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CIRCULAR NETWORKS AS EFFICIENT NASH EQUILIBRIA: TWO APPROACHES

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Abstract

The present paper analyzes a network formation problem, and in particular the existence of circular networks that constitute efficient Nash equilibria. We consider two ways in which they may arise as solutions. One in the framework presented by Bala and Goyal. In it an agent receives a payoff which is increasing in the number of agents to which he is directly or indirectly connected, while it is decreasing in the number of agents to whom he is directly connected. The other approach departs from their assumptions in two crucial aspects. On one hand, we assume that connecting to an agent pays off not only for the number of connections that the agent can provide but also for her intrinsic value. On the other hand, we assume that each path connecting two agents has an associated cost which is the sum of the number of edges it includes, and which has to be paid by each agent in the path. In both approaches it is possible to obtain circular networks as efficient Nash equilibria. But, while in Bala and Goyal's approach this is only one possibility (the other is the empty network), in our alternative approach, if the number of agents is larger than 3, it is the unique result.

1 Introduction

In the last decade, the study of the structure of interactions among agents in different contexts has grown to the point to constitute the core of a new area in the social sciences. The tools of this field, the study of *social networks*, are currently being applied in different disciplines, ranging from sociology to anthropology. and of course in economics.

The mathematics behind the analysis of social networks is provided by *graph theory*. Here the nodes are interpreted as individual agents and the edges the links over which the agents carry out exchanges (Wasserman and Faust (1994)). Game theory has been also recently applied in the analysis of networks, being the fundamental tool in the study of their normative properties, particularly their stability or efficiency (Jackson and Wolinsky (1996), Bala and Goyal (2000), Dutta and Jackson (2000, 2001)). The strategic approach focuses on the individual strategies available to the agents as well as on the corresponding individual payoff functions. The decisions the agents have to make concern their connection to other agents. The rational choices of the agents lead to Nash equilibria that may support various types of networks.

A modelling primitive is the representation of the links between agents as directed or non-directed. While the latter are important to represent situations in which the direction of flow of utility goods is less important or irrelevant (Dutta and Mutuswami (1997)), the former are important to distinguish which agent initiated the connection and the direction of flow of the ensuing flow (Bala and Goyal (2000), Dutta and Jackson (2001)).

In this paper we consider networks only as directed graphs with one-way flows. We denote by "information" the good that flows in the networks. Each agent is endowed with some amount of information. This is important for the alternative approach of Larrosa and Tohmé (2002) in which, contrary to Bala and Goyal, the amount of goods to which an agent has access is also relevant and not only the amount of agents that provide them. By establishing links to other agents she can obtain the information held by them, but she has to pay a small fee to establish those links. The problem is to determine which structures can arise as strategic equilibria in the interaction among the agents and whether they are optimal or not.

In Bala and Goyal's framework (**BG**, from now on) the payoff of an agent is given by the number of agents to which she has a direct or indirect connection, while the cost is in terms of the direct links she has to establish. In the alternative presentation of Larrosa and Tohmé (2002) (**LT** in the following) the payoff represents the total information to which she has access, while the cost is now the total number of links in the paths that allow her to accede to the information. We will see that in both cases Nash networks arise as efficient solutions in the network formation game. The difference is that in our approach this is the only possible solution once the number of agents is larger than 3.

In Section 2 we will begin our analysis with a presentation of the basic model, that is valid both for BG and LT. In Section 3 we will determine the equilibria for both approaches and show how they verify some criteria of stability and optimality. Finally, in Section 4 we discuss briefly why the results in BG cannot be applied to our approach.

2 The Model

Let $N = (1, \ldots, n)$ be a set of agents. To avoid trivial results we will always assume that $n \geq 3$. If i and j are two typical members of N, a link among them, without intermediaries, originated by i and ending in j will be represented as ij. The interpretation of ij is that i establishes a contact with j that allows i to get acquainted with both the information possessed by j as well as her network of contacts. Each agent $i \in N$ has some information of her own, $I_i \in Z_+$, (i.e. represented as a nonnegative integer). As said i can have access to more information by forming links with other agents. The formation of links is costly, in time, resources and effort, but for simplicity we will assume that a link ij has a cost of 1 (in units of utility of information). By convention we assume that the information of each agent is worth the cost of establishing a link to her, i.e. that $I_i > 1$.

The agents will try to maximize the utility of the information available to them as well as to minimize the cost of connecting to other agents. In order to do this, they will be endowed with a set of *strategies*. Each strategy for $i \in N$ is a $(n-1)$-dimensional vector $g_i = \langle g_{i,1}, \ldots, g_{i,i-1}, g_{i,i+1} \ldots g_{i,n} \rangle$ where each $g_{i,j}$, for $j \neq i$, is either 0 or 1. This is interpreted as meaning that i establishes a direct link with j if $g_{i,j} = 1$ while if $g_{i,j} = 0$ there is no such direct link. The set of all i's strategies is denoted as G_i. Since we restrict our analysis to only pure strategies, $|G_i| = 2^{n-1}$. Finally, $G = G_1 \times \ldots \times G_n$ denotes the set of strategy profiles in the interaction among the agents in N.

The existence of a direct link ij indicates an asymmetric communication between i and j. That is, $g_{i,j} = 1$ indicates that i establishes a communication with j that permits i to access to j's information but no viceversa (the symmetry between i and j is restored if also $g_{j,i} = 1$). Structures with this feature are called **one-way flow** networks.

In one-way flow networks a strategy profile can be represented as a directed graph $g = (g_1 \ldots g_n)$ over N. That is, in the directed graph the elements of N are the *nodes* while any established link like $g_{i,j} = 1$ is represented by an arrow beginning in j with its head pointing to i.[1] That is, arrowheads always point toward the agent who establishes the link. It follows immediately that:

Proposition 1 *There exists a one-to-one map between directed graphs among n nodes and strategy profiles in G.*

Example 1: consider a group of four agents, $N = \{a, b, c, d\}$. A joint strategy $g = \langle g_a, g_b, g_c, g_d \rangle$ can be represented as a table:

Strategy	**a**	**b**	**c**	**d**
g_a	X	1	0	0
g_b	0	X	1	0
g_c	0	0	X	1
g_d	0	0	0	X

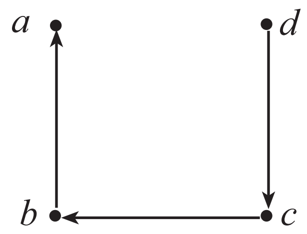
Each row is the strategy chosen by one of the agents. Columns correspond to the agents. An entry 1 in row i and column j means that the strategy of agent i prescribes to establish a link with agent j. Entries in the diagonal are crossed out since agents cannot establish links with themselves. In Figure 1 we can see the directed graph that corresponds to g.

We define $N_d^{g_i} = \{k \in N | g_{i,k} = 1\}$ as the set of agents to whom i establishes a direct link according to her strategy g_i. We say that there exists a *path* from j to i according to $g \in G$ if there exists a sequence of different[2] agents $j_0 \ldots j_m$ (with $i = j_0$ and $j = j_m$) such that $g_{j_0,j_1} = \ldots = g_{j_{m-1},j_m} = 1$. In other words, given the joint strategy g, we have that $j_1 \in N^{g_{j_0}}$, $j_2 \in N^{g_{j_1}}$, …, $j_m \in N^{g_{j_{m-1}}}$. A path from $j = j_m$ to $i = j_0$, denoted as $j \rightarrow_g i$,

[1] In order to represent the idea that when i establishes a link with j, the information flows *from j to i*.

[2] In order to avoid cycles.

Figure 1:



has a *length*, the cardinality of the sequence $j_1, j_2, \ldots, j_{m-1}, j_m$, i.e. m, which indicates the number of intermediary links between j and i. Notice that a direct link is a path of length 1.

Example 1 revisited: Given the strategy $g = \langle g_a, g_b, g_c, g_d \rangle$, we have that $N^{g_a} = \{b\}$, $N^{g_b} = \{c\}$ and $N^{g_c} = \{d\}$ while $N^{g_d} = \emptyset$. This sequence establishes a path from d to a of length 3.

We denote the set of agents accessed (directly and otherwise) by i as $N^{i;g} = \{k \in N | k \rightarrow_g i\} \cup \{i\}$. We include i in $N^{i;g}$ to indicate that i knows her own valuation, despite the fact that we assumed that there is no direct link from i to herself. Let $\mu_i : G \rightarrow \{0, ..., n \times (n-1)\}$ be the number of links in all paths that end in i, originated by agents in $N^{i;g}$ under any given joint strategy: $\mu_i(g) = |\{(j,k) \in N \times N : g_{j,k} = 1, \text{ and } \exists l \in N^{i;g} \text{ and } l \rightarrow_g i \text{ with } j,k \in l \rightarrow_g i\}|$.[3]

An additional functions is $\delta_i^d(g) = |N_d^{i;g}|$ where $N_d^{i;g} = \{j \neq i : g_{i,j} = 1\}$ is the set of agents to which i is directly linked. On the other hand, $\delta_i(g) = |N^{i;g}|$ indicates the number of agents to which i is connected, directly or indirectly.[4]

Example 2: Assume that we have $N = \{1, 2, 3, 4, 5\}$ and the strategy $g = \langle g_1, g_2, g_3, g_4, g_5 \rangle$ given by the following table:

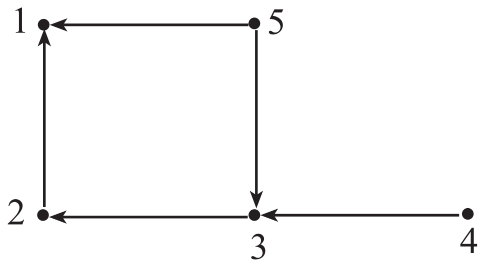
Strategy	**1**	**2**	**3**	**4**	**5**
g_1	X	1	0	0	1
g_2	0	X	1	0	0
g_3	0	0	X	1	1
g_4	0	0	0	X	0
g_5	0	0	0	0	X

Figure 2 shows the corresponding network. We have that $N^{1;g} = \{1, 2, 3, 4, 5\}$, $N^{2;g} = \{2, 3, 4, 5\}$, $N^{3;g} = \{3, 4, 5\}$ while $N^{4;g} = \{4\}$ and $N^{5;g} = \{5\}$. That is, under g we have that 1 can access to the information of all the agents while 4 and 5 have access only to their own information. The number of links required to obtain the information are $\mu_1(g) = 5$, $\mu_2(g) = 3$ and $\mu_3(g) = 2$, while $\mu_4(g) = \mu_5(g) = 0$. On the other hand, we have that $\delta_1(g) = \{1, 2, 3, 4, 5\} = 5$, $\delta_2(g) = \{2, 3, 4, 5\} = 4$,$\delta_3(g) = \{3, 4, 5\} = 3$, $\delta_4(g) = \{4\} = 1$ and $\delta_5(g) = \{5\} = 1$, while $\delta_1^d(g) = \{2, 5\} = 2$, $\delta_2^d(g) = \{3\} = 1$, $\delta_3^d(g) = \{4, 5\} = 2$, $\delta_4^d(g) = \emptyset = 0$ and $\delta_5^d(g) = \emptyset = 0$.

[3] Notice that there may be more than one path from j to i.

[4] In the original presentation of BG, δ_i^d is denoted μ_i^d while δ_i is μ_i. We changed it, in order to avoid confusions.

Figure 2:



To make this framework a game, we have to define the payoffs to agents. Let $\Pi_i : G \rightarrow R$ be a generic payoff function. We will consider two versions, Π_i^{BG} and Π_i^{LT}. The first version is:

$$\Pi_i^{BG}(g) \equiv \delta_i(g) \;\; - \delta_i^d(g)c$$

where c is the cost of establishing each link. Which we assume 1 in the examples. That is, i's payoff is just the number of agents whose information can be accessed by her, less the cost of the direct links that are established according to g.

They generalize this particular form of payoff function by means of a function

$$\Phi(\delta_i(g), \delta_i^d(g))$$

increasing in the first argument and decreasing in the second.

On the other hand, in the LT framework:

$$\Pi_i^{LT}(g) \equiv \sum_{j \in N^{i;g}} I_j - \mu_i(g)$$

That is, i's payoff is just the sum of all the information that can be accessed by her, less the cost of the paths reaching her that are established according to g (recall that each link is assumed to have a unit cost). The intuition here is that i gets a payoff from accessing to more information but at the same time she has to pay a "fee" for each of the links on the paths to the sources of information.

Example 2 revisited: Suppose the information owned by the agents is: $I_1 = 2$, $I_2 = 2$, $I_3 = 4$, $I_4 = 3$ and $I_5 = 3$. Then, under strategy g we have that for BG:

$$\Pi_1^{BG} = \delta_1(g) - \delta_1^d(g) = 5 \;-\; 2 = 3$$
$$\Pi_2^{BG} = \delta_2(g) - \delta_2^d(g) = 4 \;-\; 1 = 3$$
$$\Pi_3^{BG} = \delta_3(g) - \delta_3^d(g) = 3 \;-\; 2 = 1$$
$$\Pi_4^{BG} = \delta_4(g) - \delta_4^d(g) = 1 \;-\; 0 = 1$$
$$\Pi_5^{BG} = \delta_5(g) - \delta_5^d(g) = 1 \;-\; 0 = 1$$

while in LT:

$$\Pi_1^{LT}(g) = I_1 + \cdots + I_5 - \mu_1(g) = 2+2+4+3+3 \ - \ 5 = 9$$
$$\Pi_2^{LT}(g) = I_2 + \cdots + I_5 - \mu_2(g) = 2+4+3+3 \ - \ 3 = 9$$
$$\Pi_3^{LT}(g) = I_3 + \cdots + I_5 - \mu_3(g) = 4+3+3 \ - \ 2 = 8$$
$$\Pi_4^{LT}(g) = I_4 - \mu_4(g) = 3 \ - \ 0 = 3$$
$$\Pi_5^{LT}(g) = I_5 - \mu_5(g) = 3 \ - \ 0 = 3.$$

We can notice here that, for example, if $g_{1,5} = 0$, 1 could improve her Π_1^{LT} payoff (i.e. obtaining 10 instead of 9) because she would still have access to I_5 but using one link less. The same is true for Π_1^{BG} which would go up from 3 to 4.

But, if for instance 3 does not contact 5, 1 will improve her payoff under Π_1^{LT} from 9 to 10, while with Π_1^{BG} she still remains in 3.

For each $g \in G$, agent i obtains a structure $N^{i;g}$ and her payoff depends critically on the type of directed graph that corresponds to $N^{i;g}$ as summarized in the following proposition:

Proposition 2 *Given two joint strategies g and $g^{'}$, $\Pi_i^{BG}(g) \geq \Pi_i^{BG}(g^{'})$ iff the corresponding graphs $N^{i;g}$ and $N^{i,g^{'}}$ are such that:*

$$\delta_i(g) - \delta_i(g^{'}) \geq \delta_{id}(g) - \delta_{id}(g^{'})$$

while $\Pi_i^{LT}(g) \geq \Pi_i^{LT}(g^{'})$ iff the corresponding graphs $N^{i;g}$ and $N^{i,g^{'}}$ are such that:

$$\sum_{j \in N^{i;g}} I_j - \sum_{j \in N^{i;g^{'}}} I_j \geq \mu_i(g) - \mu_i(g^{'}).$$

Proof: *Trivial.* □

This result conveys the intuition that the goal of a rational agent is to get as much information as possible connecting to as few agents as possible (BG) or traversing as few links as possible (LT).

3 Equilibrium and Optimality

Given a network $g \in G$,[5] let g_{-i} be the directed graph obtained by removing all of agent i's direct links. Then, g can be written as $g = g_i \oplus g_{-i}$ where $\oplus$ indicates that g is formed by the union of the links of g_i and those in g_{-i}. A strategy g_i is said the *best response* of agent i to g_{-i} if

$$\Pi_i(g_i \oplus g_{-i}) \geq \Pi_i(g_i^{'} \oplus g_{-i})$$

for all $g_i^{'} \in G_i$

Example 3: *Consider again the case of $N = \{1, 2, 3, 4, 5\}$, where $I_1 = 2$, $I_2 = 2$, $I_3 = 4$, $I_4 = 3$ and $I_5 = 3$. Let g_{-1} be described by the following table:*

[5] According to Proposition 1 we identify a joint strategy g with its corresponding directed graph.

Figure 3:

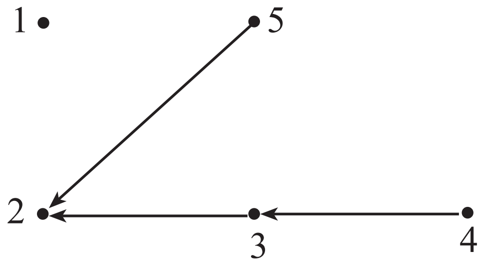
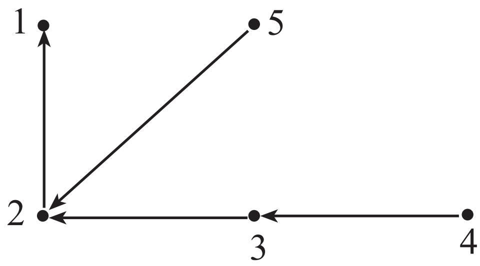


Figure 4:



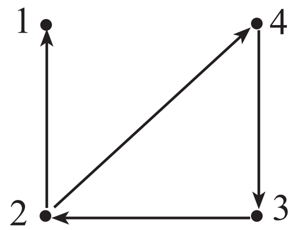
Strategy	**1**	**2**	**3**	**4**	**5**
g_2	0	X	1	0	1
g_3	0	0	X	1	0
g_4	0	0	0	X	0
g_5	0	0	0	0	X

See Figure 3 for the situation faced by 1.

She has to decide to whom establish a connection. A possibility is to remain isolated, but that would give her a BG-payoff of only 1 or a LT-payoff of 2. Alternatively, she could connect to as many of the other agents as she likes. But some connections may be redundant in terms of the gain in information. Such redundancy, in turn, would mean a higher cost for the same information. So, for instance, to connect both to 3 and 4, would ensure 1 to have access to the information of 3 and 4. The number of links required would be 3. The BG-payoff would be $3\ -\ 2 = 1$, while the LT- payoff is then $2 + 4 + 3\ -\ 3 = 6$. She could, instead, connect only to 3, since she would still get hold of the information of 3 and 4 but it would require only 2 links, i.e., her BG-payoff would be 3 1 = 2, while the LT-payoff would be $2 + 4 + 3\ -\ 2 = 7$. A bit of reflection shows that the best answer for 1 would be to connect only to the agent with the higher payoff under g_{-1}. That is, to agent 2, who has a BG-payoff of $4\ -\ 2 = 2$ and a LT-payoff of $2 + 4 + 3 + 3\ -\ 3 = 9$. Then, 1 will reach the information of $2, 3, 4$ and 5, requiring 4 links. That is, her LT-payoff would be of 10 while her BG-payoff would be $5\ -\ 1 = 4$. Figure 4 shows the resulting network.

The set of best responses to g_{-i} is $BR_i(g_{-i})$. A network $g = \langle g_1, \ldots, g_n \rangle$ is said to be a

Figure 5:



Nash network if for each i, $g_i \in BR_i(g_{-i})$ i.e. if g (as a joint strategy) is a Nash equilibrium. In order to determine the structure of Nash networks let us give a few more definitions that will allow us to describe some additional properties of networks.

Given a network g, a set $C \subset N$ is called a *component* of g if for every pair of agents i and j in C ($i \neq j$) we have that $j \in N^{i;g}$ and there does not exist $C^{'}$, $C \subset C^{'}$ for which this is true. A component C it said to be *minimal* if C is not a component anymore once a link $g_{i,j} = 1$ between two agents i and j in C is cut off, i.e. if $g_{i,j} = 0$.

Example 4:*If* $N = \{1, 2, 3, 4\}$*, consider the following network* g*, represented in Figure* 5*:*

Strategy	**1**	**2**	**3**	**4**
g_1	X	1	0	0
g_2	0	X	1	0
g_3	0	0	X	1
g_4	0	1	0	X

Clearly $C = \{2, 3, 4\}$ is a component, since $N^{2;g} = N^{3;g} = N^{4;g} = \{2, 3, 4\}$ and if we consider $N = C \cup \{1\}$, N is not a component, since 1 does not belong to $N^{2;g}$, $N^{3;g}$ or $N^{4;g}$. On the other hand, C is minimal, since if we cut off any of the links 23, 34 or 42 some of the agents are no longer reachable for at least one agent in C. So, for instance, if 23 is cut off, in the new network $g^{'}$ we have that $N^{2,g^{'}} = \{2\}$.

A network g it says to be *connected* if it supports a unique component. If that unique component is minimal, g it says to be *minimally connected*. A network that is not connected it says to be *disconnected*. A *circular network* is one in which the agents can be labelled (by means of a function $l : N \rightarrow N$) as $\{l(1), \ldots, l(n)\}$ and $g_{l(1),l(2)} = g_{l(2),l(3)} = \ldots = g_{l(n-1),l(n)} = g_{l(n),l(1)} = 1$ and there are no other links.

Then, with all these elements at hand we can state the following results:

Lemma 1 • *(BG): A strict Nash network is either empty or circular.*

- *(LT): A strict Nash network is circular.*

Proof: (BG): Bala and Goyal (2000), Proposition 2.1.
(LT): Larrosa and Tohmé (2002), Lemma 1. □

In fact, both results can be shown to be equivalent, when c the cost of establishing a direct link in BG is assumed to be $c = 1$. On the other hand, if $c > n - 1$, the unique strict Nash network in BG is the empty network. According to this result, if $c \leq 1$ a stable outcome in the strategic interaction of agents is the circular network [6]. We claim that it is stable because there is no incentives, once the circular structure arises, to cut-off links and form new ones, because the new configuration may at best achieve the same payoffs to the agents. This argument raises the question of the *optimality* of the outcome. That is, is there another configuration that may ensure better payoffs to the agents? Before answering negatively this question, let us introduce two different notions of optimality that may be worth to consider. One represents the notion of *social welfare* ensured by a network. Formally, let $W : G \rightarrow Z$ defined as $W(g) = \sum_{i=1}^{n} \Pi_i(g)$ for $g \in G$. A network g is said *efficient* if $W(g) \geq W(g^{'})$ for all $g^{'} \in G$.

On the other hand, we have the notion of *Pareto optimality*. A network g is said Pareto optimal if there does not exist another network $g^{'}$ such that for each $i \in N$, $\Pi_i(g^{'}) \geq \Pi_i(g)$ and for at least one i, $\Pi_i(g^{'}) > \Pi_i(g)$.

We have then the following result:

Proposition 3 *A strict Nash as defined network by (LT) and (BG) is both efficient and Pareto optimal.*

This result is true for both approaches, but while in LT it means that the circular network is efficient and Pareto optimal, in BG it means the same only if $c \leq 1$, while if $c > n - 1$ it indicates that the empty network is efficient and Pareto optimal.

4 Discussion

We presented in this paper a model of network formation as a non-cooperative game where agents decide to whom to link by comparing the net benefits from their actions. We introduced BG payoffs and LT ones. In the LT framework we dropped the possibility of having an empty network as a Nash equilibrium. Moreover, it can be easily seen that the comparison between LT and BG yields some surprising differences. On one hand, payoffs in BG are negatively affected by costly direct links. To translate this into the language of LT, the unique direct link established by each agent in a circular network has a cost of n. While with this cost LT still yields the circular network as outcome, in BG we would have the empty network as the strict Nash network. Therefore, results, albeit similar in some sense, can be reinterpreted as having very different meanings.

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[6]Watts (2002) showed that a circular network is also a stable outcome when considering non-myopic agents and first link's cost exceeds its benefits. She used a sequential game with discount factors attached to game rounds. Non-myopic agents could foresee the cooperative gains of create links avoiding the attractor of empty network equilibrium which is not a socially optimum outcome.

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