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Variable population manipulations of reallocation rules in economies with single-peaked preferences*

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Abstract

In a one-commodity economy with single-peaked preferences and individual endowments, we study different ways in which reallocation rules can be strategically distorted by affecting the set of active agents. We introduce and characterize the family of *monotonic* reallocation rules and show that each rule in this class is *withdrawal-proof* and *endowments-merging-proof*, at least one is *endowments-splitting-proof* and that no such rule is *pre-delivery-proof.*

Journal of Economic Literature Classification Numbers: D63, D71, D82
Keywords: single-peakedness, withdrawal-proofness, endowments-merging-proofness, endowments-splitting-proofness, pre-delivery-proofness.

1 Introduction

In the context of a variable population model of an economy consisting of one non-disposable commodity and agents with individual endowments of that commodity, we study different ways in which reallocation rules, i.e., systematic ways of selecting reallocations for each possible configuration of agents' preferences and endowments, can be strategically distorted by affecting the set of active agents. We limit our analysis

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to the case where agents' preferences are single-peaked: up to some critical level, called the peak, an increase in an agent's consumption raises her welfare; beyond that level, the opposite holds. This model has been extensively studied (see, for example, Bonifacio, 2015; Klaus et al., 1997, 1998). We allow for variable population as in Moreno (1996, 2002).[1] To illustrate this type of problem, consider the distribution of a task (e.g., teaching hours) among the members of a group with concave disutility of labor (which induces single-peaked preferences over the time they dedicate to work). From one period to the next one, external factors (e.g., research and administrative duties) might affect preferences and a reallocation of the time assigned to each agent in the first period (taken as benchmark for the second period) could benefit everyone. Another application of this model is a pollution problem in which countries have different rates of pollution and could trade via money transfers their pollution quotas.

Our analysis will be conducted over reallocation rules which are *own-peak-only* (the sole information collected by the rule from an agent's preference to determine her reallocation is her peak amount) and meet the *endowments lower bound* (no agent is made worse off than at her endowment). Two monotonicity properties are appealing in this model. First, a population monotonicity. Since variable population is allowed, it is natural to ask for a monotonicity condition requiring the arrival of new agents to affect all agents present before the change in the same direction. Adding the proviso that agents entering the economy do not change the sign of the excess demand of the economy, we get the property of *one-sided population monotonicity* (see Thomson, 1995b). Second, a resource monotonicity. If, in case of excess demand, the individual endowments decrease (or increase in case of excess supply), then no individual is better off after the change. We call this property *one-sided endowments monotonicity* (see Thomson, 1994b).

Our first result is a characterization of the family of reallocation rules that satisfy the four previously mentioned properties (Theorem 1). This family of reallocation rules, which we call "monotonic", resembles the family of weakly sequential reallocation rules presented by Bonifacio (2015) in that each rule in the family can be described by a procedure that, at each stage, guarantees levels of consumption to the agents that are adjusted throughout the process.

Following Thomson (2014), we examine robustness of monotonic reallocation rules to various types of manipulations by affecting the set of active agents. The manipulations we consider are the following:

(i) Instead of participating, an agent withdraws with her endowment. The rule is applied without her. She then trades with one of the agents that did participate the resources they control between the two of them in such a way that both end up better off.

[1] One-commodity economies with single-peaked preferences and *a social endowment* are studied, for example, by Sprumont (1991) assuming a fixed population. The first studies of the extension of this model to allow for variable populations are Thomson (1994a, 1995b).

(ii) Two agents can merge their endowments, and one of them withdraw. The rule is applied without this second agent. The agent who stays may be assigned an amount that can be divided between the two in such a way that both become at least as well off as they would have been without the manipulation, and at least one of them is better off.

(iii) An agent may split her endowment with some outsider (an agent with no endowment). The rule is applied and the guest then transfers her assignment to the agent who invited her in. The first agent may prefer her final assignment to what she would have received without the manipulation.

(iv) An agent may make a pre-delivery to some other agent of the trade the latter would be assigned if she participated. The rule is applied without the second agent. At her final assignment, the first agent may be better off than she would have been without the manipulation.

A rule immune to the first type of manipulation is called *withdrawal-proof*, and a rule immune to the second type of manipulation is *endowments-merging-proof*. It turns out that all monotonic rules satisfy both properties (Corollary 1 and Lemma 4). Immunity to the third type of manipulation is called *endowments-splitting-proofness*, and is satisfied by the proportional reallocation rule (Remark 2). Finally, a rule immune to the last type of manipulation is *pre-delivery-proof*. No monotonic reallocation rule satisfies this property (Corollary 2).

The rest of the paper is organized as follows. In Section 2 the model and some basic properties of reallocation rules are presented. In Section 3, monotonic reallocation rules are defined and characterized. The different variable population manipulations are discussed in Section 4. Final comments are gathered in Section 5.

2 Preliminaries

2.1 Model

We consider the set of natural numbers $\mathbb{N}$ as the set of **potential agents.** Denote by $\mathcal{N}$ the collection of all finite subsets of $\mathbb{N}$. Each $i \in \mathbb{N}$ is characterized by an endowment $\omega_i \in \mathbb{R}_+$ of the good and a continuous preference relation R_i defined over $\mathbb{R}_+$. Call P_i and I_i to the strict preference and indifference relations associated with R_i, respectively. We assume that agents' preferences are **single-peaked**, i.e., each R_i has a unique maximum $p(R_i) \in \mathbb{R}_+$ such that, for each pair $\{x_i, x_i'\} \subset \mathbb{R}$, we have $x_i P_i x_i'$ as long as either $x_i' < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < x_i'$ holds. Denote by $\mathcal{R}$ the domain of single-peaked preferences defined on $\mathbb{R}_+$. Given $N \in \mathcal{N}$, an **economy** consists of a profile of preferences $R \in \mathcal{R}^N$ and an individual endowments[2] vector $\omega = (\omega_i)_{i \in N} \in \mathbb{R}^N_{++}$ and is denoted by $e = (R, \omega)$. If $S \subset N$ and $R \in \mathcal{R}^N$, let $R_S = (R_j)_{j \in S}$ denote the restriction of R to S. We often write $N \setminus S$ by $-S$. Using similar notation for the vector of endowments, $e' = (R'_S, R_{-S}, \omega'_S, \omega_{-S})$ stands for the economy where the preference and endowment of agent $i \in S$ are R'_i and ω'_i, and those of agent $i \notin S$ are R_i and ω_i. Let $\mathcal{E}^N$ be the domain of economies with agents in N. Given $e = (R, \omega) \in \mathcal{E}^N$, let $z(e) = \sum_{j \in N}(p(R_j) - \omega_j)$. If $z(e) \geq 0$ we say that economy e has **excess demand** whereas if $z(e) < 0$ we say that economy e has **excess supply**. Let $\mathcal{E} = \bigcup_{N \in \mathcal{N}} \mathcal{E}^N$ denote the set of all potential economies. For each $N \in \mathcal{N}$ and each $e \in \mathcal{E}^N$, let $Z(e) = \{z \in \mathbb{R}^N_+ : \sum_{j \in N} z_j = \sum_{j \in N} \omega_j\}$ be the set of **reallocations** for economy e, and let $Z = \bigcup_{e \in \mathcal{E}} Z(e)$. A **reallocation rule** is a function $\varphi : \mathcal{E} \to Z$ such that $\varphi(e) \in Z(e)$ for each $e \in \mathcal{E}$. For each $N \in \mathcal{N}$, each $i \in N$, and each $e \in \mathcal{E}^N$, let $\Delta\varphi_i(e) = \varphi_i(e) - \omega_i$ be **agent i's net trade at e**.

2.2 Basic properties

The next informational simplicity property states that if an agent unilaterally changes her preference for another one with the same peak, then her allotment remains unchanged.

Own-peak-only: For each $e = (R, \omega) \in \mathcal{E}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$ such that $p(R'_i) = p(R_i)$, if $e' = (R'_i, R_{N \setminus \{i\}}, \omega)$ then $\varphi_i(e') = \varphi_i(e)$.

This property is weaker than the "peak-only" property,[3] that has been imposed in a number of axiomatic studies. Analyzing the uniform rule, Sprumont (1991) derives the own-peak-only property from other axioms (see also Ching, 1992, 1994).

The following property requires respecting ownership of the resource, and also can be seen as giving incentive to participate in the exchange process. It says that no agent can get a reallocation that she finds worse than her endowment.

Endowments lower bound: For each $e = (R, \omega) \in \mathcal{E}^N$, and each $i \in N$, $\varphi_i(e) R_i \omega_i$.

Next, we present our two monotonicity properties. The first one requires that as population enlarges, and the new resources and preferences considered are not as disruptive as to modify the status of the economy from excess demand to excess supply or vice versa, the welfare of each of the initially present agents should move in the same direction.

One-sided population monotonicity: For each $e = (R, \omega) \in \mathcal{E}^N$, each $N' \subset N$, and each $e' = (R_{N'}, \omega_{N'}) \in \mathcal{E}^{N'}$, $z(e)\, z(e') \geq 0$ implies either $\varphi_i(e) R_i \varphi_i(e')$ for each $i \in N'$

[2] Throughout the paper we assume that individual endowments of all agents are always strictly positive.

[3] **Peak-only:** For each $e = (R, \omega) \in \mathcal{E}^N$ and each $R' \in \mathcal{R}$ such that $p(R'_i) = p(R_i)$ for each $i \in N$, if $e' = (R', \omega)$ then $\varphi(e') = \varphi(e)$.

or $\varphi_i(e')R_i\varphi_i(e)$ for each $i \in N'$.

The second one requires all agents to benefit from a favorable change in the amount to allocate. Given two vectors $x, y \in \mathbb{R}^N$, define $x \geqslant y$ if and only if $x_i \geq y_i$ for each $i \in N$.

One-sided endowments monotonicity: For each $e = (R, \omega) \in \mathcal{E}^N$, and each $\omega' \in \mathbb{R}^N$ such that $\omega' \geqslant \omega$, if $e' = (R, \omega')$, then $z(e') \geq 0$ implies $\varphi_i(e')R_i\varphi_i(e)$ for each $i \in N$, and $z(e) \leq 0$ implies $\varphi_i(e)R_i\varphi_i(e')$ for each $i \in N$.

The usual Pareto optimality property states that, for each economy, the reallocation selected by the rule should be such that there is no other reallocation that all agents find at least as desirable and at least one agent prefers. In this model, it is equivalent to the following same-sidedness condition:

Efficiency: For each $e = (R, \omega) \in \mathcal{E}^N$, $z(e) \geq 0$ implies $\varphi_i(e) \leq p(R_i)$ for each $i \in N$, and $z(e) \leq 0$ implies $\varphi_i(e) \geq p(R_i)$ for each $i \in N$.

Lemma 1 *Each own-peak-only and one-sided endowments monotonic reallocation rule that meets the endowments lower bound is efficient.*

Proof. Let φ be an *own-peak-only, one-sided endowments monotonic* rule that meets the *endowments lower bound*, and assume φ is not *efficient*. Then, there is $e = (R, \omega) \in \mathcal{E}^N$ such that φ is not same-sided at e. Without loss of generality, assume $z(e) \geq 0$ (the other case is similar). Thus, there is $i \in N$ such that $\varphi_i(e) > p(R_i)$. This implies that

$$\varphi_i(e) \leq \omega_i. \tag{1}$$

To see that (1) holds, first assume $p(R_i) \leq \omega_i < \varphi_i(e)$. By single-peakedness, $\omega_i P_i \varphi_i(e)$, contradicting the *endowments lower bound*. Second, assume $\omega_i < p(R_i)$. Let $\widetilde{R}_i \in \mathcal{R}$ be such that $p(\widetilde{R}_i) = p(R_i)$, and $\omega_i \widetilde{P}_i \varphi_i(e)$ and let $\widetilde{e} = (\widetilde{R}_i, R_{-i}, \omega)$. By the *own-peak-only* property, $\varphi_i(\widetilde{e}) = \varphi_i(e)$. Hence, $\omega_i \widetilde{P}_i \varphi_i(\widetilde{e})$, contradicting the *endowments lower bound*. Therefore, (1) holds. Next, let $\omega'_i \in \mathbb{R}_+$ be such that $\omega'_i \leq p(R_i)$, and let $e' = (R, \omega'_i, \omega_{-i})$. By the *endowments lower bound* and the *own-peak-only* property, $\omega'_i \leq \varphi_i(e') \leq p(R_i)$. Let $\overline{R}_i \in \mathcal{R}$ be such that $p(\overline{R}_i) = p(R_i)$ and $\varphi_i(e')\overline{P}_i\varphi_i(e)$, and let $\overline{e} = (\overline{R}_i, R_{-i}, \omega'_i, \omega_{-i})$. By the *own-peak-only* property, $\varphi_i(\overline{e})\overline{P}_i\varphi_i(e)$, contradicting *one-sided endowments monotonicity* since $z(\overline{e}) \geq 0$. □

The following result will be useful in the rest of the paper.

Lemma 2 *Let φ be an efficient and own-peak-only reallocation rule that meets the endowments lower bound. Let $e = (R, \omega) \in \mathcal{E}^N$ and $i \in N$. If either $z(e) \geq 0$ and $p(R_i) \leq \omega_i$, or $z(e) \leq 0$ and $p(R_i) \geq \omega_i$, then $\varphi_i(e) = p(R_i)$.*

Proof. Let φ satisfy the properties in the lemma and let $e \in \mathcal{E}^N$ and $i \in N$. Assume $z(e) \geq 0$ and $p(R_i) \leq \omega_i$. Since $z(e) \geq 0$, by *efficiency*, $\varphi_i(e) \leq p(R_i)$. If $p(R_i) = \omega_i$,

$\varphi_i(e) = p(R_i)$ by the *endowments lower bound*. Suppose, then, that $p(R_i) < \omega_i$ and $\varphi_i(e) < p(R_i)$. Let $R'_i \in \mathcal{R}$ and $x_i \in \mathbb{R}_+$ be such that $p(R'_i) = p(R_i)$, $\varphi_i(e) < x_i < p(R_i)$ and $x_i I'_i \omega_i$. Let $e' = (R'_i, R_{N\setminus\{i\}}, \omega)$. By the *own-peak-only* property, $\varphi_i(e') = \varphi_i(e)$. Then, $\omega_i P'_i \varphi_i(e')$, contradicting the *endowments lower bound*. A similar reasoning establishes the same conclusion when $z(e) \leq 0$ and $p(R_i) \geq \omega_i$. □

3 Monotonic reallocation rules

In this section we present a well-behaved class of reallocation rules. These reallocation rules, which we call "monotonic", resemble the weakly sequential reallocation rules presented by Bonifacio (2015) in that each rule can be described by an easy step-by-step procedure that, at each stage, guarantees levels of consumption to the agents that are adjusted throughout the process.[4]

3.1 Definition

For each $N \in \mathcal{N}$ and each $e = (R, \omega) \in \mathcal{E}^N$, let $Q(e) \equiv \{q \in \mathbb{R}^N : \sum_{j\in N} q_j = 0 \text{ and } \omega + q \geqslant 0\}$ be the possible net trades of endowments in economy e and let $Q \equiv \bigcup_{e\in\mathcal{E}} Q(e)$. Next, define $\mathcal{Q} \equiv \{(q,e) \in Q \times \mathcal{E} : q \in Q(e)\}$. Each element of $\mathcal{Q}$ specifies a net trade in a particular economy. A monotonic reallocator is a function that, for each $N \in \mathcal{N}$ and each economy $e = (R, \omega) \in \mathcal{E}^N$, starting from the individual endowments of the agents (i.e., from a net trade q_i^0 equal to zero for each agent $i \in N$), generates iteratively a sequence of net trades $q^0, q^1, \ldots, q^{|N|-1}, \ldots$ Its iterative application is constrained to follow monotonic features.

Definition 1 *A* ***monotonic reallocator*** *is a function $g : \mathcal{Q} \to \mathcal{Q}$ such that $g(q,e) \in \mathcal{Q}(e)$ and, for each $N \in \mathcal{N}$, and each $e = (R, \omega) \in \mathcal{E}^N$, if $(q^t, e) = g(q^{t-1}, e) \equiv g^t(0, e)$, then:*[5]

(i) for each $i \in N$, and each $t \geq 1$,

$$q_i^t = p(R_i) - \omega_i \text{ whenever } \begin{cases} z(e) \geq 0 \text{ and } p(R_i) \leq \omega_i + q_i^{t-1} \\ z(e) < 0 \text{ and } p(R_i) \geq \omega_i + q_i^{t-1}. \end{cases}$$

(ii) for each $i \in N$, and each $t \geq 1$,

$$q_i^t \geq q_i^{t-1} \text{ whenever } z(e) \geq 0 \text{ and } p(R_i) > \omega_i + q_i^{t-1}$$

$$q_i^t \leq q_i^{t-1} \text{ whenever } z(e) < 0 \text{ and } p(R_i) < \omega_i + q_i^{t-1}.$$

[4] Weakly sequential reallocation rules, in turn, follow closely the definition of the sequential rules presented in Barberà et al. (1997) for economies with a social endowment to be allotted.

[5] Here g^t denotes g compose with itself t times.

(iii) for each $\widetilde{e} = (R, \widetilde{\omega}) \in \mathcal{E}^N$ *such that* $\widetilde{\omega} \geqslant \omega$ *and* $(\widetilde{q}^{|N|-1}, \widetilde{e}) = g^{|N|-1}(0, \widetilde{e})$,

$$\widetilde{\omega} + \widetilde{q}^{|N|-1} \geq \omega + q^{|N|-1} \text{ whenever } z(e) \leq 0 \text{ or } z(\widetilde{e}) \geq 0.$$

(iv) for each $\widetilde{N} \subset N$, *each* $\widetilde{e} = (R_{\widetilde{N}}, \omega_{\widetilde{N}}) \in \mathcal{E}^{\widetilde{N}}$, *and* $(\widetilde{q}^{|\widetilde{N}|-1}, \widetilde{e}) = g^{|\widetilde{N}|-1}(0, \widetilde{e})$,

$$\left[\widetilde{q}_i^{|\widetilde{N}|-1} - q_i^{|N|-1}\right]\left[\widetilde{q}_j^{|\widetilde{N}|-1} - q_j^{|N|-1}\right] \geq 0 \text{ whenever } \begin{cases} z(e) \geq 0 \text{ and } z(\widetilde{e}) \geq 0 \\ z(e) \leq 0 \text{ and } z(\widetilde{e}) \leq 0 \end{cases}$$

for each $\{i, j\} \subset \widetilde{N}$.

Let us put in words the above definition for the case of excess demand (this is, when $z(e) \geq 0$). The first two conditions relate to the behavior of the net trades of an economy throughout the iterations of g. Condition (i) says that if at stage $t-1$ agent i's peak is not higher than her endowment *plus* her net trade, i.e. $p(R_i) \leq \omega_i + q_i^{t-1}$, then agent i's net trade is set at $p(R_i) - \omega_i$ from stage t onward. Condition (ii) establishes that if at stage $t-1$ agent i's peak is higher than her endowment *plus* her net trade, i.e. $p(R_i) > \omega_i + q_i^{t-1}$, then her net trade should not decrease from stage $t-1$ to stage t, i.e. $q_i^t \geq q_i^{t-1}$. The last two conditions relate to the behavior of the iterations of g between two different economies. Condition (iii) states that, in another economy $\widetilde{e}$ with the same agents and preferences where no agent has lower endowment and the increase in the resources is not disruptive, i.e. $z(\widetilde{e}) \geq 0$, the resources available to each agent in the last stage of the iterations cannot be smaller than the resources available to each agent in the last stage of the iterations in the original economy. Finally, Condition (iv) says that for any subeconomy with excess demand, if the net trade of one agent in the last stage of the iteration is not smaller (bigger) than the net trade that same agent gets in the original economy, then the net trade of each of the other agents in the subeconomy should not be smaller (bigger) than the net trade that agent gets in the original economy either.

Remark 1 Note that, as there are $|N|$ agents in the economy, at most $|N|-1$ adjustments take place. Therefore, $(q^{|N|+t-1}, e) = g^{|N|+t-1}(0, e)$ implies $q^{|N|+t-1} = q^{|N|-1}$ for each $t \geq 1$.

Each monotonic reallocator induces a reallocation rule in a straightforward way:

Definition 2 *A reallocation rule* $\varphi : \mathcal{E} \to Z$ *is* ***monotonic*** *if there is a monotonic reallocator* $g : \mathcal{Q} \to \mathcal{Q}$ *such that, for each* $N \in \mathcal{N}$ *and each* $e \in \mathcal{E}^N$, $(q^{|N|-1}, e) = g^{|N|-1}(0, e)$ *implies* $\Delta\varphi(e) = q^{|N|-1}$.

A prominent member of the class of monotonic reallocation rules is the uniform reallocation rule, first studied by Thomson (1995a) (see also Klaus et al., 1997, 1998), that adapts the celebrated uniform rule characterized by Sprumont (1991) to the model with individual endowments:

Uniform reallocation rule, u: for each $N \in \mathcal{N}$, each $e \in \mathcal{E}^N$, and each $i \in N$,

$$u_i(e) = \begin{cases} \min\{p(R_i), \omega_i + \lambda(e)\} & \text{if } z(e) \geq 0 \\ \max\{p(R_i), \omega_i - \lambda(e)\} & \text{if } z(e) < 0 \end{cases}$$

where $\lambda(e) \geq 0$ and solves $\sum_{j \in N} u_j(e) = \sum_{j \in N} \omega_j$.

Within the class of monotonic reallocation rules, the uniform reallocation rule is the only one that supports *envy-free* redistributions, meaning by this that for no $N \in \mathcal{N}$, $e \in \mathcal{E}^N$, and pair of agents $\{i, j\} \subset N$ such that $\omega_i - \Delta u_j(e) \in \mathbb{R}_+$, we have $(\omega_i - \Delta u_j(e)) P_i u_i(e)$ (see Moreno, 2002, Theorem 1).

To see that the uniform reallocation rule is a monotonic reallocation rule, given $e = (R, \omega) \in \mathcal{E}^N$ and $q^0 = (0, \ldots, 0)$, consider the monotonic reallocator $g : \mathcal{Q} \to \mathcal{Q}$ defined as follows. If $(q^t, e) = g(q^{t-1}, e)$ then, for each $i \in N$ and each $t = 1, \ldots, |N| - 1$,

$$q_i^t = \begin{cases} \min\{p(R_i) - \omega_i, \lambda^t\} & \text{if } z(e) \geq 0 \\ \max\{p(R_i) - \omega_i, \lambda^t\} & \text{if } z(e) < 0 \end{cases}$$

where $\lambda^0 = 0$,

$$\lambda^t = \sum_{j=0}^{t-1} \lambda^j + \frac{\sum_{j \in N^t} (\omega_j + \lambda^{t-1} - p(R_j))}{|N \setminus (\cup_{s=1}^t N^s)|},$$

and

$$N^t = \begin{cases} \{j \in N : p(R_i) \leq \omega_j + q_j^{t-1}\} & \text{if } z(e) \geq 0 \\ \{j \in N : p(R_i) > \omega_j + q_j^{t-1}\} & \text{if } z(e) < 0 \end{cases}$$

It is easy to see that $(q^{|N|-1}, e) = g^{|N|-1}(0, e)$ implies $q^{|N|-1} = \Delta u(e)$ for each $N \in \mathcal{N}$ and each $e \in \mathcal{E}^N$.

Example 1 Consider $e = (R, \omega)^{\{1,2,3,4\}}$ with $p(R_1) = 0$, $p(R_2) = 2$, $p(R_3) = 3.5$, and $p(R_4) = 10$; and $\omega_1 = 9$, $\omega_2 = 1$, $\omega_3 = 0$, and $\omega_4 = 2$. Then, as $z(e) = 15.5 - 12 > 0$,

$$\lambda^1 = 3 \text{ and thus } q^1 = (-9, 3, 3, 3),$$

$$\lambda^2 = 4 \text{ and thus } q^2 = (-9, 1, 4, 4),$$

$$\lambda^3 = 4.5 \text{ and thus } q^3 = (-9, 1, 3.5, 4.5).$$

Therefore, $u(e) = (0, 2, 3.5, 6.5)$. ◊

Another monotonic reallocation rule, that will be analyzed in Section 4.3, is the proportional reallocation rule that we present next.

Proportional reallocation rule, φ^p: for each $N \in \mathcal{N}$, each $e \in \mathcal{E}^N$, and each $i \in N$,

$$\varphi_i^p(e) = \begin{cases} \min\{p(R_i), \lambda(e)\omega_i\} & \text{if } z(e) \geq 0 \\ \max\{p(R_i), \lambda(e)\omega_i\} & \text{if } z(e) \leq 0 \end{cases}$$

where $\lambda(e) \geq 1$ and solves $\sum_{j \in N} \varphi_j^p(e) = \sum_{j \in N} \omega_j$.[6]

3.2 Characterization

The next result states that the class of monotonic rules is characterized by the *own-peak-only* property, the *endowments lower bound, one-sided endowments monotonicity* and *one-sided population monotonicity:*

Theorem 1 *A reallocation rule satisfies the own-peak-only property, the endowments lower bound, one-sided endowments monotonicity, and one-sided population monotonicity if and only if it is a monotonic reallocation rule.*

Proof. ($\Longrightarrow$) Let φ be an *own-peak-only*, *one-sided endowments monotonic*, and *one-sided population monotonic* reallocation rule that meets the *endowments lower bound*. By Lemma 1, φ is also *efficient*. The monotonic reallocator $g : \mathcal{Q} \to \mathcal{Q}$ is constructed as follows. Given $(q^{t-1}, e) \in \mathcal{Q}$, define q^t such that $(q^t, e) = g(q^{t-1}, e)$ as

$$q^t = \varphi(e^t) - \omega \tag{2}$$

where economy $e^t = (R, \omega^t) \in \mathcal{E}^N$ is such that, for each $i \in N$,

$$\omega_i^t = \begin{cases} p(R_i) & \text{if } z(e)[p(R_i) - \omega_i^{t-1} - q_i^{t-1}] \leq 0 \\ \omega_i^{t-1} + q_i^{t-1} & \text{otherwise,} \end{cases}$$

with $\omega_i^0 = \omega_i$ and $q_i^0 = 0$.

Let us assume that $e = (R, \omega) \in \mathcal{E}^N$ is such that $z(e) \geq 0$. The other case is similar. We need to see that $\Delta\varphi(e) = q^{|N|-1}$ where $q^{|N|-1}$ is such that $(q^{|N|-1}, e) = g^{|N|-1}(0, e)$. In order to do this, for each $t = 1, \ldots, |N| - 1$, let q^t be such that $(q^t, e) = g(q^{t-1}, e) = g^t(0, e)$ (notice that $q^0 = 0$).

Claim 1: Let $t \in \{1, \ldots, |N| - 1\}$. If $p(R_i) \geq \omega_i^{t-1} + q_i^{t-1}$ for each $i \in N$, then $q^t = \Delta\varphi(e)$. Consider first the case $t = 1$. As $q^0 = 0$, by the hypothesis $p(R_i) \geq \omega_i$ for each $i \in N$. The *endowments lower bound, efficiency,* and feasibility imply $\varphi(e) = \omega$, and therefore $\Delta\varphi(e) = 0$. Note that, since in e^1 no agent's peak is less than her endowment and $z(e^1) \geq 0$, by the same reasoning as before $\varphi(e^1) = \omega$. Thus, $q^1 = \omega - \omega = 0$. Next, assume the claim is true for each $t < T$. Then $q_i^{T-1} = 0$ for each $i \in N$ and, again, since in e^T no agent's peak is less than her endowment and $z(e^T) \geq 0$, we get $q_i^T = 0 = \Delta\varphi(e)$. This proves the claim.

Claim 2: Let $t \in \{1, \ldots, |N| - 1\}$. If $i \in N$ is such that $p(R_i) \leq \omega_i^{t-1} + q_i^{t-1}$, then $q_i^t = \Delta\varphi_i(e)$. Let $i \in N$ be such that $p(R_i) \leq \omega_i^{t-1} + q_i^{t-1}$. First, notice that when $t = 1$,

[6] Note that, since individual endowments are always strictly positive, this rule is well-defined.

$p(R_i) \leq \omega_i^0 + q_i^0$ implies, as $q_i^0 = 0$ and $\omega_i^0 = \omega_i$, that $p(R_i) \leq \omega_i$. Then, by Lemma 2, $\varphi_i(e) = p(R_i)$ and, therefore,

$$\Delta\varphi_i(e) = p(R_i) - \omega_i. \tag{3}$$

Next, let $t \in \{1, \dots, |N|\}$. Since $\omega_i^t = p(R_i)$ and $z(e^t) \geq 0$, by Lemma 2 applied to economy e^t, we have $\varphi_i(e^t) = p(R_i)$ and then $q_i^t = p(R_i) - \omega_i$. Hence, by (3), $q_i^t = \Delta\varphi_i(e)$. This proves the claim.

Claims 1 and 2 show that if $q^{|N|-1}$ is such that $(q^{|N|-1}, e) = g^{|N|-1}(0, e)$, then $q^{|N|-1} = \Delta\varphi(e)$. It remains to be checked that function g satisfies conditions (i)-(iv) in Definition 1. Condition (i) is clear by Claims 1 and 2. Condition (ii) follows from the next claim.

Claim 3: for each $t = 1, \dots, |N| - 1$, if $i \in N$ is such that $p(R_i) > \omega_i^{t-1} + q_i^{t-1}$, then $q_i^t \geq q_i^{t-1}$. Let $i \in N$ be such that $p(R_i) > \omega_i^{t-1} + q_i^{t-1}$. Consider first the case $t = 1$. Since $q^0 = 0$ and $\omega_i^0 = \omega_i$, by the hypothesis $p(R_i) > \omega_i$. Then, $\omega_i^1 = \omega_i$. This implies, as $z(e^1) \geq 0$, that $\varphi_i(e^1) \geq \omega_i^1$ by the *endowments lower bound*. Hence, $q_i^1 = \varphi_i(e^1) - \omega_i \geq 0 = q_i^0$ and thus $q_i^1 \geq q_i^0$. Next, assume the claim is true for each $t < T$. Then $q_i^{T-1} \geq q_i^{T-2} \geq \ldots \geq q_i^0 = 0$. Since $p(R_i) > \omega_i^{T-1} + q_i^{T-1}$ implies $\omega_i^T = \omega_i^{T-1} + q_i^{T-1}$ and $z(e^T) \geq 0$, by the *endowments lower bound* $\varphi_i(e^T) \geq \omega_i^T = \omega_i + \sum_{k=1}^{T-1} q_i^k$. Then,

$$q_i^T = \varphi_i(e^T) - \omega_i \geq \sum_{k=1}^{T-1} q_i^k \geq q_i^{T-1}.$$

This proves the claim.

Condition (iii) follows from the definition of g and *one-sided endowments monotonicity* of φ, whereas condition (iv) is consequence of the definition of g and *one-sided population monotonicity* of φ.

($\Longleftarrow$) Let φ be a monotonic reallocation rule. Then there exists a monotonic reallocator g such that, for each $e = (R, \omega) \in \mathcal{E}^N$, if $q^{|N|-1}$ is such that $(q^{|N|-1}, e) = g^{|N|-1}(0, e)$, then $\Delta\varphi(e) = q^{|N|-1}$. We will consider only the case $z(e) \geq 0$, since an analogous argument can be used in the case $z(e) < 0$. Next, we prove that φ is *efficient*,[7] *one-sided endowments monotonic*, and *one-sided population monotonic*.

Efficiency: We need to show that $\varphi_i(e) \leq p(R_i)$ for each $i \in N$. Suppose $\varphi_i(e) \neq p(R_i)$. Then $\omega_i + q_i^{|N|-1} \neq p(R_i)$. If $q_i^{|N|-1} > p(R_i) - \omega_i$, by (i) in Definition 1 we have $q_i^{|N|} = p(R_i) - \omega_i$. Then, $q_i^{|N|-1} \neq q_i^{|N|}$, contradicting Remark 1. Thus, $q_i^{|N|-1} \leq p(R_i) - \omega_i$ implying $\Delta\varphi_i(e) = q_i^{|N|-1} \leq p(R_i) - \omega_i$ and $\varphi_i(e) \leq p(R_i)$.

One-sided endowments monotonicity: Let $\tilde{e} = (R, \tilde{\omega}) \in \mathcal{E}^N$ be such that $\tilde{\omega} \geqslant \omega$ and $z(\tilde{e}) \geq 0$, let $\tilde{q}^{|N|-1}$ be such that $(\tilde{q}^{|N|-1}, \tilde{e}) = g^{|N|-1}(0, \tilde{e})$ and consider $i \in N$ such that $p(R_i) > \tilde{\omega}_i$. By condition (iii) in Definition 1, $\tilde{\omega}_i + \tilde{q}_i^{|N|-1} \geq \omega_i + q_i^{|N|-1}$. Then, $\varphi_i(\tilde{e}) \geq \varphi_i(e)$ and, as by *efficiency* $p(R_i) \geq \varphi_i(\tilde{e})$, we have $\varphi_i(\tilde{e}) R_i \varphi_i(e)$.

[7] This is used to proof the two monotonicity properties.

One-sided population monotonicity: Let $\widetilde{N} \subset N$ and $\widetilde{e} = (R_{\widetilde{N}}, \omega_{\widetilde{N}}) \in \mathcal{E}^{\widetilde{N}}$ be such that $z(\widetilde{e}) \geq 0$. Take $\{i, j\} \subseteq \widetilde{N}$. By condition (iv) in Definition 1, $[\widetilde{q}_i^{|\widetilde{N}|} - q_i^{|N|}][\widetilde{q}_j^{|\widetilde{N}|} - q_j^{|N|}] \geq 0$. Assume, without loss of generality, that $\widetilde{q}_i^{|\widetilde{N}|} \geq q_i^{|N|}$. Then, $\widetilde{q}_j^{|\widetilde{N}|} \geq q_j^{|N|}$. This implies $\varphi_i(\widetilde{e}) \geq \varphi_i(e)$ and $\varphi_j(\widetilde{e}) \geq \varphi_j(e)$. As $z(\widetilde{e}) \geq 0$, by *efficiency*, $p(R_i) \geq \varphi_i(\widetilde{e})$ and $p(R_j) \geq \varphi_j(\widetilde{e})$. Thus, $\varphi_i(\widetilde{e}) R_i \varphi_i(e)$ and $\varphi_j(\widetilde{e}) R_i \varphi_j(e)$.

To complete the proof, notice that φ satisfies the *own-peak-only* property because g does, and meets the *endowments lower bound* because, for each agent, the adjustment process at each step guarantees an amount at least as good as the individual endowment. □

The independence of the axioms involved in the characterization of Theorem 1 is analyzed in Appendix A.

4 Variable population manipulations

In this section, we analyze each of the four properties of immunity to manipulation presented in the introduction and its relations with the family of monotonic reallocation rules.

4.1 Withdrawal-proofness

Consider an economy and suppose that an agent withdraws with her endowment and the reallocation rule is applied without her. It could be the case that the amount that some other agent received in the reallocation together with the endowment of the agent that withdrew could be re-divided between the two of them in such a way that both agents get (strictly) better off with respect to the assignments they would have obtained if the first agent had not withdrawn. We require immunity to this sort of behavior:

Withdrawal-proofness: For each $e = (R, \omega) \in \mathcal{E}^N$, each $\{i, j\} \subset N$ and each $(x_i, x_j) \in \mathbb{R}^2_+$ such that $x_i + x_j = \varphi_i(e') + \omega_j$, where $e' = (R_{N \setminus \{j\}}, \omega_{N \setminus \{j\}})$, it is not the case that $x_k P_k \varphi_k(e)$ for each $k \in \{i, j\}$.

Each *efficient*, *own-peak-only*, and *one-sided population monotonic* reallocation rule that meets the *endowments lower bound* satisfies this property.

Lemma 3 *Each efficient, own-peak-only, and one-sided population monotonic reallocation rule that meets the endowments lower bound is withdrawal-proof.*

Proof. Let φ satisfy the hypothesis of the Theorem. By Lemma 1, φ is also *efficient*. Assume φ is not *withdrawal-proof*. Then, there are $e = (R, \omega) \in \mathcal{E}^N$, $\{i, j\} \subset N$, and $(x_i, x_j) \in \mathbb{R}^2_+$ such that, if $e' = (R_{N \setminus \{j\}}, \omega_{N \setminus \{j\}})$, then

$$x_i + x_j = \varphi_i(e') + \omega_j, \tag{4}$$

and

$$x_k P_k \varphi_k(e) \text{ for each } k \in \{i, j\}. \tag{5}$$

Assume $z(e) \geq 0$. The case $z(e) \leq 0$ can be handled similarly. By (5), $z(e) > 0$. By *efficiency*, $\varphi_k(e) \leq p(R_k)$ for each $k \in N$. By (5), $\varphi_k(e) < x_k$ for each $k \in \{i, j\}$ and therefore, by (4),

$$\varphi_i(e) + \varphi_j(e) < x_i + x_j = \varphi_i(e') + \omega_j. \tag{6}$$

Claim: there is $k^\star \in N \setminus \{i, j\}$ such that $\varphi_{k^\star}(e') < \varphi_{k^\star}(e)$. Otherwise,

$$\sum_{k \in N \setminus \{i,j\}} \varphi_k(e') \geq \sum_{k \in N \setminus \{i,j\}} \varphi_k(e) \tag{7}$$

and, since $\sum_{k \in N \setminus \{j\}} \omega_k = \sum_{k \in N \setminus \{j\}} \varphi_k(e')$, by (6) and (7) we have

$$\sum_{k \in N} \omega_k = \sum_{k \in N \setminus \{j\}} \varphi_k(e') + \omega_j > \sum_{k \in N} \varphi_k(e) = \sum_{k \in N} \omega_k,$$

which is absurd. This proves the Claim.

Now, by the Claim and *efficiency*, $\varphi_{k^\star}(e') < \varphi_{k^\star}(e) \leq p(R_{k^\star})$. This implies

$$\varphi_{k^\star}(e) P_{k^\star} \varphi_{k^\star}(e') \tag{8}$$

and also, by *efficiency*, $z(e') \geq 0$. By (5), $\varphi_j(e) \neq p(R_j)$ holds. Then, by Lemma 2, $\omega_j \leq p(R_j)$; and by the *endowments lower bound*, $\varphi_j(e) \geq \omega_j$. It follows from this and (6) that $0 \leq \varphi_j(e) - \omega_j < \varphi_i(e') - \varphi_i(e)$. Therefore, $\varphi_i(e') > \varphi_i(e)$. As $z(e') \geq 0$, by *efficiency*, $\varphi_i(e') \leq p(R_i)$. Thus, $\varphi_i(e) < \varphi_i(e') \leq p(R_i)$ and

$$\varphi_i(e') P_i \varphi_i(e). \tag{9}$$

Note that, as $z(e) \geq 0$ and $z(e') \geq 0$, (8) and (9) contradict *one-sided population monotonicity*. We conclude that φ is *withdrawal-proof*. □

As a consequence of the previous result and Theorem 1, the whole class of *monotonic* reallocation rules precludes this kind of manipulation.

Corollary 1 *Each monotonic reallocation rule is withdrawal-proof.*

4.2 Endowments-merging-proofness

Another manipulation involving variable population is the following. Consider an economy and a pair of agents in that economy. One of those agents gives her endowment to the other and withdraws. The reallocation rule is applied without the first agent and with the second agent's enlarged endowment. The allocation that the second agent obtains could be divided between the two agents in such a way that each agent is at least as well off as she would have been if the merging had not taken place, and at least one of them is better off. We require immunity to this sort of behavior:

Endowments-merging-proofness: For each $e = (R, \omega) \in \mathcal{E}^N$, each $\{i, j\} \subset N$ and each $(x_i, x_j) \in \mathbb{R}^2_+$ such that $x_i + x_j = \varphi_i(e')$, where $e' = (R_{N\setminus\{j\}}, \omega'_i, \omega_{N\setminus\{i,j\}})$ and $\omega'_i = \omega_i + \omega_j$, it is not the case that $x_k R_k \varphi_k(e)$ for each $k \in \{i, j\}$, and $x_k P_k \varphi_k(e)$ for at least one $k \in \{i, j\}$.

Each rule in the class of monotonic reallocation rules precludes such manipulations.

Lemma 4 *Each monotonic reallocation rule is endowments-merging-proof.*

Proof. Let φ be a monotonic reallocation rule. By Theorem 1, φ is *one-sided endowments monotonic*. By Lemma 1, φ is also *efficient*. Assume φ is not *endowments-merging-proof.* Then, there are $e = (R, \omega) \in \mathcal{E}^N$, $\{i, j\} \subset N$, and $(x_i, x_j) \in \mathbb{R}^2_+$ such that, if $e' = (R_{N\setminus\{j\}}, \omega'_i, \omega_{N\setminus\{i,j\}})$, then

$$x_i + x_j = \varphi_i(e'), \tag{10}$$

$$x_k R_k \varphi_k(e) \text{ for each } k \in \{i, j\}, \tag{11}$$

and

$$x_k P_k \varphi_k(e) \text{ for at least one } k \in \{i, j\}. \tag{12}$$

Assume $z(e) \geq 0$. By (12), $z(e) > 0$. By *efficiency*, $\varphi_k(e) \leq p(R_k)$ for each $k \in N$. By (11) and (12), $x_k \geq \varphi_k(e)$ for each $k \in \{i, j\}$ and $x_k > \varphi_k(e)$ for at least one $k \in \{i, j\}$. Therefore, by (10),

$$\varphi_i(e') = x_i + x_k > \varphi_i(e) + \varphi_j(e). \tag{13}$$

Claim 1: there is $k^\star \in N \setminus \{i, j\}$ such that $\varphi_{k^\star}(e') < \varphi_{k^\star}(e)$. Otherwise,

$$\sum_{k \in N\setminus\{i,j\}} \varphi_k(e') \geq \sum_{k \in N\setminus\{i,j\}} \varphi_k(e) \tag{14}$$

and, since $\sum_{k \in N\setminus\{j\}} \omega_k = \sum_{k \in N\setminus\{j\}} \varphi_k(e')$, by (13) and (14) we have

$$\sum_{k \in N} \omega_k = \sum_{k \in N\setminus\{j\}} \varphi_k(e') > \sum_{k \in N} \varphi_k(e) = \sum_{k \in N} \omega_k,$$

which is absurd.

Now, by Claim 1 and *efficiency*, $\varphi_{k^\star}(e') < \varphi_{k^\star}(e) \leq p(R_{k^\star})$, which implies $z(e') \geq 0$. Let $e'' = (R_{N\setminus\{j\}}, \omega_{N\setminus\{j\}})$. As $z(e') \geq 0$, it follows that $z(e'') \geq 0$. By Corollary 1, φ is *withdrawal-proof*, which implies that

$$\varphi_i(e) + \varphi_j(e) \geq \varphi_i(e'') + \omega_j. \tag{15}$$

Since $(\omega'_i, \omega_{N\setminus\{i,j\}}) \geqslant \omega_{N\setminus\{j\}}$, by *one-sided endowments monotonicity*, $\varphi_k(e') R_k \varphi_k(e'')$ for each $k \in N \setminus \{j\}$. Then *efficiency* implies

$$\varphi_k(e') \geq \varphi_k(e'') \text{ for each } k \in N \setminus \{j\}. \tag{16}$$

Combining (13) and (15) we obtain

$$\varphi_i(e') > \varphi_i(e'') + \omega_j. \tag{17}$$

Claim 2: there is $k^{\star\star} \in N \setminus \{i,j\}$ such that $\varphi_{k^{\star\star}}(e'') > \varphi_{k^{\star\star}}(e')$. Otherwise,

$$\sum_{k \in N \setminus \{i,j\}} \varphi_k(e'') \leq \sum_{k \in N \setminus \{i,j\}} \varphi_k(e') \tag{18}$$

and, since $\sum_{k \in N \setminus \{j\}} \omega_k = \sum_{k \in N \setminus \{j\}} \varphi_k(e'')$, by (17) and (18) we have

$$\sum_{k \in N} \omega_k = \omega_j + \sum_{k \in N \setminus \{j\}} \varphi_k(e'') < \sum_{k \in N \setminus \{j\}} \varphi_k(e') = \sum_{k \in N} \omega_k,$$

which is absurd.

Therefore, by Claim 2, there is $k^{\star\star} \in N \setminus \{i,j\}$ such that $\varphi_{k^{\star\star}}(e'') > \varphi_{k^{\star\star}}(e')$. This contradicts (16). We conclude that φ is *endowments-merging-proof.* □

4.3 Endowments-splitting-proofness

Consider an economy and assume that an agent in the economy transfers some of her endowment to another agent that was not initially present; the rule is applied, and the guest transfers her assignment to the agent who invited her in. The first agent could obtain an amount that she prefers to her initial assignment. We require immunity to this type of behavior:

Endowments-splitting-proofness: For each $e = (R, \omega) \in \mathcal{E}^N$, each $i \in N$, each $j \notin N$, each $R_j \in \mathcal{R}$, and each $(\omega'_i, \omega'_j) \in \mathbb{R}^2_+$ such that $\omega'_i + \omega'_j = \omega_i$, we have $\varphi_i(e) R_i [\varphi_i(e') + \varphi_j(e')]$, with $e' = (R, R_j, \omega'_i, \omega_{N \setminus \{i\}}, \omega'_j)$.

Not all monotonic reallocation rules satisfy this property. The following example shows that the uniform reallocation rule violates *endowments-splitting-proofness.*

Example 2 Let $e = (R, \omega) \in \mathcal{E}^{\{1,2,3\}}$ be such that $p(R_1) = 4$, $p(R_2) = 0$, $p(R_3) = \omega_1 = \omega_2 = 2$ and $\omega_3 = 1$. Then, $u_1(e) = 3$, $u_2(e) = 0$, and $u_3(e) = 2$. Next, let $R_4 \in \mathcal{R}$ be such that $p(R_4) = 4$ and let $\omega'_4 = 1$. Consider the economy $e' = (R, R_4, \omega'_1, \omega_{\{2,3\}}, \omega'_4) \in \mathcal{E}^{\{1,2,3,4\}}$ with $\omega'_1 = 1$ (notice that $\omega_1 = \omega'_4 + \omega'_1$). It follows that $u_1(e') = u_3(e') = u_4(e') = \frac{5}{3}$, $u_2(e') = 0$, and $u_1(e') + u_4(e') = \frac{10}{3} P_1 3 = u_1(e)$. This implies that u is not *endowments-splitting-proof.* ◊

Priority reallocation rules[8] violate the property as well.

[8]Given a linear $\prec$ order over the set of potential agents $\mathbb{N}$, the *priority reallocation rule* $\varphi^{\prec}$ for economies with excess demand (supply) satiates all suppliers (demanders) and demanders (suppliers) according to order $\prec$, respecting the endowments lower bound. For economies with excess supply, a symmetric procedure is performed. It is easy to see that such reallocation rules are monotonic in our sense.

Example 3 Consider $\prec$ as the usual "less than" order in $\mathbb{N}$. Let $e = (R, \omega) \in \mathcal{E}^{\{1,3,4\}}$ be such that $p(R_1) = 0$, $\omega_1 = 4$, $p(R_3) = p(R_4) = 6$, and $\omega_3 = \omega_4 = 2$. Then, $\varphi_1^{\prec}(e) = 0$, $\varphi_3^{\prec}(e) = 6$, and $\varphi_4^{\prec}(e) = 2$. Next, let $R_2 \in \mathcal{R}$ be such that $p(R_2) = 4$ and let $\omega_2 = 1$. Consider the economy $e' = (R, R_2, \omega_{\{1,3\}}, \omega_2, \omega_4') \in \mathcal{E}^{\{1,2,3,4\}}$ with $\omega_4' = 1$ (notice that $\omega_4 = \omega_4' + \omega_2 = 2$). It follows that $\varphi_1^{\prec}(e') = 0$, $\varphi_2^{\prec}(e') = 4$, $\varphi_3^{\prec}(e') = 3$, and $\varphi_4^{\prec}(e') = 1$. However, $\varphi_2^{\prec}(e') + \varphi_4^{\prec}(e') = 4 + 1 = 5 \; P_4 \; 2 = \varphi_4^{\prec}(e)$. This implies that $\varphi^{\prec}$ is not *endowments-splitting-proof.* ◊

However, the proportional reallocation rule is immune to endowments' splitting:

Remark 2 *The proportional reallocation rule is endowments-splitting-proof.*

Proof. Suppose φ^p is not *endowments-splitting-proof.* Then, there are $e = (R, \omega) \in \mathcal{E}^N$, $i \in N$, $j \notin N$, $R_j \in \mathcal{R}$, and $(\omega_i', \omega_j') \in \mathbb{R}_+^2$ with $\omega_i' + \omega_j' = \omega_i$ such that, if $e' = (R, R_j, \omega_i', \omega_{N \setminus \{i\}}, \omega_j')$, then

$$[\varphi_i^p(e') + \varphi_j^p(e')] P_i \varphi_i^p(e). \tag{19}$$

Consider first the case $z(e) \geq 0$. By (19), $\varphi_i^p(e) < p(R_i)$ and therefore

$$\lambda(e)\omega_i = \varphi_i^p(e) < \varphi_i^p(e') + \varphi_j^p(e'). \tag{20}$$

Since $z(e') \geq z(e) \geq 0$,

$$\varphi_i^p(e') + \varphi_j^p(e') \leq \lambda(e')\omega_i' + \lambda(e')\omega_j' = \lambda(e')\omega_i. \tag{21}$$

By (20) and (21),

$$\lambda(e) < \lambda(e'). \tag{22}$$

It follows that there is $k \in N \setminus \{i\}$ such that $\varphi_k^p(e') < \varphi_k^p(e)$. Otherwise, there is a violation of feasibility by (20). Then, $\lambda(e')\omega_k = \varphi_k^p(e') < \varphi_k^p(e) \leq \lambda(e)\omega_k$, which implies $\lambda(e') < \lambda(e)$, contradicting (22). If $z(e) \leq 0$ and $z(e') \leq 0$, the proof is similar to the previous one. Assume then that $z(e) \leq 0$ and $z(e') \geq 0$. By (19), $\varphi_i^p(e) > p(R_i) \geq \varphi_i^p(e')$. This implies the existence of $k \in N \setminus \{i\}$ such that $\varphi_k^p(e) < \varphi_k^p(e')$. But then,

$$p(R_k) \leq \varphi_k^p(e) < \varphi_k^p(e') \leq p(R_k),$$

which is a contradiction. Therefore, φ^p is *endowments-splitting-proof.* □

4.4 Pre-delivery-proofness

Consider now the case in which one agent makes a "pre-delivery" to some other agent of the trade that this second agent would be assigned if she had participated with everyone else. After the rule is applied, the first agent may end up with an amount she prefers to his assignment if she had not carried out the pre-delivery. We require immunity to this sort of behavior.

Pre-delivery-proofness: For each $e = (R, \omega) \in \mathcal{E}^N$ and each $\{i, j\} \subset N$ such that $\omega_i + \omega_j - \varphi_j(e) \geq 0$, $\varphi_i(e) R_i \varphi_i(e')$ where $e' = (R_{N\setminus\{j\}}, \omega_i', \omega_{N\setminus\{i,j\}})$ and $\omega_i' = \omega_i + \omega_j - \varphi_j(e)$.

As the following proposition shows, every *efficient* and *own-peak-only* reallocation rule that meets the *endowments lower bound* violates *pre-delivery-proofness.*

Theorem 2 *No efficient and own-peak-only reallocation rule that meets the endowments lower bound is pre-delivery-proof.*

Proof. Let reallocation rule φ satisfy the hypothesis of the Proposition. Let $e = (R, \omega) \in \mathcal{E}^{\{1,2,3\}}$ be such that $0 < p(R_1) = \omega_2 = \omega_3 < \omega_1 < p(R_2) = p(R_3)$. Then $z(e) > 0$ and, as $p(R_1) < \omega_1$, by Lemma 2 we have $\varphi_1(e) = p(R_1)$. By feasibility, there is $i^\star \in \{2, 3\}$ such that $\varphi_{i^\star}(e) < \omega_1$. Assume, without loss of generality, that $i^\star = 2$. Let $\omega_2' = \omega_2 + \omega_1 - \varphi_1(e)$. Then $\omega_2' = \omega_1$. Consider now the economy $e' = (R_{\{2,3\}}, \omega_2', \omega_3)$. It follows that $z(e') > 0$. By *efficiency,* $\varphi_2(e') \leq p(R_2)$, and since $p(R_2) > \omega_2'$, by the *endowments lower bound* we have $\varphi_2(e') \geq \omega_2' = \omega_1$. By feasibility then, $\varphi_2(e') = \omega_1$. Therefore, $\varphi_2(e) < \omega_1 = \varphi_2(e') < p(R_2)$, which implies $\varphi_2(e') P_2 \varphi_2(e)$. Thus, φ is not *pre-delivery-proof.* □

Of course, the previous result extends to the whole class of monotonic reallocation rules.

Corollary 2 *No monotonic reallocation rule is pre-delivery-proof.*

5 Final comments

We conclude with some remarks. One may ask whether the definition of *withdrawal-proofness* can be made with just one of the agents involved in the manipulation strictly improving. However, not even the uniform reallocation rule satisfies this variant, as the following example shows:

Example 4 Consider $e = (R, \omega)^{\{1,2,3,4\}}$ with $p(R_1) = p(R_4) = 1$, $p(R_2) = 4$, $p(R_3) = 3$, and $\omega_1 = \omega_4 = 3$, $\omega_2 = \omega_3 = 1$. Then, $z(e) > 0$ and $u_1(e) = u_4(e) = 1$, $u_2(e) = u_3(e) = 3$. When agent 4 withdraws, if $e' = (R_{\{1,2,3\}}, \omega_{\{1,2,3\}})$, then $u_1(e') = 1$, $u_2(e') = u_3(e') = 2$. So $x_2 = 4 \; P_2 \; 3 = u_2(e)$, $x_4 = 1 \; R_1 \; 1 = u_4(e)$, and $x_2 + x_4 = 5 = u_2(e') + \omega_4$. ◊

In Thomson (2014), the property of *withdrawal-proofness* is presented in this variant: one of the agents involved in the manipulation can be indifferent between the amount she gets from the rule and the amount she gets after the manipulation is performed. However, in all the impossibility examples presented there both agents get strictly better off, so our version does not hold in those examples (classical multi-commodity exchange model with homothetic and quasi-linear preferences) either.

Finally, it is worth mentioning that the results obtained in our paper are in sharp contrast with findings in models with several goods and classical preferences. The Walrasian reallocation rule is neither *withdrawing-proof*, nor *endowments-merging-proof*, nor *endowments-splitting-proof*. These negative results are obtained by Thomson (2014) in two classical subdomains: (i) the domain of economies in which preferences are homothetic and strictly convex, and individual endowments are proportional, and (ii) the domain of economies in which preferences are quasi-linear and strictly convex. The Walrasian reallocation rule, however, is *pre-delivery-proof* on the classical domain (see Thomson, 2014). "Constrained dictatorial rules", defined by maximizing the welfare of a particular agent subject to each of the others finding their assignment at least as desirable as their endowment, satisfy none of these various requirements either (see Thomson, 2022).

A Independence of axioms in Theorem 1

In order to study the independence of axioms in the characterization of Theorem 1, next we consider several reallocation rules. For each $N \in \mathcal{N}$ and each $e \in \mathcal{E}^N$, let $N^+(e) = \{i \in N : p(R_i) > \omega_i\}$ be the set of *demanders* of e. Agents in $N \setminus N^+$ are called *suppliers*. Let $\mathcal{S}(e) \equiv \sum_{j \in N \setminus N^+}(\omega_j - p(R_j))$.

Given a linear $\preceq$ order over the set of potential agents $\mathbb{N}$, the *priority reallocation rule* $\varphi^{\preceq}$ for each economy with excess demand (supply) satiates all suppliers (demanders) and demanders (suppliers) according to order $\preceq$, respecting the endowments lower bound. So if $e = (R, \omega) \in \mathcal{E}^N$ is such that $z(e) \geq 0$,

$$\varphi_i^{\preceq}(e) = \begin{cases} p(R_i) & \text{if } i \in N \setminus N^+(e) \\ \min\left\{p(R_i), \omega_i + \mathcal{S}(e) - \sum_{j \in N^+ : j \prec i} \Delta\varphi_j(e)\right\} & \text{otherwise} \end{cases}$$

In case $z(e) < 0$, the rule is defined similarly. Priority reallocation rules allow the definition of a reallocation rule which is not *one-sided population monotonic*.

Reallocation rule $\overline{\varphi}$: for each $N \in \mathcal{N}$, each $e \in \mathcal{E}^N$, and each $i \in N$,

$$\overline{\varphi}_i(e) = \begin{cases} \varphi_i^{\preceq}(e) & \text{if } |N| \text{ is odd} \\ \varphi_i^{\succeq}(e) & \text{if } |N| \text{ is even} \end{cases}$$

where $\succeq$ is the dual of $\preceq$.

Next, let us recall the celebrated *uniform* rule, first characterized by Sprumont (1991).

Uniform rule, φ^u: for each $N \in \mathcal{N}$, each $e \in \mathcal{E}^N$, and each $i \in N$,

$$\varphi_i^u(e) = \begin{cases} \min\{p(R_i), \lambda(e)\} & \text{if } z(e) \geq 0 \\ \max\{p(R_i), \lambda(e)\} & \text{if } z(e) < 0 \end{cases}$$

where $\lambda(e)$ and solves $\sum_{j \in N} \varphi_j^u(e) = \sum_{j \in N} \omega_j$.

Since this rule does not take into account individual endowments, it trivially does not meet the *endowments lower bound*.

The following rule satiates as many agents as possible. For economies with excess demand, demanders are satiated according to their claims. First minimal demands are satiated uniformly. If there is some supply left, then the next smallest demands are satiated, and so on. This reallocation rule is not *one-sided endowments monotonic*.

Maximally satiating reallocation rule $\varphi^{\max}$: For each $N \in \mathcal{N}$ and each $e \in \mathcal{E}^N$ such that $z(e) \geq 0$, partition $N^+(e)$ into subsets $N_1, N_2, \ldots, N_s$ such that (i) for each $t \in \{1, \ldots, s\}$, $p(R_i) - \omega_i = p(R_j) - \omega_j$ for each $i, j \in N_t$, and (ii) $p(R_i) - \omega_i < p(R_j) - \omega_j$ if $i \in N_r$, $j \in N_s$, and $r < s$. Then,

$$\varphi_i^{\max}(e) = p(R_i)$$

if $i \in N \setminus N^+$, and

$$\varphi_i^{\max}(e) = \min\left\{p(R_i), \omega_i + \frac{1}{|N_t|}\left(\mathcal{S}(e) - \sum_{j \in \cup_{r=1}^{t-1} N_r} \Delta\varphi_j(e)\right)\right\}$$

if $i \in N_t$ and $t \in \{1, \ldots, s\}$. The formula when $z(e) < 0$ is obtained similarly.

	Own-peak-only	Endow LB	OS endow mon	OS pop mon
$\overline{\varphi}$	+	+	+	−
$\varphi^{\max}$	+	+	−	+
φ^u	+	−	+	+
?	−	+	+	+

Table 1: *Independence of axioms in the characterization of Theorem 1.*

Each one of the previously presented rules satisfies all properties of the characterization in Theorem 1 except one. This is shown in Table 1. For example, reallocation rule φ^u satisfies both monotonicity properties and the *own-peak-only* property, but does not meet the *endowments lower bound*. Whether there is a reallocation rule that satisfies all properties except the *own-peak-only* property remains an open question.

References

BARBERÀ, S., M. O. JACKSON, AND A. NEME (1997): "Strategy-proof allotment rules," *Games and Economic Behavior*, 18, 1–21.

BONIFACIO, A. G. (2015): "Bribe-proof reallocation with single-peaked preferences," *Social Choice and Welfare*, 44, 617–638.

CHING, S. (1992): "A simple characterization of the uniform rule," *Economics letters*, 40, 57–60.

——— (1994): "An alternative characterization of the uniform rule," *Social Choice and Welfare*, 11, 131–136.

KLAUS, B., H. PETERS, AND T. STORCKEN (1997): "Reallocation of an infinitely divisible good," *Economic Theory*, 10, 305–333.

——— (1998): "Strategy-proof division with single-peaked preferences and individual endowments," *Social Choice and Welfare*, 15, 297–311.

MORENO, B. (1996): "The uniform rule in economies with single-peaked preferences, endowments and population-monotonicity," Working Paper, Instituto Valenciano de Investigaciones Económicas.

——— (2002): "Single-peaked preferences, endowments and population-monotonicity," *Economics Letters*, 75, 87–95.

SPRUMONT, Y. (1991): "The division problem with single-peaked preferences: a characterization of the uniform allocation rule," *Econometrica*, 59, 509–19.

THOMSON, W. (1994a): "Consistent solutions to the problem of fair division when preferences are single-peaked," *Journal of Economic Theory*, 63, 219–245.

——— (1994b): "Resource-monotonic solutions to the problem of fair division when preferences are single-peaked," *Social Choice and Welfare*, 11, 205–223.

——— (1995a): "Axiomatic analysis of generalized economies with single-peaked preferences," Mimeo.

——— (1995b): "Population-monotonic solutions to the problem of fair division when preferences are single-peaked," *Economic Theory*, 5, 229–246.

——— (2014): "New variable-population paradoxes for resource allocation," *Social Choice and Welfare*, 42, 255–277.

——— (2022): "Constrained dictatorial rules are subject to variable-population paradoxes," Mimeo.