \tau)$ and $\hat{B}(\hat{L}_{\tau_2}; \tau)$, we will not obtain correct asymptotic coverage. Instead, we use the numerical bootstrap of Hong and Li (2018, 2020). For a given τ, we write the map in (24) as

$$\psi(F_a, F_Y, \hat{U}_{\tau_1}, \hat{L}_{\tau_2}; \tau) = \begin{pmatrix} F_a^{-1}(\tau - \delta \hat{U}_{\tau_1} - F_Y^{-1}(\tau) \\ F_a^{-1}(\tau - \delta \hat{L}_{\tau_2} - F_Y^{-1}(\tau) \end{pmatrix} \quad (26)$$

The idea is that we can approximate $(\mathbb{G}_{U_{\tau_1}}(\tau), \mathbb{G}_{L_{\tau_2}}(\tau))'$ using a standard bootstrap for $\hat{F}_a$ and $\hat{F}_Y$ and a numerical bootstrap for $\hat{U}_{\tau_1}$ and $\hat{L}_{\tau_2}$. First, fix $\hat{U}_{\tau_1}$ and $\hat{L}_{\tau_2}$, and let $\hat{F}_a^*$ and $\hat{F}_Y^*$ be the bootstrap counterparts of $\hat{F}_a$ and $\hat{F}_Y$. Then, define

$$\hat{\psi}^{\bullet}_{aY}(\hat{F}_a^*, \hat{F}_Y^*; \tau) = \sqrt{n} \left(\psi(\hat{F}_a^*, \hat{F}_Y^*, \hat{U}_{\tau_1}, \hat{L}_{\tau_2}; \tau) - \psi(\hat{F}_a, \hat{F}_Y, \hat{U}_{\tau_1}, \hat{L}_{\tau_2}; \tau) \right). \tag{27}$$

Now, fix $\hat{F}_a$ and $\hat{F}_Y$, let $\theta(\tau_1)^*$ and $\hat{\theta}(\tau_2)^*$ be the bootstrap counterparts of $\hat{\theta}(\tau_1)$ and $\hat{\theta}(\tau_2)$, and define the perturbed parameters

$$\hat{\theta}(\tau_1)^p := \hat{\theta}(\tau_1) + \varepsilon_n \sqrt{n} (\hat{\theta}(\tau_1)^* - \hat{\theta}(\tau_1)), \tag{28}$$

and

$$\hat{\theta}(\tau_2)^p := \hat{\theta}(\tau_2) + \varepsilon_n \sqrt{n} (\hat{\theta}(\tau_2)^* - \hat{\theta}(\tau_2)), \tag{29}$$

where the sequence ε_n is constrained to satisfy $\varepsilon_n \to 0$ and $\varepsilon_n \sqrt{n} \to \infty$, as $n \to \infty$. In the empirical application we set $\varepsilon_n = n^{-1/3}$. Define, the perturbed version of U_{τ_1} and L_{τ_2} as

$$\begin{pmatrix} \hat{U}^p_{\tau_1} \\ \hat{L}^p_{\tau_2} \end{pmatrix} = \begin{pmatrix} \min\{\max\{0, \hat{\theta}(\tau_1)^p\}, 1\} \\ \max\{\min\{0, \hat{\theta}(\tau_2)^p\}, -1\} \end{pmatrix}. \tag{30}$$

Then, define

$$\hat{\psi}^{\bullet}_{UL}(\hat{U}^p_{\tau_1}, \hat{L}^p_{\tau_2}; \tau) = \frac{1}{\varepsilon_n} \left(\psi(\hat{F}_a, \hat{F}_Y, \hat{U}^p_{\tau_1}, \hat{L}^p_{\tau_2}; \tau) - \psi(\hat{F}_a, \hat{F}_Y, \hat{U}_{\tau_1}, \hat{L}_{\tau_2}; \tau) \right). \tag{31}$$

The approximation to $(\mathbb{G}_{U_{\tau_1}}(\tau), \mathbb{G}_{L_{\tau_2}}(\tau))'$ is given by the distribution of

$$\hat{\psi}^{\bullet}_{aY}(\hat{F}_a^*, \hat{F}_Y^*; \tau) + \hat{\psi}^{\bullet}_{UL}(\hat{U}^p_{\tau_1}, \hat{L}^p_{\tau_2}; \tau),$$

which, in turn, is approximated by the simulated procedure below.

Algorithm 3 (Bootstrap for $\hat{B}(\hat{U}_{\tau_1}; \tau)$ and $\hat{B}(\hat{L}_{\tau_2}; \tau)$)**.**

1. *Given the data* $\{Y_i, D_i, D_{\delta,i}\}_{i=1}^n$*, compute* $\psi(\hat{F}_a, \hat{F}_Y, \hat{\theta}(\tau_1), \hat{\theta}(\tau_2); \tau)$ *given in* (26)*.*

2. *Obtain B bootstrap samples of size n from* $\{Y_i, D_i, D_{\delta,i}\}_{i=1}^n$*.*

3. *For* $b = 1, \ldots, B$*, following* (27)*, compute*

$$\hat{\psi}^{\bullet}_{aY}(\hat{F}_a^b, \hat{F}_Y^b; \tau) = \sqrt{n} \left(\psi(\hat{F}_a^b, \hat{F}_Y^b, \hat{U}_{\tau_1}, \hat{L}_{\tau_2}; \tau) - \psi(\hat{F}_a, \hat{F}_Y, \hat{U}_{\tau_1}, \hat{L}_{\tau_2}; \tau) \right). \tag{32}$$

4. *For* $b = 1, \ldots, B$*, following* (21)*, compute* $\hat{\theta}(\tau_1)^b$ *and* $\hat{\theta}(\tau_2)^b$*. Following* (28) *and* (29)*, compute the*

perturbed parameters as

$$\hat{\theta}(\tau_1)^{p,b} := \hat{\theta}(\tau_1) + \varepsilon_n \sqrt{n}(\hat{\theta}(\tau_1)^b - \hat{\theta}(\tau_1)),$$

and

$$\hat{\theta}(\tau_2)^{p,b} := \hat{\theta}(\tau_2) + \varepsilon_n \sqrt{n}(\hat{\theta}(\tau_2)^b - \hat{\theta}(\tau_2))$$

Following (30) *compute*

$$\begin{pmatrix} \hat{L}^{p,b}_{\tau_2} \\ \hat{U}^{p,b}_{\tau_1} \end{pmatrix} = \begin{pmatrix} \max\{\min\{0, \hat{\theta}(\tau_2)^{p,b}\}, -1\} \\ \min\{\max\{0, \hat{\theta}(\tau_1)^{p,b}\}, 1\} \end{pmatrix}.$$

5. *For $b = 1, \ldots, B$, following* (31), *compute*

$$\hat{\psi}^{\bullet}_{UL}(\hat{U}^{p,b}_{\tau_1}, \hat{L}^{p,b}_{\tau_2}; \tau) = \frac{1}{\varepsilon_n}\left(\psi(\hat{F}_a, \hat{F}_Y, \hat{U}^{p,b}_{\tau_1}, \hat{L}^{p,b}_{\tau_2}; \tau) - \psi(\hat{F}_a, \hat{F}_Y, \hat{U}_{\tau_1}, \hat{L}_{\tau_2}; \tau)\right).$$

6. *For $b = 1, \ldots, B$, define*

$$\hat{\psi}'(b, \tau) = \hat{\psi}^{\bullet}_{aY}(\hat{F}^b_a, \hat{F}^b_Y; \tau) + \hat{\psi}^{\bullet}_{UL}(\hat{U}^{p,b}_{\tau_1}, \hat{L}^{p,b}_{\tau_2}; \tau). \tag{33}$$

7. *Obtain the* $(100 \times \alpha/2)\%$ *and* $(100 \times (1 - \alpha/2))\%$ *percentiles from the first coordinate of* (33). *These are denoted* $\xi_{\alpha/2,\tau,U_{\tau_1}}$ *and* $\xi_{1-\alpha/2,\tau,U_{\tau_1}}$.
8. *Obtain the* $(100 \times \alpha/2)\%$ *and* $(100 \times (1 - \alpha/2))\%$ *percentiles from the second coordinate of* (33). *These are denoted* $\xi_{\alpha/2,\tau,L_{\tau_2}}$ *and* $\xi_{1-\alpha/2,\tau,L_{\tau_2}}$.

The $1 - \alpha$ confidence intervals are then computed as

$$\begin{aligned} \mathcal{CI}(B(U_{\tau_1}; \tau), \alpha) &= \left[\hat{B}(\hat{U}_{\tau_1}; \tau) - \frac{\xi_{1-\alpha/2,\tau,U_{\tau_1}}}{\sqrt{n}}, \hat{B}(\hat{U}_{\tau_1}; \tau) - \frac{\xi_{\alpha/2,\tau,U_{\tau_1}}}{\sqrt{n}}\right], \\ \mathcal{CI}(B(L_{\tau_2}; \tau), \alpha) &= \left[\hat{B}(\hat{L}_{\tau_2}; \tau) - \frac{\xi_{1-\alpha/2,\tau,L_{\tau_2}}}{\sqrt{n}}, \hat{B}(\hat{L}_{\tau_2}; \tau) - \frac{\xi_{\alpha/2,\tau,L_{\tau_2}}}{\sqrt{n}}\right]. \end{aligned}$$

The simultaneous $1 - \alpha$ confidence intervals, by the Bonferroni correction,[13] are given by the Cartesian product

$$\mathcal{CI}(B(U_{\tau_1}; \tau), B(L_{\tau_2}; \tau), \alpha) = \mathcal{CI}(B(U_{\tau_1}; \tau), \alpha/2) \times \mathcal{CI}(B(L_{\tau_2}; \tau), \alpha/2).$$

[13] If we want simultaneous $1 - \alpha$ confidence intervals, for each coordinate the confidence intervals must be constructed at the $1 - \alpha/2$ level.

Alternatively, simultaneous $1-\alpha$ confidence intervals can be constructed using a lower confidence interval for $B(U_{\tau_1};\tau)$: $\hat{B}(\hat{U}_{\tau_1};\tau)-\frac{\xi_{1-\alpha/2,\tau,U_{\tau_1}}}{\sqrt{n}}$, and upper confidence interval for $B(L_{\tau_2};\tau)$: $\hat{B}(\hat{L}_{\tau_2};\tau)-\frac{\xi_{\alpha/2,\tau,L_{\tau_2}}}{\sqrt{n}}$.

We can construct uniform confidence bands for $\tau\in(\delta,1-\delta)$ in the following way.

Algorithm 4 (Bootstrap for $\hat{B}(\hat{U}_{\tau_1};\cdot)$ and $\hat{B}(\hat{L}_{\tau_2};\cdot)$)**.**

1. *Given a grid of values* $\{\tau_k\}_{k=1}^K\subset(\delta,1-\delta)$*, following* (33)*, compute for* $b=1,\ldots,B$

$$\hat{\psi}'(b)=\max_{k=1,\ldots,K}\left|\hat{\psi}^{\bullet}_{aY}(\hat{F}^b_a,\hat{F}^b_Y;\tau_k)+\hat{\psi}^{\bullet}_{UL}(\hat{U}^{p,b}_{\tau_1},\hat{L}^{p,b}_{\tau_2};\tau_k)\right| \tag{34}$$

2. *Obtain the* $(100\times(1-\alpha))\%$ *percentile from the first coordinate of* (34)*. This is denoted* $\xi_{1-\alpha,U_{\tau_1}}$.

3. *Obtain the* $(100\times(1-\alpha))\%$ *percentile from the second coordinate of* (34)*. This is denoted* $\xi_{1-\alpha,L_{\tau_2}}$.

The one-sided or two-sided $1-\alpha$ confidence bands are computed as before.

4.3 Quantile Breakdown Frontier: Marginal Effect

The quantile breakdown frontier for the sign of the marginal effect is given by (see (20)) the map $\tau\mapsto F_{Y|D=0}(F_Y^{-1}(\tau))-F_{Y|D=1}(F_Y^{-1}(\tau))$, and the estimated counterpart is $\hat{\theta}(\tau)=\hat{F}_{Y|D=0}(\hat{F}_Y^{-1}(\tau))-\hat{F}_{Y|D=1}(\hat{F}_Y^{-1}(\tau))$, where

$$\hat{F}_{Y|D=0}(y):=\frac{\sum_{i=1}^n\mathbb{1}\{Y_i\le y\}(1-D_i)}{\sum_{i=1}^n(1-D_i)},$$

and

$$\hat{F}_{Y|D=1}(y):=\frac{\sum_{i=1}^n\mathbb{1}\{Y_i\le y\}D_i}{\sum_{i=1}^n D_i}.$$

As before, we want to investigate the weak convergence of $\sqrt{n}(\hat{\theta}-\theta)$ in $\ell^\infty(0,1)$:

$$\sqrt{n}(\hat{\theta}-\theta)=\sqrt{n}\left(\hat{F}_{Y|D=0}\circ\hat{F}_Y^{-1}-\hat{F}_{Y|D=1}\circ\hat{F}_Y^{-1}-\left(F_{Y|D=0}\circ F_Y^{-1}-F_{Y|D=1}\circ F_Y^{-1}\right)\right).$$

Recall that the bounds on the marginal effect can be computed for any $\tau\in(0,1)$, as opposed to the global effect, where we are constrained to $\tau\in(\delta,1-\delta)$. The main assumption is

Assumption 8 (Functional CLT)**.** *The following multivariate functional central limit theorem holds*

$$\sqrt{n}\begin{pmatrix}\hat{F}_Y-F_Y\\ \hat{F}_{Y|D=0}-F_{Y|D=0}\\ \hat{F}_{Y|D=1}-F_{Y|D=1}\end{pmatrix}\rightsquigarrow\begin{pmatrix}\mathbb{G}_Y\\ \mathbb{G}_0\\ \mathbb{G}_1\end{pmatrix},$$

where $\mathbb{G}_Y$*,* $\mathbb{G}_0$*, and* $\mathbb{G}_1$ *are Brownian bridges in* $\ell^\infty(\mathcal{Y})$*, where* $\mathcal{Y}$ *is the common support of* Y*,* $Y|D=0$*, and* $Y|D=1$*.*

The next assumption is needed to establish the Hadamard differentiability of the composition map, and the quantile process.

Assumption 9 (Conditions for Hadamard Differentiability)**.**

1. *The distribution functions $F_{Y|D=0}(y)$ and $F_{Y|D=1}(y)$ are differentiable, with uniformly continuous and bounded derivatives on their support $\mathcal{Y}$. The derivatives are $f_{Y|D=0}(y)$ and $f_{Y|D=1}(y)$ respectively.*

2. *The support $\mathcal{Y}$ is the compact set $[y_l, y_u]$.*

3. *$F_Y(y)$ is continuously differentiable on $\mathcal{Y}$ with strictly positive derivative f_Y.*

Theorem 8 (Asymptotic Distribution of QBF for Marginal Effect)**.** *Under Assumptions 8 and 9*

$$\sqrt{n}(\hat{\theta} - \theta) = \sqrt{n}\left(\hat{F}_{Y|D=0} \circ \hat{F}_Y^{-1} - \hat{F}_{Y|D=1} \circ \hat{F}_Y^{-1} - \left(F_{Y|D=0} \circ F_Y^{-1} - F_{Y|D=1} \circ F_Y^{-1}\right)\right)$$
$$\rightsquigarrow \mathbb{G}_{0,Y} - \mathbb{G}_{1,Y},$$

where, for $d = 0, 1$, $\mathbb{G}_{d,Y} := \mathbb{G}_d \circ F_Y^{-1} - f_{Y|D=d} \circ F_Y^{-1} \cdot \frac{\mathbb{G}_Y \circ F_Y^{-1}}{f_Y \circ F_Y^{-1}}$ are tight Gaussian elements of $\ell^\infty(0,1)$.

Confidence intervals/bands can be constructed following the same procedures outlined in Algorithms 1 and 2, because by Theorem 23.9 in van der Vaart (1998), the empirical bootstrap is valid. We skip the details to avoid repetition.

5 Empirical application: What do unions do?

There is an extensive literature that studies unions and inequality. A recent contribution by Farber et al. (2020) contains a review of the literature. In our empirical application, in particular, we look at how unions affect the distribution of wages for *all* workers. Unions can have a variety of effects on the distribution of wages. As argued by Freeman (1980), unions can raise the wages of unionized workers relative to non-unionized workers, possibly through more bargaining power. So, if higher paid workers unionize, the dispersion of wages can increase, but if lower paid workers unionize, the dispersion of wages can decrease. Furthermore, within a given industry, the union can reduce the dispersion of wages by standardizing the wages. This will impact the distribution of wages more or less depending on the size of the industry and the wages it pays.

A key difficulty in identifying the causal effect of unions on wages is that selection into unions is non-random. Hence, any measurement of the union premium, the difference in wages between similar union and nonunion workers, will be biased for the causal effect. Indeed, this has been a long standing concern of labor economists.[14] With respect to selection into unions, Card (1996)

[14]Indeed, the opening words of Card (1996) are:

> *Despite a large and sophisticated literature there is still substantial disagreement over the extent to which differences in the structure of wages between union and nonunion workers represent an effect of trade unions, rather than a consequence of the nonrandom selection of unionized workers.*

argues that unionized workers with low observed skills, tend to have high unobserved skills. The reverse happens with high skilled unionized workers: they tend to have low unobservable skills. Due to this selection bias, it might be impossible for a policy maker to device a policy where the *newly* unionized workers are selected in a way such that they are drawn from the distribution of the *already* unionized workers.

Using the techniques developed in this paper, we are going to consider the effect of both globally and marginally expanding union coverage. We will explicitly allow for non-random selection into unions. Moreover, as opposed to Firpo, Fortin and Lemieux (2009), we will not assume distributional invariance: the distribution of the *newly* unionized workers can be different from the distribution of the *already* unionized workers. That is, we do not use any imputation method to impute the union premium of the newly unionized workers

Following Freeman (1980), Card (2001) and Card, Lemieux and Riddell (2004) we consider a two sector economy. Each worker has a well-defined pair of potential (log) wages: $Y_i(1)$ for the unionized sector and $Y_i(0)$ for the nonunionized sector. Under Assumption 2, and for any policy D_δ, we have the following classification of individuals:

	$D_\delta = 0$	$D_\delta = 1$
$D = 0$	*nonunionized*	*newly unionized*
$D = 1$	-	*unionized*

The relevant unobserved distribution is then $F_{Y(1)|\textit{newly unionized}}$: the union wages of the newly unionized workers. So, we look at departures of $F_{Y(1)|\textit{newly unionized}}$ from $F_{Y(1)|\textit{unionized}}$, which is observed. This difference is what we refer to as the policy selection bias.

Using the data in Firpo, Fortin and Lemieux (2009) we estimate the quantile breakdown frontier for marginal and global effects of different type of policies on the distribution of real log hourly wages. We use the 1983-1985 Outgoing Rotation Group (ORG) Supplement of the Current Population Survey. Our sample consists of 266,956 observations on U.S. males. See Lemieux (2006) for more details about the data.

The unionization rate in the dataset is 0.26. Figure 10 shows the typical hump-shaped pattern of the unionization rates by quantiles of the distribution of wages. For lower quantiles, unionization rates are quite low. They peak in the past the middle of the distribution and then drop at the higher quantiles. We will analyze a randomized policy and a non-randomized policy. In the first case, we will analyze the policy that marginally increases unionization by selecting workers at random. We will look at the quantile breakdown frontier for the sign of the marginal effect. That is, we set $g = 0$ and look at whether the marginal effect is positive or negative. Figure 11 shows the result for a grid of $\tau \in (0.1, 0.9)$, along with 95% pointwise confidence intervals and uniform confidence bands. We can see that along almost all quantiles, the quantile breakdown frontier is positive, and it peak at around 0.27 for $\tau = 0.4$. This means that if the selection bias due to U is greater than 0.27, then the conclusion $M_\tau > 0$ does not hold for any τ.

In the second case, we will analyze a non-randomized policy. Consider a 10% increase in the unionization rate by unionizing workers whose wages are below the $.10/(1-p)$-quantile ≈ 0.14-quantile of the wages of the nonunionized sector. In the notation of this paper, we have $D = 1$ if a worker is unionized, $D_\delta = 1$ if a worker is unionized under the policy, Y is (log) wage, and $\delta = 0.1$. That is, D_δ is given by

$$D_\delta = \begin{cases} 1 & \text{if } D = 1 \\ 1 & \text{if } D = 0 \text{ and } Y \leq F^{-1}_{Y|D=0}(0.14) \\ 0 & \text{otherwise} \end{cases}$$

This guarantees that the unionization rate increases by roughly 10%. Indeed, the mean of D_δ is now 0.36. Figure 12 shows the quantile breakdown frontiers for $g_\tau = 0.1$ for a grid of $\tau \in (0.1, 0.9)$. This is the empirical counterpart of the right side of Figure 5. Pointwise confidence intervals (shaded) and uniform confidence bands (dashed) are also shown, both at the 95% level. Since the dependent variable is log wages, $g_\tau = 0.1$ amounts to a 10% change in wages for a given quantile.

For lower quantiles, if we want the policy to result in an increase of wages *higher* than 10%, then the departure from point identification is given by U_τ in the positive part of the curve: for example, for the 20$^{\text{th}}$ quantile, $U_{.2} \approx 0.45$. For higher quantiles, if we want the policy to result in changes of wages *lower* than 10%, then the maximum departure from point identification, L_τ, is given by the negative part of the curve. For example, for the 80$^{\text{th}}$ quantile, $L_{.8} \approx -0.36$. In terms of our notation, if we are interested in the conclusions $G_{.2,D_{.1}} > 0.1$ and $G_{.8,D_{.1}} < 0.1$, then the robust region is

$$\{(L, U) : -1 \leq L \leq -0.36 \text{ and } 0 \leq U \leq 0.45\}$$

Recall that U and L come from Assumption 3:

$$L \leq F_{Y(1)|\textit{newly unionized}}(y) - F_{Y(1)|\textit{unionized}}(y) \leq U.$$

So if we are interested in the 20$^{\text{th}}$ and 80$^{\text{th}}$ quantile, we need

$$-0.36 \leq F_{Y(1)|\textit{newly unionized}}(y) - F_{Y(1)|\textit{unionized}}(y) \leq 0.45. \tag{35}$$

for the conclusions to hold. This does not rule out either direction of first-order stochastic domination, but it does put a bound on it. Since $F_{Y(1)|\textit{unionized}}(y)$ can be estimated, then simulation exercises can be carried out on possible CDFs that satisfy (35), *i.e.*, they are not too far away from the empirical counterpart of $F_{Y(1)|\textit{unionized}}(y)$. Figure 13 shows the estimated bounds for the global effect when setting $L_{.8} \approx -0.36$ and $U_{.2} \approx 0.45$. For $\tau = 0.2$, we can see that the identified region lies above 0.10, and for $\tau = 0.8$, the identified region lies below 0.10. Pointwise confidence intervals (shaded) and uniform confidence bands (dashed) are also shown, both at the 95% level.

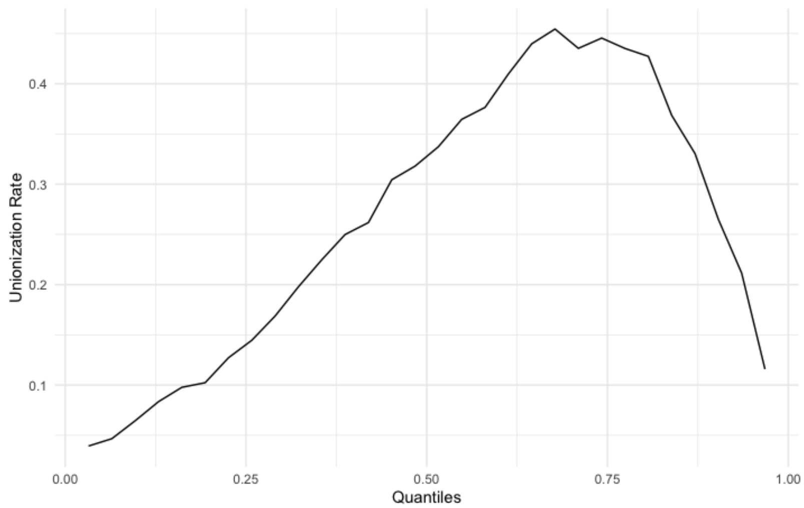


Figure 10: *Unionization rates by quantiles of the distribution of wages.*

We repeat the same exercise for the global effect, this time for $g = 0.05$. We keep $\delta = 0.1$. The quantile breakdown frontier and the bounds on the global effect can be seen in Figures 14 and 15. At the 20^{th} quantile, $U_{.2} \approx 0.67$, while at the 80^{th} quantile, $L_{.8} \approx -0.05$. This means that the hypothesis $G_{.8,D_{.1}} < 0.05$ is not very robust: any policy selection bias above given L in $[-1, -0.05)$ result in identification regions for $G_{.8,D_{.1}}$ that contain values greater than 0.05.

Figures 13 and 15 show that, because of the continuity of the quantile breakdown frontier, when we focus on conclusions at the 20^{th} and 80^{th} quantiles, we are also deriving bounds for the global effect at other quantiles. Thus, in Figure 13, we can see that the global effect, which is consistent with $G_{.2,D_{.1}} > 0.1$ and $G_{.8,D_{.1}} < 0.1$ is positive up to $\tau = 0.6$. In other words, the combinations of L and U that ensure that $G_{.2,D_{.1}} > 0.1$ and $G_{.8,D_{.1}} < 0.1$, imply that $G_{\tau,D_{.1}} > 0$ for $\tau \in (0.1, 0.6)$.

6 Conclusion

In this paper we show how to perform a sensitivity analysis on the effect of counterfactual policies on the quantiles of an outcome of interest. We focus on counterfactual policies which increase the proportion of treated individuals and obtain partial identified sets for both global and marginal effects on the unconditional quantiles. In the former, the increase δ in the proportion is fixed, while in the latter goes to 0. By dropping the standard distributional invariance assumption, we are able to broaden the scope of policies that can be analyzed. Our partial identification results are used to perform a sensitivity analysis based on the departure from point identification. The sensitivity analysis is greatly simplified by the introduction of the quantile breakdown frontier, a curve that quantifies the maximum amount of selection bias compatible with a given conclusion at each quantile. A further use of the quantile breakdown frontier, is to bound the global effect curve in order for it to be consistent with a set of desired conclusions.

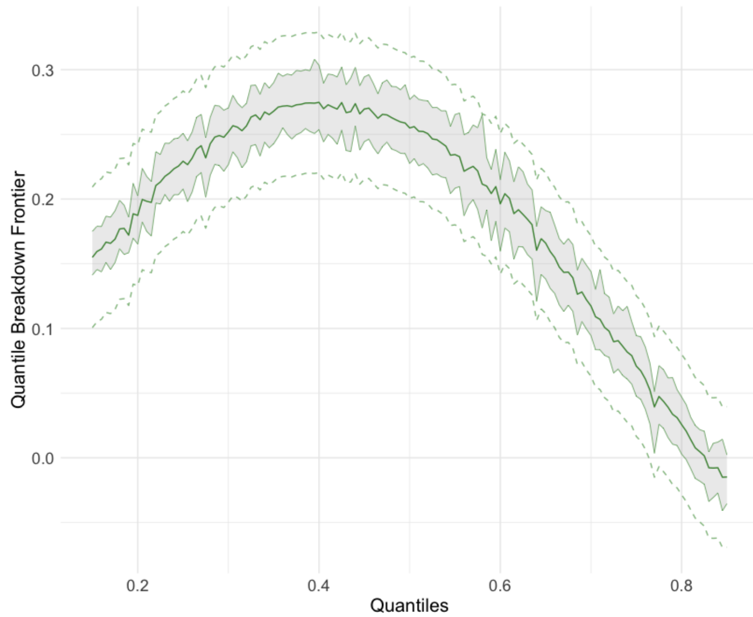


Figure 11: *Quantile Breakdown Frontier for the sign of the marginal effect. 95% confidence intervals (shaded) and 95% confidence bands (dashed).*

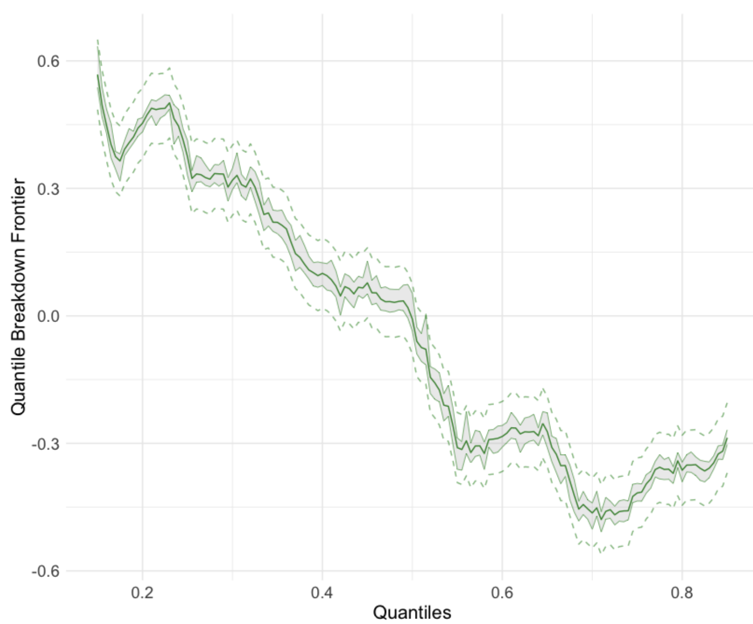


Figure 12: *Quantile Breakdown Frontier for the global effect and $g = 0.1$. 95% confidence intervals (shaded) and 95% confidence bands (dashed).*

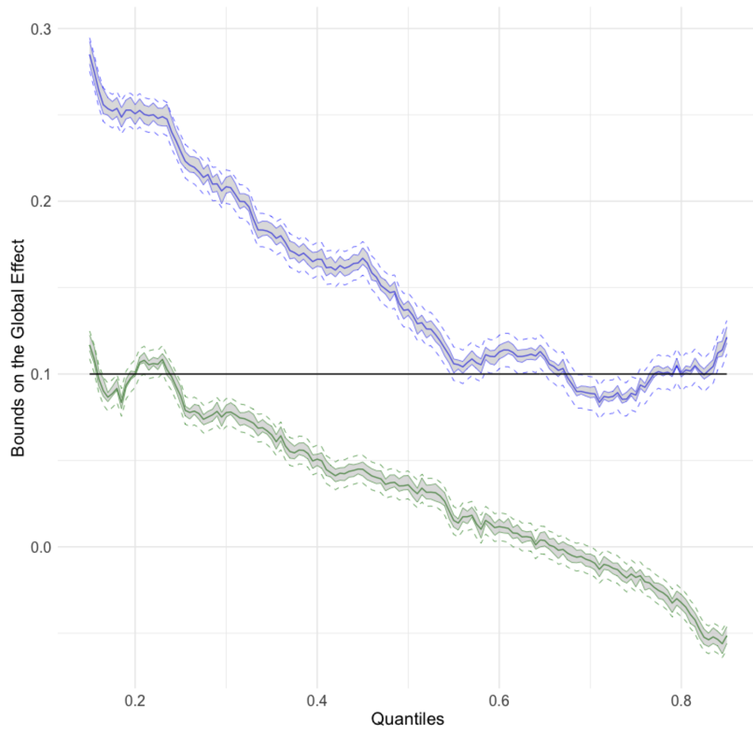


Figure 13: *Bounds on the global effect for* $L_{.8} \approx -0.36$ *and* $U_{.2} \approx 0.45$ *and* $g = 0.1$*. 95% confidence intervals (shaded) and 95% confidence bands (dashed).*

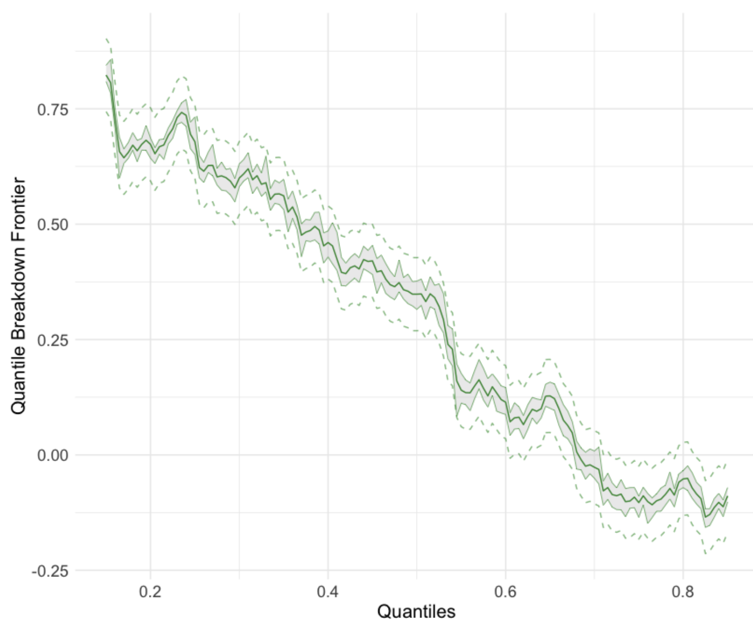


Figure 14: *Quantile Breakdown Frontier for the global effect and* $g = 0.05$*. 95% confidence intervals (shaded) and 95% confidence bands (dashed).*

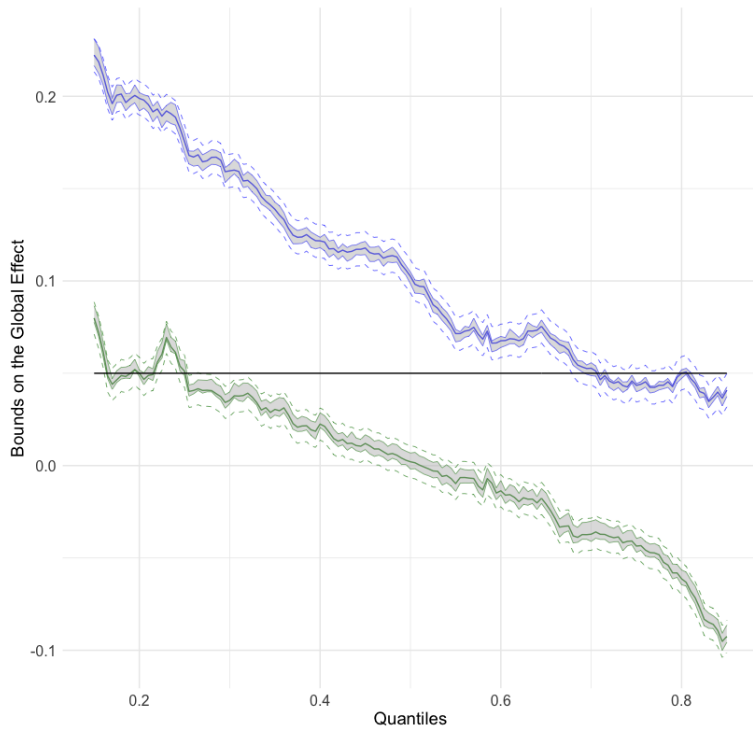


Figure 15: *Bounds on the global effect for* $L_{.8} \approx -0.05$ *and* $U_{.2} \approx 0.67$ *and* $g = 0.05$*. 95% confidence intervals (shaded) and 95% confidence bands (dashed).*

Our empirical application takes another look at the relationship between unions and inequality. In particular, we perform a sensitivity analysis on a policy that increases unionization by 10%. This is done by selecting nonunionized workers who are below a certain threshold of income. We then look at the effect of this policy on the 20$^{\text{th}}$ and 80$^{\text{th}}$ quantiles of the distribution of wages. We are interested in the following conclusion: the change in the 20$^{\text{th}}$ quantile of wages is greater than 10%, while the change at the 80$^{\text{th}}$ quantile is less than 10%. We derive the values of selection bias consistent with the conclusion. Our results show that this policy is consistent with moderate values of selection bias.

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Appendices

A Proofs

Proof of Theorem 1. Using the fact that $Y_{D_\delta} = D_\delta Y(1) + (1 - D_\delta)Y(0)$, we have

$$\begin{aligned} F_{Y_{D_\delta}}(y) &= \Pr(D=0, D_\delta=0)F_{Y(0)|D=0,D_\delta=0}(y) + \Pr(D=0, D_\delta=1)F_{Y(1)|D=0,D_\delta=1}(y) \\ &+ \Pr(D=1, D_\delta=0)F_{Y(0)|D=1,D_\delta=0}(y) + \Pr(D=1, D_\delta=1)F_{Y(1)|D=1,D_\delta=1}(y). \end{aligned}$$

Under Assumption 2, the probability weights are

$$\begin{aligned} \Pr(D=0, D_\delta=0) &= 1 - p - \delta, \\ \Pr(D=0, D_\delta=1) &= \delta, \\ \Pr(D=1, D_\delta=0) &= 0, \\ \Pr(D=1, D_\delta=1) &= p. \end{aligned}$$

Therefore, we rewrite $F_{Y_{D_\delta}}(y)$ as

$$\begin{aligned} F_{Y_{D_\delta}}(y) &= (1 - p - \delta)F_{Y(0)|D=0,D_\delta=0}(y) + \delta F_{Y(1)|D=0,D_\delta=1}(y) \\ &+ pF_{Y(1)|D=1,D_\delta=1}(y), \end{aligned}$$

We add and subtract $\delta F_{Y(1)|D=1,D_\delta=1}(y)$, to get

$$F_{Y_{D_\delta}}(y) = F_a(y) + \delta \left[F_{Y(1)|D=0,D_\delta=1}(y) - F_{Y(1)|D=1,D_\delta=1}(y) \right].$$

□

Proof of Theorem 3. Let $\Gamma_\tau[F]$ be the τ-quantile of F. The Hadamard derivative at F is (See Lemma 21.3 in van der Vaart (1998))

$$\Gamma'_{\tau,F}[h] = -\frac{h(F^{-1}(\tau))}{f(F^{-1}(\tau))}.$$

for any $h \in D[-\infty, \infty]$ continuous at $F^{-1}(\tau)$.[15] We write the marginal effect as

$$
\begin{aligned}
M_{\tau,\mathcal{D}} &= \lim_{\delta \downarrow 0} \frac{\Gamma_\tau\left[F_{Y_{D_\delta}}\right] - \Gamma_\tau[F_Y]}{\delta} \\
&= \lim_{\delta \downarrow 0} \frac{\Gamma_\tau\left[F_{Y_{D_0}} + \delta\left(\frac{F_{Y_{D_\delta}} - F_{Y_{D_0}}}{\delta}\right)\right] - \Gamma_\tau[F_Y]}{\delta} \\
&= \lim_{\delta \downarrow 0} \frac{\Gamma_\tau\left[F_Y + \delta\left(\frac{F_{Y_{D_\delta}} - F_Y}{\delta}\right)\right] - \Gamma_\tau[F_Y]}{\delta} \\
&= \Gamma'_{\tau,F_Y}[\dot{F}_{Y,\mathcal{D}}] \\
&= \frac{\dot{F}_{Y,\mathcal{D}}(F_Y^{-1}(\tau))}{f_Y(F_Y^{-1}(\tau))}.
\end{aligned}
$$

The third equality follows from $F_{Y_{D_0}} = F_Y$. The fourth equality follows from

$$
\lim_{\delta \downarrow 0} \sup_{y \in \mathcal{Y}} \left| \frac{F_{Y_{D_\delta}}(y) - F_Y(y)}{\delta} - \dot{F}_{Y,\mathcal{D}}(y) \right| = 0,
$$

which is required by Lemma 21.3 in van der Vaart (1998).

□

Proof of Remark 5. We want to show that $F_a^{-1}(\tau - \delta) - F_Y^{-1}(\tau) \leq 0$ and $F_a^{-1}(\tau + \delta) - F_Y^{-1}(\tau) \geq 0$. Manipulating equation (4) in Theorem 1 we can obtain that F_a and F_Y are related by[16]

$$
F_a(y) = F_Y(y) + \delta\left[F_{Y(1)|D=0,D_\delta=1}(y) - F_{Y(1)|D=1,D_\delta=1}(y)\right].
$$

Since $-1 \leq F_{Y(1)|D=0,D_\delta=1}(y) - F_{Y(1)|D=1,D_\delta=1}(y) < 1$, then

$$
F_Y(y) - \delta \leq F_a(y) \leq F_Y(y) + \delta.
$$

Therefore, we have

$$
F_a^{-1}(F_Y(y) - \delta) \leq y \leq F_a^{-1}(F_Y(y) + \delta).
$$

Since the previous display is valid for any $y \in \mathcal{Y}$, we set $y = F_Y^{-1}(\tau)$. This implies that

$$
F_a^{-1}(\tau - \delta) \leq F_Y^{-1}(\tau) \leq F_a^{-1}(\tau + \delta).
$$

[15] For $[a, b] \subset [-\infty, \infty]$, $D[a, b]$ is the Skorohod space: the set of all real-valued cadlag functions: right continuous with left limits everywhere in $[a, b]$. $D[a, b]$ is equipped with the uniform norm: $\|x\|_\infty := \sup_{t \in [a,b]} |x(t)|$.

[16] This follows from noting that $F_Y(y) = (1 - p - \delta)F_{Y|D=0,D_\delta=0}(y) + \delta F_{Y|D=0,D_\delta=1}(y) + pF_{Y|D=1,D_\delta=1}(y)$, while by (4), $F_a(y) = (1 - p - \delta)F_{Y|D=0,D_\delta=0}(y) + (p + \delta)F_{Y|D=1,D_\delta=1}(y)$. So, if we add and subtract $\delta F_{Y|D=0,D_\delta=1}(y)$, we obtain $F_a(y) = F(y) + \delta\left[F_{Y|D=1,D_\delta=1}(y) - F_{Y|D=0,D_\delta=1}(y)\right]$.

Thus, we have that $F_a^{-1}(\tau-\delta) - F_Y^{-1}(\tau) \leq 0$ and $F_a^{-1}(\tau+\delta) - F_Y^{-1}(\tau) \geq 0$. $\square$

Proof of Theorem 4. The lower bound is the limit when δ goes to 0 of

$$\frac{F_a^{-1}(\tau-\delta U) - F_Y^{-1}(\tau)}{\delta}. \tag{A.1}$$

We will show that this limit exists and compute its value. Recall that by (1) we can simplify the apparent distribution in (4) to

$$F_a(y) = (1-p-\delta)F_{Y|D=0}(y) + (p+\delta)F_{Y|D=1}(y)$$

We will write $F_{a,\delta}(y)$ to make explicit the fact that the apparent distribution depends on δ. Define

$$g(\delta_1,\delta_2) = F_{a,\delta_1}^{-1}(\tau-\delta_2 U)$$

to emphasize the double role played by δ. The map $\delta_1 \mapsto g(\delta_1,\delta_2)$ for a fixed δ_2 is the composition

$$\delta_1 \in \mathbb{R} \overset{h}{\mapsto} F_{a,\delta_1} \in D[-\infty,\infty] \overset{\Gamma}{\mapsto} F_{a,\delta_1}^{-1}(\tau-\delta_2 U) \in \mathbb{R}.$$

For $[a,b] \subset [-\infty,\infty]$, $D[a,b]$ is the set of all real-valued cadlag functions: right continuous with left limits everywhere in $[a,b]$. $D[a,b]$ is equipped with the uniform norm $\|\cdot\|_\infty$. The first map $h : \delta_1 \in \mathbb{R} \mapsto F_{a,\delta_1} \in D[-\infty,\infty]$ has Hadamard derivative given by $F_{Y|D=1}(y) - F_{Y|D=0}(y)$, while the second map has Hadamard derivative given by (See Lemma 21.3 in van der Vaart (1998))

$$\Gamma'_{F_{a,\delta_1}}[G] = -\frac{G(F_{a,\delta_1}^{-1}(\tau-\delta_2 C))}{f_{a,\delta_1}(F_{a,\delta_1}^{-1}(\tau-\delta_2 U))}.$$

for $G \in D[-\infty,\infty]$ continuous at $F_{a,\delta_1}^{-1}(\tau-\delta_2 C)$. Then, the derivative of the composite map $\delta_1 \mapsto \Gamma \circ h(\delta_1)$ is $\Gamma'_{F_{a,\delta_1}}[h'(\delta_1)]$, which is for a $\delta_2 = 0$

$$\frac{\partial F_{a,\delta_1}^{-1}(\tau)}{\partial \delta_1} = -\frac{F_{Y|D=1}(F_{a,\delta_1}^{-1}(\tau)) - F_{Y|D=0}(F_{a,\delta_1}^{-1}(\tau))}{f_{a,\delta_1}(F_{a,\delta_1}^{-1}(\tau))}.$$

which is continuous at $\delta_1 = 0$. The derivative of the second map $\delta_2 \mapsto g(\delta_1,\delta_2)$, for a fixed $\delta_1 = 0$, can be obtained via the identity

$$F_{a,\delta_1}\left(F_{a,\delta_1}^{-1}(\tau-\delta_2 U)\right) = \tau-\delta_2 U.$$

Differentiating through with respect to δ_2, we obtain

$$\frac{\partial F_a^{-1}(\tau - \delta_2 U)}{\partial \delta_2} = -\frac{U}{f_Y(F_Y^{-1}(\tau - \delta_2 U))}.$$

which is continuous with respect to δ_2.

Therefore, both partial derivatives of the map $(\delta_1, \delta_2) \mapsto g(\delta_1, \delta_2)$ exist and are continuous, hence the limit in (A.1) exists and is equal to

$$\begin{aligned}\lim_{\delta\to 0} \frac{F_a^{-1}(\tau - \delta U) - F_Y^{-1}(\tau)}{\delta} &= \left.\frac{\partial g(\delta_1, 0)}{\partial \delta_1}\right|_{\delta_1=0} + \left.\frac{\partial g(0, \delta_2)}{\partial \delta_2}\right|_{\delta_2=0} \\ &= -\frac{F_{Y|D=1}(F_Y^{-1}(\tau)) - F_{Y|D=0}(F_Y^{-1}(\tau))}{f_Y(F_Y^{-1}(\tau))} - \frac{U}{f_Y(F_Y^{-1}(\tau))}.\end{aligned}$$

For the upper bound, we have the analogous result

$$\lim_{\delta\to 0} \frac{F_a^{-1}(\tau - \delta L) - F_Y^{-1}(\tau)}{\delta} = -\frac{F_{Y|D=1}(F_Y^{-1}(\tau)) - F_{Y|D=0}(F_Y^{-1}(\tau))}{f_Y(F_Y^{-1}(\tau))} - \frac{L}{f_Y(F_Y^{-1}(\tau))}.$$

□

Proof of Theorem 5. We introduce some new notation related to Assumption 6. Let $\mathbb{D}_\delta \subset \ell^\infty(\mathcal{Y})$ denote the set of all restrictions of distribution functions on $\mathbb{R}$ to $[F_Y^{-1}(\delta) - \varepsilon, F_Y^{-1}(1-\delta) + \varepsilon]$. Additionally, $\mathbb{C}_\delta$ is the set of continuous functions on $[F_Y^{-1}(\delta) - \varepsilon, F_Y^{-1}(1-\delta) + \varepsilon]$. Also, $\mathbb{UC}(\mathcal{Y})$ is the set of uniformly continuous functions defined on $\mathcal{Y}$.

The estimator of the apparent counterfactual distribution F_a is given by

$$\hat{F}_a(y) = (1 - \hat{p} - \delta)\hat{F}_{Y|D=0,D_\delta=0}(y) + (\hat{p} + \delta)\hat{F}_{Y|D=1,D_\delta=1}(y)$$

The apparent counterfactual can be written as the map $\mathbb{D}(\mathcal{Y})^2 \times (0,1) \mapsto \mathbb{D}(\mathcal{Y})$ given by

$$\begin{aligned}\psi(F_{Y|D=0,D_\delta=0}, F_{Y|D=1,D_\delta=1}, p) &= (1 - p - \delta)F_{Y|D=0,D_\delta=0} + (p + \delta)F_{Y|D=1,D_\delta=1} \\ &= (1-\delta)F_{Y|D=0,D_\delta=0} + \delta F_{Y|D=1,D_\delta=1} + (F_{Y|D=1,D_\delta=1} - F_{Y|D=0,D_\delta=0})p.\end{aligned}$$

This map is linear, so the Hadamard derivative tangentially to $\ell^\infty(\mathcal{Y})^2 \times (0,1)$ at $(F_{Y|D=0,D_\delta=0}, F_{Y|D=1,D_\delta=1}, p)$ is the map

$$\psi'_{F_{Y|D=0,D_\delta=0}, F_{Y|D=1,D_\delta=1}, p}(h_1, h_2, h_3) = (1-\delta)h_1 + \delta h_2 + (F_{Y|D=1,D_\delta=1} - F_{Y|D=0,D_\delta=0})h_3.$$

By the functional Delta method (see Theorem 20.8 in van der Vaart (1998)) and Assumption

5, we have

$$\sqrt{n}(\hat{F}_a - F_a) = \sqrt{n}(\psi(\hat{F}_{Y|D=0,D_\delta=0}, \hat{F}_{Y|D=1,D_\delta=1}, \hat{p}) - \psi(F_{Y|D=0,D_\delta=0}, F_{Y|D=1,D_\delta=1}, p))$$
$$\rightsquigarrow \mathbb{G}_a := (1-\delta)\mathbb{G}_{0,0} + \delta\mathbb{G}_{1,1} + (F_{Y|D=1,D_\delta=1} - F_{Y|D=0,D_\delta=0})\mathbb{Z}_p.$$

and convergence takes place in $\ell^\infty(\mathcal{Y})$. The random element $\mathbb{G}_a$ is Gaussian. Indeed, for any $y \in \mathcal{Y}$

$$\mathbb{G}_a(y) = (1-\delta)\mathbb{G}_{0,0}(y) + \delta\mathbb{G}_{1,1}(y) + (F_{Y|D=1,D_\delta=1}(y) - F_{Y|D=0,D_\delta=0}(y))\mathbb{Z}_p$$

is a linear combination of normal random variables.

Now we deal with

$$\sqrt{n}(\hat{\theta} - \theta) = -\frac{1}{\delta}\sqrt{n}\left(\hat{F}_a(\hat{F}_Y^{-1} + g) - F_a(F_Y^{-1} + g)\right)$$

This can be written as the composition of two maps. The first one is $\phi : \mathbb{D}(\mathcal{Y}) \times \mathbb{D}_\delta \mapsto \mathbb{D}(\mathcal{Y}) \times \ell^\infty(\delta, 1-\delta)$ given by $\phi(H_1, H_2) \mapsto (H_1, H_2^{-1})$. The second one is $\psi : \mathbb{D}(\mathcal{Y}) \times \ell^\infty(\delta, 1-\delta) \mapsto \ell^\infty(\delta, 1-\delta)$ given by $\psi(H_1, H_2) \mapsto H_1 \circ (H_2 + g)$. Thus

$$\psi \circ \phi(F_a, F_Y) = F_a(F_Y^{-1} + g).$$

By Assumption 6 and Lemma 21.4(i) in van der Vaart (1998), ϕ has Hadamard derivative at (F_a, F_Y) tangentially to $\ell^\infty(\mathcal{Y}) \times \mathbb{C}_\delta$ given by the map

$$\phi'_{(F_a,F_Y)}(h_1, h_2) = \left(h_1, -\frac{h_2 \circ F_Y^{-1}}{f_Y \circ F_Y^{-1}}\right).$$

The second map $\psi : \mathbb{D}(\mathcal{Y}) \times \ell^\infty(\delta, 1-\delta) \mapsto \ell^\infty(\delta, 1-\delta)$ is given by $\psi(H_1, H_2) \mapsto H_1 \circ (H_2 + g)$. It has Hadamard derivative tangentially to $\mathbb{UC}(\mathcal{Y}) \times \ell^\infty(\delta, 1-\delta)$ at any H_1 such that its derivative h_1 is bounded and uniformly continuous on $\mathcal{Y}$, and any H_2. To see, this we combine Lemmas 3.9.25 and 3.9.27 in van der Vaart and Wellner (1996). Let $\alpha_t \to \alpha$ and $\beta_t \to \beta$ in $\mathbb{D}(\mathcal{Y})$ and $\ell^\infty(\delta, 1-\delta)$ respectively, as $t \to 0$.

$$\begin{aligned}
&\frac{\psi(H_1 + t\alpha_t, H_2 + t\beta_t) - \psi(H_1, H_2)}{t} - \alpha \circ (H_2 + g) - h_1 \circ (H_2 + g) \cdot \beta \\
&= \frac{H_1 \circ (H_2 + g + t\beta_t) + t\alpha_t \circ (H_2 + g + t\beta_t) - H_1 \circ (H_2 + g)}{t} - \alpha \circ (H_2 + g) - h_1 \circ (H_2 + g) \cdot \beta \\
&= (\alpha_t - \alpha) \circ (H_2 + g + t\beta_t) + \alpha \circ (H_2 + g + t\beta_t) - \alpha \circ (H_2 + g) \\
&+ \frac{H_1 \circ (H_2 + g + t\beta_t) - H_1 \circ (H_2 + g)}{t} - h_1 \circ (H_2 + g) \cdot \beta
\end{aligned}$$

The first term, $(\alpha_t - \alpha) \circ (H_2 + g + t\beta_t)$, converges to 0 in $\mathbb{D}(\mathcal{Y})$ (that is, uniformly) because convergence of $\alpha_t \to \alpha$ is uniform. The second term, $\alpha \circ (H_2 + g + t\beta_t) - \alpha \circ (H_2 + g)$, converges to 0 in $\mathbb{D}(\mathcal{Y})$ because α is uniformly continuous on $\mathcal{Y}$. For the last term, fix a $\tau \in (\delta, 1-\delta)$. By the mean-value theorem

$$
\begin{aligned}
&\frac{H_1(H_2(\tau)+g+t\beta_t(\tau)) - H_1(H_2(\tau)+g)}{t} - h_1(H_2(\tau)+g)\cdot\beta(\tau) \\
&= h_1(\varepsilon_{\tau,t})\beta_t(\tau) - h_1(H_2(\tau)+g)\cdot\beta(\tau) \\
&= h_1(\varepsilon_{\tau,t})(\beta_t(\tau)-\beta(\tau)) + (h_1(\varepsilon_{\tau,t}) - h_1(H_2(\tau)+g))\cdot\beta(\tau)
\end{aligned}
$$

The first term, $h_1(\varepsilon_{\tau,t})(\beta_t(\tau)-\beta(\tau))$, converges uniformly to 0 because h_1 is bounded on $\mathcal{Y}$, and β_t converges uniformly to β. The second term converges to 0 uniformly because h_1 is uniformly continuous on $\mathcal{Y}$.

Hence, by Assumption 6, ψ has Hadamard derivative at (F_a, F_Y^{-1}) tangentially to $\mathbb{UC}(\mathcal{Y}) \times \ell^\infty(\delta, 1-\delta)$ given by the map

$$
\psi'_{(F_a,F_Y^{-1})}(h_1,h_2) = h_1 \circ (F_Y^{-1}+g) + f_a \circ (F_Y^{-1}+g)\cdot h_2.
$$

We use the chain rule (see Theorem 20.9 in van der Vaart (1998)) to conclude that $\psi \circ \phi$ has Hadamard derivative at (F_a, F_Y) tangentially to $\mathbb{UC}(\mathcal{Y}) \times \mathbb{C}_\delta$ given by the map

$$
\begin{aligned}
(\psi\circ\phi)'_{(F_a,F_Y)}(h_1,h_2) &= \psi'_{\phi(F_a,F_Y)} \circ \phi'_{(F_a,F_Y)}(h_1,h_2) \\
&= \psi'_{(F_a,F_Y^{-1})} \circ (h_1, -h_2(F_Y^{-1})/f_Y(F_Y^{-1})) \\
&= h_1 \circ (F_Y^{-1}+g) - f_a \circ (F_Y^{-1}+g)\frac{h_2 \circ F_Y^{-1}}{f_Y \circ F_Y^{-1}}.
\end{aligned}
$$

By the functional Delta method (see Theorem 20.8 in van der Vaart (1998)) and the continuous mapping theorem (because of the $-1/\delta$ factor), we have that

$$
\begin{aligned}
\sqrt{n}(\hat{\theta}-\theta) &= -\frac{1}{\delta}\sqrt{n}\left(\hat{F}_a \circ (\hat{F}_Y^{-1}+g) - F_a \circ (F_Y^{-1}+g)\right) \\
&\rightsquigarrow -\frac{1}{\delta}(\psi\circ\phi)'_{(F_a,F_Y)}(\mathbb{G}_a, \mathbb{G}_Y)) \\
&:= \mathbb{G}_\theta = -\frac{1}{\delta}\mathbb{G}_a \circ (F_Y^{-1}+g) + \frac{1}{\delta} f_a \circ (F_Y^{-1}+g)\frac{\mathbb{G}_Y \circ F_Y^{-1}}{f_Y \circ F_Y^{-1}}.
\end{aligned}
$$

To see that $\mathbb{G}_\theta$ is indeed Gaussian, we evaluate it at $\tau \in (\delta, 1-\delta)$ to get

$$
\mathbb{G}_\theta(\tau) = -\frac{1}{\delta}\mathbb{G}_a(F_Y^{-1}(\tau)+g) + \frac{1}{\delta} f_a(F_Y^{-1}(\tau)+g)\frac{\mathbb{G}_Y(F_Y^{-1}(\tau))}{f_Y(F_Y^{-1}(\tau))},
$$

which is a linear combination of two normal random variables: $\mathbb{G}_a(F_Y^{-1}(\tau)+g)$ and $\mathbb{G}_Y(F_Y^{-1}(\tau))$. $\square$

Proof of Theorem 6. The map given in (22) is

$$\phi(H) = \begin{pmatrix} \min\{\max\{0, H(\tau_1)\}, 1\} \\ \max\{\min\{0, H(\tau_2)\}, -1\} \end{pmatrix}.$$

is the composition of an evaluation map $\theta \in \ell^\infty(\delta, 1-\delta) \mapsto (\theta(\tau_1), \theta(\tau_2))$ and of the max/min composition. The evaluation map is linear, hence fully Hadamard differentiable. The composition of max/min is Hadamard directional differentiable by the chain rule for Hadamard directional differentiable maps (see Proposition 3.6 in Shapiro (1990); Lemma C2 of Masten and Poirier (2020)). Hence, another application of the chain rule yields that $\phi(H)$ is Hadamard directional differentiable at any $H \in \ell^\infty(\delta, 1-\delta)$ tangentially to $\ell^\infty(\delta, 1-\delta)$. By direct computation, the derivative, for any $h \in \ell^\infty(\delta, 1-\delta)$, is given by the map

$$\phi'_H(h) = \begin{pmatrix} h(\tau_1)\mathbb{1}_{\{0<H(\tau_1)<1\}} + \max(0, h(\tau_1))\mathbb{1}_{\{H(\tau_1)=0\}} + \min(0, h(\tau_1))\mathbb{1}_{\{H(\tau_1)=1\}} \\ h(\tau_2)\mathbb{1}_{\{-1<H(\tau_2)<0\}} + \min(0, h(\tau_2))\mathbb{1}_{\{H(\tau_2)=0\}} + \max(0, h(\tau_2))\mathbb{1}_{\{H(\tau_2)=-1\}} \end{pmatrix}. \tag{A.2}$$

Combining (A.2) with Theorem 2.1 in Fang and Santos (2019) and the result of Theorem 5, we arrive at

$$\sqrt{n}\begin{pmatrix} \hat{U}_{\tau_1} - U_{\tau_1} \\ \hat{L}_{\tau_2} - L_{\tau_2} \end{pmatrix} = \sqrt{n}(\phi(\hat{\theta}) - \phi(\theta)) \rightsquigarrow \phi'_\theta(\mathbb{G}_\theta),$$

where

$$\phi'_\theta(\mathbb{G}_\theta) = \begin{pmatrix} \mathbb{G}_\theta(\tau_1)\mathbb{1}_{\{0<\theta(\tau_1)<1\}} + \max(0, \mathbb{G}_\theta(\tau_1))\mathbb{1}_{\{\theta(\tau_1)=0\}} + \min(0, \mathbb{G}_\theta(\tau_1))\mathbb{1}_{\{\theta(\tau_1)=1\}} \\ \mathbb{G}_\theta(\tau_2)\mathbb{1}_{\{-1<\theta(\tau_2)<0\}} + \min(0, \mathbb{G}_\theta(\tau_2))\mathbb{1}_{\{\theta(\tau_2)=0\}} + \max(0, \mathbb{G}_\theta(\tau_2))\mathbb{1}_{\{\theta(\tau_2)=-1\}} \end{pmatrix}.$$

□

Proof of Theorem 7. Recall that by (22)

$$\phi(\theta) = \begin{pmatrix} U_{\tau_1} \\ L_{\tau_2} \end{pmatrix},$$

This map is not fully differentiable with respect to θ, only directional differentiable. Now, for fixed τ, consider the map

$$\psi(F_a, F_Y, \theta, \tau) = \begin{pmatrix} F_a^{-1}(\tau - \delta\phi(\theta)_1) - F_Y^{-1}(\tau) \\ F_a^{-1}(\tau - \delta\phi(\theta)_2) - F_Y^{-1}(\tau) \end{pmatrix}, \tag{A.3}$$

where $\phi(\theta)_1$ and $\phi(\theta)_2$ are the first and second coordinates of $\phi(\theta)$ respectively. We want to find the distribution of

$$\sqrt{n}\left(\psi(\hat{F}_a, \hat{F}_Y, \hat{\theta}, \tau) - \psi(F_a, F_Y, \theta, \tau)\right)$$

Recall the notation introduced before: $\mathbb{D}_\delta \subset \ell^\infty(\mathcal{Y})$ denotes the set of all restrictions of distribution functions on $\mathbb{R}$ to $[F_Y^{-1}(\delta) - \varepsilon, F_Y^{-1}(1-\delta) + \varepsilon]$ for some $\varepsilon > 0$. Additionally, $\mathbb{C}_\delta$ is set of continuous functions on $[F_Y^{-1}(\delta) - \varepsilon, F_Y^{-1}(1-\delta) + \varepsilon]$.

Consider the map from $\mathbb{D}_\delta^2 \times \ell^\infty(\delta, 1-\delta) \mapsto \ell^\infty(\delta, 1-\delta)^2 \times [0,1] \times [-1,0]$ given by

$$m(H_1, H_2, H_3) = (H_1^{-1}, H_2^{-1}, \phi(H_3)_1, \phi(H_3)_2), \tag{A.4}$$

for ϕ defined in (22). Now consider the map from $\ell^\infty(\delta, 1-\delta)^2 \times [0,1] \times [-1,0] \mapsto \ell^\infty(\delta, 1-\delta)^2$ given by

$$q(H_1, H_2, H_3, H_4) = \begin{pmatrix} H_1(\cdot - \delta H_3) - H_2(\cdot) \\ H_1(\cdot - \delta H_4) - H_2(\cdot) \end{pmatrix}. \tag{A.5}$$

We can see that ψ in (A.3) is the composition

$$\psi(F_a, F_Y, \theta, \cdot) = q \circ m(F_a, F_Y, \theta).$$

By Assumptions 6 and 7, Lemma 21.4(i) in van der Vaart (1998), Theorem 5 and the chain rule for Hadamard directional differentiable maps, the map m is Hadamard directional differentiable (see Proposition 3.6 in Shapiro (1990); Lemma C2 of Masten and Poirier (2020)) at $(F_a, F_Y, \theta(\tau_1), \theta(\tau_2))$ tangentially to $\mathbb{C}_\delta^2 \times \ell^\infty(\delta, 1-\delta)$, with derivative given by the map

$$m'_{(F_a, F_Y, \theta)}(h_1, h_2, h_3) = \left(-\frac{h_1 \circ F_a^{-1}}{f_a \circ F_a^{-1}}, -\frac{h_2 \circ F_Y^{-1}}{f_Y \circ F_Y^{-1}}, \phi'_\theta(h_3)_1, \phi'_\theta(h_3)_2 \right) \tag{A.6}$$

where the map $h \mapsto \phi'_H(h)$ is given in (A.2), and $\phi'_H(h)_1$ and $\phi'_H(h)_2$ are the first and second coordinates respectively.

The map $q(H_1, H_2, H_3, H_4)$ in (A.5) has Hadamard derivative at $(F_a^{-1}, F_Y^{-1}, U_{\tau_1}, L_{\tau_2})$ tangentially to $\mathbb{UC}(\delta, 1-\delta) \times \ell^\infty(\delta, 1-\delta) \times [0,1] \times [-1,0]$ given by the map

$$q'_{(F_a^{-1}, F_Y^{-1}, U_{\tau_1}, L_{\tau_2})}(h_1, h_2, h_3, h_4) = \begin{pmatrix} h_1(\cdot - \delta U_{\tau_1}) - \frac{\delta h_4}{f_a \circ F_a^{-1}(\cdot - \delta U_{\tau_1})} - h_2(\cdot) \\ h_1(\cdot - \delta L_{\tau_2}) - \frac{\delta h_3}{f_a \circ F_a^{-1}(\cdot - \delta L_{\tau_2})} - h_2(\cdot) \end{pmatrix}.$$

We use the chain rule to conclude that the map $q \circ m$ has Hadamard directional derivative at

(F_a, F_Y, θ) tangentially to $\mathbb{C}^2_\delta \times \ell^\infty(\delta, 1-\delta)$ given by the map

$$\begin{aligned}
(q \circ m)'_{(F_a,F_Y,\theta)}(h_1,h_2,h_3) &= q'_{m(F_a,F_Y,\theta)} \circ m'_{(F_a,F_Y,\theta)}(h_1,h_2,h_3) \\
&= q'_{(F_a^{-1},F_Y^{-1},L_{\tau_2},U_{\tau_1})} \\
&\circ \left(-\frac{h_1 \circ F_a^{-1}}{f_a \circ F_a^{-1}}, -\frac{h_2 \circ F_Y^{-1}}{f_Y \circ F_Y^{-1}}, \phi'_\theta(h_3)_1, \phi'_\theta(h_3)_2 \right) \\
&= \begin{pmatrix} -\frac{h_1 \circ F_a^{-1}(\cdot - \delta U_{\tau_1})}{f_a \circ F_a^{-1}(\cdot - \delta U_{\tau_1})} - \frac{\delta \phi'_\theta(h_3)_2}{f_a \circ F_a^{-1}(\cdot - \delta U_{\tau_1})} - \frac{h_2 \circ F_Y^{-1}}{f_Y \circ F_Y^{-1}(\cdot)} \\ -\frac{h_1 \circ F_a^{-1}(\cdot - \delta L_{\tau_2})}{f_a \circ F_a^{-1}(\cdot - \delta L_{\tau_2})} - \frac{\delta \phi'_\theta(h_3)_1}{f_a \circ F_a^{-1}(\cdot - \delta L_{\tau_2})} - \frac{h_2 \circ F_Y^{-1}}{f_Y \circ F_Y^{-1}(\cdot)} \end{pmatrix}.
\end{aligned}$$

Using Assumption 5, Theorem 5 and Theorem 2.1 in Fang and Santos (2019), we conclude that

$$\begin{aligned}
\sqrt{n}\left(\psi(\hat{F}_a, \hat{F}_Y, \hat{\theta}, \cdot) - \psi(F_a, F_Y, \theta, \cdot)\right) &\rightsquigarrow (q \circ m)'_{(F_a,F_Y,\theta)}(\mathbb{G}_a, \mathbb{G}_Y, \mathbb{G}_\theta) \\
&= \begin{pmatrix} -\frac{\mathbb{G}_a \circ F_a^{-1}(\cdot - \delta U_{\tau_1})}{f_a \circ F_a^{-1}(\cdot - \delta U_{\tau_1})} - \frac{\delta \phi'_\theta(\mathbb{G}_\theta)_2}{f_a \circ F_a^{-1}(\cdot - \delta U_{\tau_1})} - \frac{\mathbb{G}_Y \circ F_Y^{-1}(\cdot)}{f_Y \circ F_Y^{-1}(\cdot)} \\ -\frac{\mathbb{G}_a \circ F_a^{-1}(\cdot - \delta L_{\tau_2})}{f_a \circ F_a^{-1}(\cdot - \delta L_{\tau_2})} - \frac{\delta \phi'_\theta(\mathbb{G}_\theta)_1}{f_a \circ F_a^{-1}(\cdot - \delta L_{\tau_2})} - \frac{\mathbb{G}_Y \circ F_Y^{-1}(\cdot)}{f_Y \circ F_Y^{-1}(\cdot)} \end{pmatrix},
\end{aligned}$$

and convergence takes place in $\ell^\infty(\delta, 1-\delta) \times \ell^\infty(\delta, 1-\delta)$.

□

Proof of Theorem 8. For $d = 0$ or $d = 1$, we find the asymptotic distribution of $\sqrt{n}(\hat{F}_{Y|D=d} \circ \hat{F}_Y^{-1} - F_{Y|D=d} \circ F_Y^{-1})$. Consider first the map $\psi : \mathbb{D}(\mathcal{Y})^2 \to \mathbb{D}(\mathcal{Y}) \times \ell^\infty(0,1)$, given by $\psi(H_1, H_2) = (H_1, H_2^{-1})$. Here, $\mathbb{D}(\mathcal{Y})$ is the set of all restrictions of distribution functions on $\mathbb{R}$ to $\mathcal{Y} = [y_l, y_u]$, such that they give mass 1 to $(y_l, y_u]$. Also, $\mathbb{C}(\mathcal{Y})$ is the set of all (uniformly) continuous functions defined on $\mathcal{Y}$.

By Lemma 21.4.(ii) in van der Vaart (1998), and Assumption 9, ψ is Hadamard differentiable tangentially to $\ell^\infty(\mathcal{Y}) \times \mathbb{C}(\mathcal{Y})$ at $(F_{Y|D=d}, F_Y)$, with derivative given by the map

$$\psi'_{(F_{Y|D=d},F_Y)}(h_1,h_2) = \left(h_1, -\frac{h_2 \circ F_Y^{-1}}{f_y \circ F_Y^{-1}} \right).$$

Now, consider the map $\phi : \mathbb{D}(\mathcal{Y}) \times \ell^\infty(0,1) \to \ell^\infty(0,1)$ given by $\phi(H_1, H_2) = H_1 \circ H_2^{-1}$. By Lemmas 3.9.25 and 3.9.27 in van der Vaart and Wellner (1996), and Assumption 9, ϕ has Hadamard derivative at $(F_{Y|D=d}, F_Y^{-1})$ tangentially to $\mathbb{UC}(\mathcal{Y}) \times \ell^\infty(0,1)$ given by the map

$$\phi'_{(F_{Y|D=d},F_Y^{-1})}(h_1,h_2) = h_1 \circ F_Y^{-1} + f_{Y|D=d} \circ F_Y^{-1} \cdot h_2.$$

We use the chain rule (see Theorem 20.9 in van der Vaart (1998)) to conclude that $\phi \circ \psi$ has

Hadamard derivative at $(F_{Y|D=d}, F_Y)$ tangentially to $\mathbb{UC}(\mathcal{Y}) \times \mathbb{C}(\mathcal{Y})$ given by the map

$$
\begin{aligned}
(\phi \circ \psi)'_{(F_{Y|D=d}, F_Y)}(h_1, h_2) &= \phi'_{\phi(F_{Y|D=d}, F_Y)} \circ \psi'_{(F_{Y|D=d}, F_Y)}(h_1, h_2) \\
&= \phi'_{(F_{Y|D=d}, F_Y^{-1})} \circ (h_1, -h_2 \circ F_Y^{-1} / f_Y \circ F_Y^{-1}) \\
&= h_1 \circ F_Y^{-1} - f_{Y|D=d} \circ F_Y^{-1} \cdot \frac{h_2 \circ F_Y^{-1}}{f_Y \circ F_Y^{-1}}.
\end{aligned}
$$

By the functional Delta method (see Theorem 20.8 in van der Vaart (1998)) we have that

$$
\begin{aligned}
\sqrt{n}(\hat{F}_{Y|D=d} \circ \hat{F}_Y^{-1} - F_{Y|D=d} \circ F_Y^{-1}) &\rightsquigarrow (\phi \circ \psi)'_{(F_{Y|D=d}, F_Y)}(\mathbb{G}_d, \mathbb{G}_Y)) \\
\mathbb{G}_{d,Y} &:= \mathbb{G}_d \circ F_Y^{-1} - f_{Y|D=d} \circ F_Y^{-1} \cdot \frac{\mathbb{G}_Y \circ F_Y^{-1}}{f_Y \circ F_Y^{-1}}.
\end{aligned}
$$

By the continuous mapping theorem

$$
\begin{aligned}
\sqrt{n}(\hat{\theta} - \theta) &= \sqrt{n}\left(\hat{F}_{Y|D=0} \circ \hat{F}_Y^{-1} - \hat{F}_{Y|D=1} \circ \hat{F}_Y^{-1} - \left(F_{Y|D=0} \circ F_Y^{-1} - F_{Y|D=1} \circ F_Y^{-1}\right)\right) \\
&\rightsquigarrow \mathbb{G}_{0,Y} - \mathbb{G}_{1,Y}.
\end{aligned}
$$

□

B A Closer Look at the Marginal Effect

This appendix contains example that show the difficult of justifying the existence of the marginal effect. The first example shows that the distributional invariance assumption of Firpo, Fortin and Lemieux (2007, 2009) identifies the marginal effect.

Example B.1 (Distributional Invariance)**.** *If we assume distributional invariance: for $d = 0, 1$, then $F_{Y_{D_\delta}|D_\delta=d}(y) = F_{Y|D=d}(y)$, we obtain*

$$
\begin{aligned}
F_{Y_{D_\delta}}(y) &= (p+\delta) F_{Y_{D_\delta}|D_\delta=1}(y) + (1-p-\delta) F_{Y_{D_\delta}|D_\delta=0}(y) \\
&= (p+\delta) F_{Y|D=1}(y) + (1-p-\delta) F_{Y|D=0}(y) \\
&= F_Y(y) + \delta\left(F_{Y|D=1}(y) - F_{Y|D=0}(y)\right),
\end{aligned}
$$

where the last line uses the decomposition $F_Y(y) = pF_{Y|D=1}(y) + (1-p)F_{Y|D=0}(y)$. Now,

$$
\frac{F_{Y_{D_\delta}}(y) - F_Y(y)}{\delta} = F_{Y|D=1}(y) - F_{Y|D=0}(y)
$$

which implies trivially that $\dot{F}_{Y,\mathcal{D}}(y) = F_{Y|D=1}(y) - F_{Y|D=0}(y)$. Note further that $\dot{F}_{Y,\mathcal{D}}$ is independent of $\mathcal{D}$.

The next example illustrates a case where $F_{Y_{D_0}}(y) \neq F_Y(y)$.

Example B.2 (Threshold Crossing Model). *Suppose that individuals select into treatment by $D = \mathbb{1}\{V \leq p\}$ for $V \sim U_{[0,1]}$. Consider the sequence of policies $D_\delta = \mathbb{1}\{V \leq p + \delta\}$, and $\delta \geq 0$. Then,*

$$F_{Y_{D_\delta}}(y) = \Pr(Y \leq y | V \leq p + \delta)(p + \delta) + \Pr(Y \leq y | V > p + \delta)(1 - p - \delta)$$

In this case, $F_{Y_{D_0}} = F_Y$. However, if the sequence of policies is $D_\delta = \mathbb{1}\{V > 1 - p - \delta\}$, then $F_{Y_{D_0}}$ might not coincide with F_Y.

The next three examples show that the condition of uniform differentiability may easily fail.

Example B.3 (Effect at the Margin). *For the case of a non-randomized policy that satisfies Assumption 2, equation (3) in Theorem 1 can be written as*

$$\frac{F_{Y_{D_\delta}}(y) - F_Y(y)}{\delta} = F_{Y(1)|D=0,D_\delta=1}(y) - F_{Y(0)|D=0,D_\delta=1}(y).$$

It is not immediate that the limit when δ goes to 0 of the right hand side exists pointwise for any $y \in \mathcal{Y}$. This is not obvious, since the conditioning set $D = 0, D_\delta = 1$ shrinks to a measure 0 set: those individuals whose $D = 0$ and $D = 1$.

In the case of a threshold-crossing model for D, as in $D = \mathbb{1}\left\{V \leq F_V^{-1}(p)\right\}$, and a sequence of policies such that $D_\delta = \mathbb{1}\left\{V \leq F_V^{-1}(p + \delta)\right\}$, the event $\{D = 0, D_\delta = 1\}$ is equivalent to the event $\left\{0 \leq V \leq F_V^{-1}(p + \delta)\right\}$, so we can define the limiting conditioning probability to be

$$\begin{aligned} \lim_{\delta \to 0} \frac{F_{Y_{D_\delta}}(y) - F_Y(y)}{\delta} &= \lim_{\delta \to 0} F_{Y(1)|D=0,D_\delta=1}(y) - \lim_{\delta \to 0} F_{Y(0)|D=0,D_\delta=1}(y) \\ &= F_{Y(1)|V=0}(y) - F_{Y(0)|V=0}(y), \end{aligned}$$

If the distributions $F_{Y(1)|V=0}$ and $F_{Y(0)|V=0}$ are continuous then pointwise convergence is equivalent to uniform convergence. This implies that $\dot{F}_{Y,\mathcal{D}}(y) = F_{Y(1)|V=0}(y) - F_{Y(0)|V=0}(y)$ which is the familiar result (see Martinez-Iriarte and Sun (2020)) that the marginal individuals, those whose $V = 0$, are the ones that drive the marginal effect in a threshold-crossing model. The effect is policy dependent because it entails a particular departure in the selection equation. This policy dependence is emphasized in Carneiro, Heckman and Vytlacil (2010, 2011).

Example B.4 (Unconfoundedness). *Consider the following structural model for the observed outcome $Y = h(D, X, U)$. The counterfactual outome is given by $Y_\delta = h(D_\delta, X, U)$. Kaplan (2019) analyzes a setting where both $D \perp U \| X$ and $D_\delta \perp U \| X$ for every $D_\delta \in \mathcal{D}$ hold. Let $\mathcal{X}_d$ denote the common support* *of $X|D = d$ and $X|D_\delta = d$.*

$$
\begin{aligned}
F_{Y_{D_\delta}}(y) &= (p+\delta)\int_{\mathcal{X}_1} F_{Y_{D_\delta}|D_\delta=1,X=x}(y)dF_{X|D_\delta=1}(x) + (1-p-\delta)\int_{\mathcal{X}_0} F_{Y_{D_\delta}|D_\delta=0,X=x}(y)dF_{X|D_\delta=0}(x) \\
&= (p+\delta)\int_{\mathcal{X}_1}\int_{\mathcal{Y}} 1\{h(1,x,U)\le y\}\, dF_{U|D_\delta=1,X=x}(u)dF_{X|D_\delta=1}(x) \\
&+ (1-p-\delta)\int_{\mathcal{X}_0}\int_{\mathcal{Y}} 1\{h(0,x,U)\le y\}\, dF_{U|D_\delta=0,X=x}(u)dF_{X|D_\delta=0}(x)
\end{aligned}
$$

Using $D \perp U \| X$ and $D_\delta \perp U \| X$ we get

$$
\begin{aligned}
F_{Y_{D_\delta}}(y) &= (p+\delta)\int_{\mathcal{X}_1}\int_{\mathcal{Y}} 1\{h(1,x,U)\le y\}\, dF_{U|D=1,X=x}(u)dF_{X|D_\delta=1}(x) \\
&+ (1-p-\delta)\int_{\mathcal{X}_0}\int_{\mathcal{Y}} 1\{h(0,x,U)\le y\}\, dF_{U|D=0,X=x}(u)dF_{X|D_\delta=0}(x) \\
&= (p+\delta)\int_{\mathcal{X}_1} F_{Y|D=1,X=x}(y)dF_{X|D_\delta=1}(x) \\
&+ (1-p-\delta)\int_{\mathcal{X}_0} F_{Y|D=0,X=x}(y)dF_{X|D_\delta=0}(x).
\end{aligned}
$$

We can write this last expression as

$$
\begin{aligned}
F_{Y_{D_\delta}}(y) &= F_Y(y) + p\int_{\mathcal{X}_1} F_{Y|D=1,X=x}(y)d\left(F_{X|D_\delta=1}(x) - F_{X|D=1}(x)\right) \\
&+ (1-p)\int_{\mathcal{X}_0} F_{Y|D=0,X=x}(y)d\left(F_{X|D_\delta=0}(x) - F_{X|D=0}(x)\right) \\
&+ \delta\left[\int_{\mathcal{X}_1} F_{Y|D=1,X=x}(y)dF_{X|D_\delta=1}(x) - \int_{\mathcal{X}_0} F_{Y|D=0,X=x}(y)dF_{X|D_\delta=0}(x)\right].
\end{aligned}
$$

It shows that the counterfactual distribution is identified. Under some regularity conditions, we have that, pointwise in $y \in \mathcal{Y}$

$$
\begin{aligned}
\lim_{\delta\to 0}\frac{F_{Y_{D_\delta}}(y) - F_Y(y)}{\delta} &= p\int_{\mathcal{X}_1} F_{Y|D=1,X=x}(y)\left.\frac{\partial f_{X|D_\delta=1}(x)}{\partial\delta}\right|_{\delta=0} dx \\
&+ (1-p)\int_{\mathcal{X}_0} F_{Y|D=0,X=x}(y)\left.\frac{\partial f_{X|D_\delta=0}(x)}{\partial\delta}\right|_{\delta=0} dx \\
&+ F_{Y|D=1}(y) - F_{Y|D=0}(y).
\end{aligned}
$$

The regularity conditions relate to the smoothness of the maps $\delta \mapsto F_{X|D_\delta=d}(x)$ for $d = 0, 1$. Uniform convergence is not guaranteed though.

Example B.5 (Propensity Score Manipulation)**.** *Martinez-Iriarte and Sun (2020) consider a setting where $D = \mathbb{1}\{U_D \le \mu(X)\}$, $U_D \sim U_{(0,1)}$, and a sequence of policies that satisfy $E[P_\delta(X)] = P(X) + \delta$,* *where $P(X)$ is the propensity score, $P(X) := \Pr(D = 1|X)$ and $P_\delta(X) := \Pr(D_\delta = 1|X)$. In this case, the counterfactual distribution can be written as*

$$\begin{aligned} F_{Y_\delta}(y) &= F_Y(y) + \delta E\left[\left\{F_{Y(1)|U_D,X}(y|P(X), X) - F_{Y(0)|U_D,X}(y|P(X), X)\right\}\dot{P}(X)\right] \\ &+ R(\delta), \end{aligned}$$

where, as δ goes to 0, $R(\delta)$ goes to 0 uniformly in $y \in \mathcal{Y}$. $\dot{P}(x)$ is the reaction of the propensity score to the sequence policies:

$$\dot{P}(x) := \left.\frac{\partial P_\delta(x)}{\partial \delta}\right|_{\delta=0}.$$

Using the uniformity in $y \in \mathcal{Y}$ of the remainder $R(\delta)$, we obtain

$$\dot{F}_{Y,\mathcal{D}}(y) = E\left[\left\{F_{Y(1)|U_D,X}(y|P(X), X) - F_{Y(0)|U_D,X}(y|P(X), X)\right\}\dot{P}(X)\right].$$

Here, the adjustment term $\dot{P}(x)$ will be different for different sequences $\mathcal{D}$ considered.