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Nash implementation in a many-to-one matching market *

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Abstract

In a many-to-one matching market with substitutable preferences, we analyze the game induced by a stable rule. When both sides of the market play strategically, we show that any stable rule implements, in Nash equilibrium, the individually rational matchings. Also, when only workers play strategically and firms' preferences satisfy the law of aggregated demand, we show that any stable rule implements, in Nash equilibrium, the stable matchings.

JEL classification: C78, D47.

Keywords: Stable matchings, Nash equilibrium, substitutable preferences, matching game.

1 Introduction

In this paper, we study a many-to-one matching market in which agents on one side of the market (that we call *firms*) have to be assigned to subsets of agents on the other side of the market (that we call *workers*) and the only requirement on subsets of workers that each firm's preference has to satisfy is substitutability.

In centralized markets, a centralizing board needs to collect the preferences of all agents to produce a stable matching. Normally, agents are expected to behave strategically by not revealing their true preferences in order to benefit. When this occurs, the

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matching market is a matching game. In this game, the set of players is the set of all agents and the set of strategies for each player is the set of all possible preferences that she could state.

A stable rule is a function that associates each profile of stated preferences to a stable matching under the stated preferences. This stable matching is a solution to the game. In this paper, the equilibrium concept we focus on is Nash equilibrium. Under the Nash equilibrium, an agent does not improve from deviating from their initially chosen strategy, assuming the other agents also keep their strategies unchanged.

It is well known for matching markets that there is no stable rule for which truth-telling is a dominant strategy for each agent (see Dubins and Freedman, 1981; Roth, 1982, 1985; Sotomayor, 1996, 2012; Martínez et al., 2004; Manasero and Oviedo, 2022, among others). That is, given the true preferences and a stable rule, at least one agent benefits from misrepresenting her preferences regardless of what the rest of the agents state. In particular, this agent benefits when the other agents state their true preferences. Therefore, truth-telling is not a Nash equilibrium of the game. It is expected that the stability of the equilibrium solutions, under the true preferences, be affected when agents behave strategically.

In this paper, when both sides of the market play strategically, we show that any stable rule implements, in Nash equilibrium, the individually rational matchings. In addition, when only workers play strategically and firms' preferences satisfy the "law of aggregated demand " (LAD, from now on), [1] we show that any stable rule implements, in Nash equilibrium, the stable matchings.

The contribution of this paper is to generalize the approach first presented by Sotomayor (2012) for the many-to-one matching market with responsive preferences (a more restrictive requirement than substitutability) to substitutable preferences.

In other direction, there is an extensive literature that focuses on studying the implementation of rules using Maskin's results as the main tool (see Maskin, 1977, 1999; Kara and Sönmez, 1996, 1997; Ehlers, 2004; Haake and Klaus, 2009, among others). Additionally, to study implementation, the aforementioned authors analyze the relationship between stability, monotonicity, individual rationality, and Pareto efficiency. Maskin (1977, 1999) shows that a monotonicity condition (Maskin's monotonicity) is necessary for a rule to be implementable. Also, he shows that Maskin's monotonicity and no veto power together are sufficient conditions for implementability. It is important to highlight that, unlike previous works, our results cannot be obtained through Maskin's implementation result since, although the stable rules satisfy Maskin's monotonicity, they do not satisfy no veto power (see Kara and Sönmez (1996) for more details).

[1] This property is first studied by Alkan (2002) under the name of "cardinal monotonicity". See also Hatfield and Milgrom (2005).

For the marriage market and many-to-one matching market with responsive preferences, Kara and Sönmez (1996, 1997) show that the stable rules are Nash implementable. Ehlers (2004) obtains positive implementation results in marriage markets when agents are allowed to have weak preferences. In a many-to-one matching market with contracts, Haake and Klaus (2009) show that the stable rules are Maskin monotonic and implementable. All the mentioned articles demonstrate the implementability of stable rules using some implementation conditions, eg; monotonicity (Maskin, 1999) , essential monotonicity (Yamato, 1992) or implementability condition (Moore and Repullo, 1990). Unlike this one, we focus on studying a game and identifying the strategies that are the Nash equilibrium of the game and that allow us to implement stable solutions.

In a marriage market, Alcalde (1996) studies the designing of specific mechanisms to implement stable solutions. He presents two types of mechanisms. One of them implements in undominated Nash equilibria the set of all stable matchings; the other implements optimal stable matching for one of the sides of the market via dominance resolvability. This last mechanisms, is the classic algorithm in matching theory, the Gale-Shapley mechanism.

The rest of the paper is organized as follows. Section 2 presents the model and some preliminaries. All the results of the paper are presented in Section 3. First, we show that any stable rule implements the individually rational matchings under Nash equilibrium. Second, assuming that only one side of the market plays strategically when firms' preferences satisfy substitutability and LAD, we show that any stable rule implements, in Nash equilibrium, the stable matchings. Finally, concluding remarks are gathered in Section 4.

2 Model and preliminaries

We consider a many-to-one matching market where there are two disjoint sets of agents: the set of *firms* F and the set of *workers* W. Each firm $f \in F$ has a strict preference relation P_f over the set of all subsets of W. Each worker $w \in W$ has a strict preference relation P_w over the individual firms and the prospect of being unmatched, denoted by $\emptyset$. We denote by P the preference profile for all agents: firms and workers. For each $f \in F$, we denote by R_f the weak preference over the set of all subsets of W associated with P_f; i.e., for all $S, S' \subseteq W$, SR_fS' if and only if either $S = S'$ or SP_fS'. Similarly, for each $w \in W$, R_w is the weak preference over F associated with P_w. Given the profile P, we consider that each agent $a \in F \cup W$ may misrepresent her preferences P_a, by any preferences Q_a. We denote by $Q = (P_{-a}, Q_a)$ the profile of such preferences, where P_{-a} is the subprofile obtained by removing P_a from P . A (many-to-one) matching market is denoted by (F, W, P). Since the sets F and W are kept fixed throughout the paper, we often identify the market (F, W, P) with the preference profile P. Given an agent $a \in F \cup W$, a set S in the opposite side of the market is **acceptable** for a under P if $SP_a\emptyset$ [2]. For each agent, $a \in F \cup W$, the preference relation P_a is represented by the ordered list of its acceptable sets (from most to least preferred).[3] Given a set of workers $W' \subseteq W$ and a firm $f \in F$, let $\boldsymbol{C_f^P(W')}$ (the choice set for f under P) denote firm f's most preferred subset of W' according to the preference relation P_f.

Definition 1 *A **matching** μ is a function from $F \cup W$ into $2^{F \cup W}$ such that, for each $w \in W$ and each $f \in F$:*

(i) $\mu(w) \subseteq F$ *with* $|\mu(w)| \leq 1$.

(ii) $\mu(f) \subseteq W$.

(iii) $w \in \mu(f)$ *if and only if* $\mu(w) = \{f\}$.[4]

Let $\mathcal{M}$ denote the set of all matchings. Agent $a \in F \cup W$ is **matched** if $\mu(a) \neq \emptyset$, otherwise a is **unmatched**. A matching μ is **blocked by a worker** $\boldsymbol{w}$ if $\emptyset P_w \mu(w)$; that is, worker w prefers being unemployed rather than working for firm $\mu(w)$. Similarly, μ is **blocked by a firm** $\boldsymbol{f}$ if $\mu(f) \neq C_f^P(\mu(f))$; that is, firm f wants to fire some workers in $\mu(f)$. A matching is **individually rational** if it is not blocked by any individual agent. The set of individually rational matchings for market P is denoted by $\boldsymbol{I(P)}$.

A matching μ is **blocked by a firm-worker pair** $\boldsymbol{(f, w)}$ if $w \in C_f^P(\mu(f) \cup \{w\})$, and $fP_w\mu(w)$; that is, if they are not matched through μ, firm f wants to hire w, and worker w prefers firm f rather than $\mu(w)$. A matching μ is **stable** if it is individually rational and it is not blocked by any firm-worker pair. The set of stable matchings for market P is denoted by $\boldsymbol{S(P)}$.

Blair (1988) defines a partial order over matchings in which a matching dominates another matching if each firm wishes to keep the workers hired under the first one, even if all the workers hired under the second one are also available, and do not wish to hire any new worker. Formally, given two sets of workers $S, T \in 2^W$, we write $\boldsymbol{SR_f^B T}$ when $S = C_f^P(S \cup T)$. We also write: $\boldsymbol{SP_f^B T}$ when $SR_f^B T$ and $S \neq T$. Furthermore, given two matchings μ and μ', we say that $\boldsymbol{\mu}$ **weakly Blair-dominates** $\boldsymbol{\mu'}$, and write $\boldsymbol{\mu R^B \mu'}$, when $\mu(f) R_f^B \mu'(f)$ for each $f \in F$. If $\mu R_f^B \mu'$ and $\mu \neq \mu'$, we say that $\boldsymbol{\mu}$ **Blair-dominates** $\boldsymbol{\mu'}$ and write $\boldsymbol{\mu P^B \mu'}$.

Our general framework assumes substitutability on firms' preferences. This condition, first introduced by Kelso and Crawford (1982), is the weakest requirement in

[2] In the case that $a \in W$ the set S is a single set.

[3] For instance, $P_f : w_1w_2, w_3, w_1, w_2, \emptyset$ indicates that $\{w_1, w_2\}P_f\{w_3\}P_f\{w_1\}P_f\{w_2\}P_f\emptyset$ and $P_w : f_1, f_2, f_3, \emptyset$ indicates that $\{f_1\}P_w\{f_2\}P_w\{f_3\}P_w\emptyset$.

[4] Usually, we will omit the curly brackets. For instance, instead of condition (iii) we will write: "$w \in \mu(f)$ if and only if $\mu(w) = f$".

preferences to guarantee the existence of stable matchings. A firm has substitutable preferences when she wants to continue being matched to a worker even if other workers become unavailable. Formally, a firm f's preference relation satisfies **substitutability** if, for each $w \in W$ and each $S \subseteq W$ such that $w \in S$, $w \in C_f^P(S)$ implies that $w \in C_f^P(S' \cup \{w\})$ for each $S' \subseteq S$. Moreover, if firm f's preference relation is substitutable then it holds that

$$C_f^P(S \cup S') = C_f^P\left(C_f^P(S) \cup S'\right) \tag{1}$$

for each pair of subsets S and S' of W.[5] Throughout this paper we assume that the preferences of all firms are substitutable. We denote by $\mathcal{Q}$ the domain of substitutable preferences.

The set of stable matchings under substitutable preferences is very well structured. Blair (1988) proves that this set has two lattice structures, one concerning R_F^B and the other one concerning R_W. Furthermore, it contains two distinctive matchings: the firm-optimal stable matching μ_F and the worker-optimal stable matching μ_W. The matching μ_F is unanimously considered by all firms to be the best among all stable matchings and μ_W is unanimously considered by all workers to be the best among all stable matchings, according to the respective Blair's partial orders (see Roth, 1984; Blair, 1988, for more details).

A **rule** is a function $h : \mathcal{Q} \to \mathcal{M}$ that selects for each strategic preference profile $\mathrm{Q} \in \mathcal{Q}$ a matching $h(\mathrm{Q}) \in \mathcal{M}$. A rule h is **stable** if $h(\mathrm{Q}) \in S(\mathrm{Q})$ for each $\mathrm{Q} \in \mathcal{Q}$.

Given a preference profile $P \in \mathcal{Q}$ and a stable rule h the **(matching) game** induced by P and h is denoted by (h, P). A strategic profile Q is a **Nash equilibrium** at P, if no agent can achieve a better outcome deviating from her strategy, assuming that the other agents do not deviate from the strategy Q. Under the Nash equilibrium, an agent does not improve from deviating from their initially chosen strategy, assuming the other agents also keep their strategies unchanged. Formally,

Definition 2 *Let be (h, P) the game induced by P and the stable rule h. A strategy profile* Q *is a **Nash equilibrium** at P if for each w and for each f, $h(\mathrm{Q})R_w h(\mathrm{Q}_{-w}, \widehat{\mathrm{Q}}_w)$ and $h(\mathrm{Q})R_f^B h(\mathrm{Q}_{-f}, \widehat{\mathrm{Q}}_f)$ for each strategy $\widehat{\mathrm{Q}}_w$ of worker w and for each strategy $\widehat{\mathrm{Q}}_f$ of firm f.*

A **solution concept** is a function $C : \mathcal{Q} \to 2^{\mathcal{M}}$ that selects, for each market $P \in \mathcal{Q}$, a subsets of matchings $C(P) \subseteq \mathcal{M}$. We say that the game (h, P) **implements** C in Nash equilibrium if,

i) for each Nash equilibrium Q, $h(\mathrm{Q}) \in C(P)$,

ii) for each matching $\mu \in C(P)$ there is a Nash equilibrium $\mathrm{Q}^\star$ of the game (h, P) such that $h(\mathrm{Q}^\star) = \mu$.

[5] See Proposition 2.3 in Blair (1988), for more details.

3 Results

In this section, we present our results. Next, we show that any stable matching rule implements, in Nash equilibrium, the individually rational matchings.[6] Formally,

Theorem 1 *Let $P \in \mathcal{Q}$ be a market and let $h : \mathcal{Q} \to \mathcal{M}$ be a stable rule. Then, the game (h, P) implements in Nash equilibrium $I(P)$.*

Proof. Let P be a market and h a stable rule. In order to show that the game (h, P) implements in Nash equilibrium $I(P)$ we need to prove the following items:

***i*)** Let Q be a Nash equilibrium of the game (h, P). We prove that $h(\mathsf{Q}) \in I(P)$. Assume that $h(\mathsf{Q})$ is blocked by a worker w, $\emptyset P_w h(\mathsf{Q})(w)$ then the worker can improve by choosing the strategy in which no firm is acceptable i.e., $\mathsf{Q}'_{\mathsf{w}} = \emptyset$. Thus $h(\mathsf{Q}_{-\mathsf{w}}, \mathsf{Q}'_{\mathsf{w}}) P_w h(\mathsf{Q})$, contradicting that Q is a Nash equilibrium of the game. Therefore, $h(\mathsf{Q})$ is not blocked by any worker under P. Let f be a firm, since Q is a Nash equilibrium, for each strategy Q'_f,

$$h(\mathsf{Q})(f) = C_f^P \left(h(\mathsf{Q})(f) \cup h(\mathsf{Q}_{-f}, \mathsf{Q}'_f)(f) \right). \tag{2}$$

By choosing a strategy in which no subset of workers is acceptable i.e., $\mathsf{Q}'_f = \emptyset$ since h is a stable rule, $h(\mathsf{Q}_{-f}, \mathsf{Q}'_f)(f) = \emptyset$. Hence, (2) becomes $h(\mathsf{Q})(f) = C_f^P(h(\mathsf{Q})(f))$. Therefore, $h(\mathsf{Q})$ is not blocked by any firm under P. This implies that $h(\mathsf{Q}) \in I(P)$.

***ii*)** We need to prove that each individually rational matching under P can be supported by a Nash equilibrium. In order to do so, given $\mu \in I(P)$, we define the strategic profile $\mathsf{Q}^\star = (\mathsf{Q}^\star_f, \mathsf{Q}^\star_w)$ in which each firm f declares $\mathsf{Q}^\star_f$ such that:

- ⋄ for each $S \nsubseteq \mu(f)$, $\emptyset \mathsf{Q}^\star_f S$,
- ⋄ for each $S \subseteq \mu(f)$, $S \mathsf{Q}^\star_f \emptyset$,
- ⋄ for each $S, S' \subseteq \mu(f)$, $S P_f S'$ if and only if $S \mathsf{Q}^\star_f S'$,

and each worker w declares $\mathsf{Q}^\star_w = \mu(w)$.[7] Note that by definition, $\mathsf{Q}^\star_f$ is substitutable for each $f \in F$. Now, we need to prove that 1) $h(\mathsf{Q}^\star) = \mu$, and 2) $\mathsf{Q}^\star$ is a Nash equilibrium of the game.

[6]This result generalizes the result first presented by Alcalde (1996) for the marriage market.

[7]Observe that in the strategic profile $\mathsf{Q}^\star$ each firm list all subsets of $\mu(f)$ keeping the same order of the true preference P and each worker if it is assigned under μ to a firm then list this firm as the only acceptable firm, otherwise it list $\emptyset$.

1) $\boldsymbol{h(\mathrm{Q}^\star) = \mu}$. Since h is a stable rule under $\mathrm{Q}^\star$, to prove 1) is suffices to show that $S(\mathrm{Q}^\star) = \{\mu\}$. First, we show that $\mu \in S(\mathrm{Q}^\star)$. Since $\mu \in I(P)$, the definition of the preference profile $\mathrm{Q}^\star$ implies that $\mu \in I(\mathrm{Q}^\star)$. Assume that $\mu \notin S(\mathrm{Q}^\star)$. Then, there is a blocking pair (f, w) of μ under $\mathrm{Q}^\star$. Thus, $\mu(w) \neq f$, $f \mathrm{Q}^\star_w \mu(w)$ and $w \in C_f^{\mathrm{Q}^\star}(\mu(f) \cup \{w\})$. By definition of choice set, $C_f^{\mathrm{Q}^\star}(\mu(f) \cup \{w\})$ is a subset of $\mu(f) \cup \{w\}$ containing worker w and is an acceptable set for f under $\mathrm{Q}^\star$. This contradicts the definition of $\mathrm{Q}^\star_f$. Second, we claim that μ is the unique stable matching under $\mathrm{Q}^\star$. Otherwise, there is $\mu' \in S(\mathrm{Q}^\star)$ and $\mu' \neq \mu$. We assume that there is a worker $w' \in W$ such that $\mu(w') \neq \mu'(w')$. There are two cases to consider:

Case 1.1: $\boldsymbol{\mu(w') = \emptyset}$. Thus, there is $f' \in F$ such that $\mu'(w') = f'$. Definition of $\mathrm{Q}^\star_{w'}$ implies that $\emptyset \mathrm{Q}^\star_{w'} \mu'(w') = f'$, contradicting that $\mu' \in I(\mathrm{Q}^\star)$.

Case 1.2: There is $\boldsymbol{f \in F}$ such that $\boldsymbol{\mu(w') = f}$. By definition of $\mathrm{Q}^\star_f$, $\mu'(f) \subseteq \mu(f)$. Thus,

$$\mu(f) \cup \mu'(f) = \mu(f). \tag{3}$$

Individually rational of μ under P and definition of $\mathrm{Q}^\star_f$ imply that

$$\mu(f) = C_f^P(\mu(f)) = C_f^{\mathrm{Q}^\star}(\mu(f)). \tag{4}$$

Equations (3) and (4) imply that $\mu(f) = C_f^{\mathrm{Q}^\star}(\mu(f) \cup \mu'(f))$. Since $w' \in \mu(f) \setminus \mu'(f)$, the substitutability of $\mathrm{Q}^\star_f$ implies that

$$w' \in C^{\mathrm{Q}^\star_f}(\mu'(f) \cup \{w'\}). \tag{5}$$

By definition of $\mathrm{Q}^\star_{w'}$,

$$f = \mu(w') \mathrm{Q}^\star_{w'} \mu'(w'). \tag{6}$$

Hence, by (5) and (6), (f, w') is a blocking pair of μ' under $\mathrm{Q}^\star$. This contradicts that $\mu' \in S(\mathrm{Q}^\star)$. Therefore, by Cases 1.1 and 1.2, μ is the unique stable matching under $\mathrm{Q}^\star$.

2) $\boldsymbol{\mathrm{Q}^\star = (\mathrm{Q}^\star_f, \mathrm{Q}^\star_w)}$ **is a Nash equilibrium of** (h, P). Otherwise, there are two cases to consider:

Case 2.1: There is a firm $\boldsymbol{f \in F}$ and a strategy $\boldsymbol{\widehat{\mathrm{Q}}_f}$ such that $\boldsymbol{h(\mathrm{Q}^\star) R_f^B h(\mathrm{Q}^\star_{-f}, \widehat{\mathrm{Q}}_f)}$ does not hold. The strategy $(\mathrm{Q}^\star_{-f}, \widehat{\mathrm{Q}}_f)$ is denote by Q'. Thus,

$$h(\mathrm{Q}^\star)(f) \neq C_f^P(h(\mathrm{Q}^\star)(f) \cup h(\mathrm{Q}')(f)).$$

Then, there is a $w' \in C_f^P(h(\mathrm{Q}^\star)(f) \cup h(\mathrm{Q}')(f))$ such that $w' \notin h(\mathrm{Q}^\star)(f)$ and $w' \in h(\mathrm{Q}')(f)$. By 1), $h(\mathrm{Q}^\star)(f) = \mu(f)$ hence, $f \neq \mu(w')$. Recall that, $\mathrm{Q}'_{w'} =$

$Q^{\star}_{w'}$ then, $\mu(w')$ is the only acceptable firm under Q' for w'. Hence, $\emptyset Q'_{w'} f$ and $f = h(Q')(w')$. These last, contradicts that $h(Q') \in I(Q')$.

Case 2.2: There is a worker $w \in W$ and a strategy $\widehat{Q}_w$ such that $h(Q^{\star}) R_w h(Q^{\star}_{-w}, \widehat{Q}_w)$ does not hold. The strategy $(Q^{\star}_{-w}, \widehat{Q}_w)$ is denote by Q''. Thus,

$$h(Q''(w)) P_w h(Q^{\star})(w). \quad (7)$$

Let $f' \in F$ such that $f' = h(Q'')(w)$. By 1), $h(Q^{\star})(w) = \mu(w)$ and (7) becomes

$$f' P_w \mu(w).$$

Thus, $f' \neq \mu(w)$. Hence, $w \notin \mu(f')$. Recall that $Q''_{f'} = Q^{\star}_{f'}$. Since f' declares as acceptable sets to all subsets of $\mu(f')$ that are acceptable in P and $w \notin \mu(f')$ then no subset that contains w is acceptable under Q''. But $w \in h(Q'')(f')$. This contradicts that $h(Q'') \in I(Q'')$. By Cases 2.1 and 2.2, $Q^{\star}$ is a Nash equilibrium.

Therefore by ***i)*** and ***ii)***, h implements in Nash equilibrium the individually rational matchings. □

The existence of Nash equilibria follows from Theorem 1. Nash equilibrium strategies require agents on both sides of the market to misreport their true preferences and coordinate among themselves. For this reason, from now on, we assume that only workers play strategically and firms declare always their true preference. This means that each strategic profile Q is given by $Q = (P_f, Q_w)$ in which each firm chooses the true preference P_f and each worker chooses the strategy Q_w. By requiring an additional condition on firms' preferences, we can implement the stable matchings under Nash equilibrium. This additional condition is the "law of aggregate demand", which says that when a firm chooses from an expanded set, it hires at least as many workers as before. Formally,

Definition 3 *A firm f's preference relation satisfies the **law of aggregate demand (LAD)** if $S' \subseteq S \subseteq W$ implies $|C^P_f(S')| \leq |C^P_f(S)|$.*

We denote by $\mathcal{Q}_{LAD}$ the domain of substitutable preferences that satisfy LAD. The following theorem asserts that any stable rule implements, in Nash equilibrium, the stable matchings.

Theorem 2 *Let $P \in \mathcal{Q}_{LAD}$ be a market and let $h : \mathcal{Q}_{LAD} \to \mathcal{M}$ be a stable rule. Then, the game (h, P) implements in Nash equilibrium $S(P)$.*

The following is devoted to proving this theorem. In order to do so, we need to define some special stable rules: the **worker-optimal stable rule** h_W and the **firm-optimal**

stable rule h_F. Under the first one (the second one, respectively) the participants are assigned in accordance with the worker-optimal (respectively firm-optimal) stable matching under Q. Given a profile of preferences Q , we denote by $h_W(\mathsf{Q}) = \mu_W(\mathsf{Q})$ and $h_F(\mathsf{Q}) = \mu_F(\mathsf{Q})$ the worker-optimal stable matching and the firm-optimal stable matching, respectively.

A natural question is whether given a Nash equilibrium we know of any stable rule that selects a stable matching under true preferences for the equilibrium. The following lemma gives us an answer. An important fact about this result is that LAD is not needed to obtain it.[8]

Lemma 1 *Let $P \in \mathcal{Q}$ be a market. If Q is a Nash equilibrium of the game (h, P), then $h_F(\mathsf{Q}) \in S(P)$.*

Proof. Let $\mathsf{Q} = (P_f, \mathsf{Q}_w)$ be a Nash equilibrium of the game (h_F, P). By Theorem 1 we have that $h_F(\mathsf{Q}) \in I(P)$. Assume that $h_F(\mathsf{Q}) \notin S(P)$. Thus, there is a blocking pair (f, w) such that

$$w \in C_f^P\left(h_F(\mathsf{Q})(f) \cup \{w\}\right) \text{ and } f P_w h_F(\mathsf{Q})(w). \tag{8}$$

Now, we consider the strategic profile $\mathsf{Q}' = (\mathsf{Q}_{-w}, \mathsf{Q}'_w)$ where $\mathsf{Q}'_w = f, \emptyset$. There are two cases to consider:

Case 1: $h_F(\mathsf{Q}')(w) = \emptyset$. Thus,

$$f \mathsf{Q}'_w h_F(\mathsf{Q}')(w) = \emptyset. \tag{9}$$

We claim that $w \notin C_f^{\mathsf{Q}'}\left(h_F(Q')(f) \cup \{w\}\right)$. Otherwise, $w \in C_f^{\mathsf{Q}'}\left(h_F(\mathsf{Q}')(f) \cup \{w\}\right)$ and condition (9) imply that (f, w) blocks $h_F(\mathsf{Q}')$ under Q'. This contradicts that $h_F(\mathsf{Q}') \in S(\mathsf{Q}')$. Therefore,

$$w \notin C_f^{\mathsf{Q}'}\left(h_F(\mathsf{Q}')(f) \cup \{w\}\right). \tag{10}$$

Condition (10) together with $\mathsf{Q}' = (\mathsf{Q}_{-w}, \mathsf{Q}'_w)$ imply that

$$w \notin C_f^P\left(h_F(\mathsf{Q}')(f) \cup \{w\}\right). \tag{11}$$

Assume that,

$$h_F(\mathsf{Q})(f) = C_f^P\left(h_F(\mathsf{Q})(f) \cup h_F(\mathsf{Q}')(f)\right). \tag{12}$$

Thus, using repeatedly (1) and (12),

$$C_f^P\left(h_F(\mathsf{Q})(f) \cup \{w\}\right) = C_f^P\left(C_f^P\left(h_F(\mathsf{Q})(f) \cup h_F(\mathsf{Q}')(f)\right) \cup \{w\}\right)$$

$$= C_f^P\left(h_F(\mathsf{Q})(f) \cup h_F(\mathsf{Q}')(f) \cup \{w\}\right) = C_f^P\left(h_F(\mathsf{Q})(f) \cup C_f^P\left(h_F(\mathsf{Q}')(f) \cup \{w\}\right)\right).$$

[8] In this paper, the only results that need LAD are Lemma 2 and Theorem 2 .

Now, using (8)

$$w \in C_f^P \left(h_F(\mathsf{Q})(f) \cup \{w\}\right) = C_f^P \left(h_F(\mathsf{Q})(f) \cup C_f^P \left(h_F(\mathsf{Q}')(f) \cup \{w\}\right)\right).$$

This last toghether with condition (10) imply that $w \in h_F(\mathsf{Q})(f)$, contradicting this Case's hypothesis. Therefore,

$$h_F(\mathsf{Q})(f) \neq C_f^P \left(h_F(\mathsf{Q})(f) \cup h_F(\mathsf{Q}')(f)\right).$$

This means that $h_F(\mathsf{Q})(f) R_f^B h_F(\mathsf{Q}')(f)$ does not hold. Then Q is not a Nash equilibrium.

Case 2: $h_F(\mathsf{Q}')(w) = f$. By (8) we have $h_F(\mathsf{Q}')(w) P_w h_F(\mathsf{Q})(w)$, then Q is not a Nash equilibrium. Therefore, by Cases 1 and 2, Q is not a Nash equilibrium, contradicting our hypothesis.

□

The following lemma states that if Q is a Nash equilibrium then the equilibrium solution is the firm-optimal stable rule.

Lemma 2 *Let $P \in \mathcal{Q}_{LAD}$ be a market and h be a stable rule. If Q is a Nash equilibrium of the game (h, P), then $h(\mathsf{Q}) = h_F(\mathsf{Q})$.*

Proof.

Let $\mathsf{Q} = (P_f, \mathsf{Q}_w)$ be a Nash equilibrium of the game (h, P). Assume that $h(\mathsf{Q}) \neq h_F(\mathsf{Q})$. Thus, there is a firm f such that $h(\mathsf{Q})(f) \neq h_F(\mathsf{Q})(f)$. Since both are stable rules under Q and h_F is firm-optimal we have that $h_F(\mathsf{Q})(f) R_f^B h(\mathsf{Q})(f)$. By definition of Blair's order,

$$h_F(\mathsf{Q})(f) = C_f^{\mathsf{Q}} \left(h_F(\mathsf{Q})(f) \cup h(\mathsf{Q})(f)\right) \neq h(\mathsf{Q})(f). \tag{13}$$

Since $\mathsf{Q} = (P_f, \mathsf{Q}_w)$, (13) becomes

$$h_F(f) = C_f^P \left(h_F(f) \cup h(\mathsf{Q})(f)\right) \neq h(\mathsf{Q}). \tag{14}$$

We define the strategic profile $\mathsf{P}^\star = (\mathsf{Q}_{-f}, \mathsf{Q}_f^\star)$ in which each firm f declares $\mathsf{Q}_f^\star$ such that:

- ◇ for each $S \nsubseteq \mu(f)$, $\emptyset \mathsf{Q}_f^\star S$,
- ◇ for each $S \subseteq \mu(f)$, $S \mathsf{Q}_f^\star \emptyset$,
- ◇ for each $S, S' \subseteq \mu(f)$, $S P_f S'$ if and only if $S \mathsf{Q}_f^\star S'$. [9]

[9] Observe that in the strategic profile $\mathsf{P}^\star$ the firm f list all subsets of $\mu(f)$ keeping the same order of the true preference P and the other agents declare Q.

We claim that $h_F(Q) \in S(P^\star)$. Assume that $h_F(Q) \notin S(P^\star)$. Since $h_F(Q) \in I(P^\star)$, there is a blocking pair (f', w) of $h_F(Q)$ under $P^\star$ that is,

$$f' P^\star_w h_F(Q)(w) \text{ and } w \in C^{P^\star}_{f'}\left(h_F(Q)(f') \cup \{w\}\right). \tag{15}$$

If $f' \neq f$, definition of $P^\star$ implies that (15) becomes

$$f' Q_w h_F(Q)(w) \text{ and } w \in C^{Q}_{f'}\left(h_F(Q)(f') \cup \{w\}\right).$$

Thus, (f', w) blocks $h_F(Q)$ under Q, contradicting that $h_F(Q) \in S(Q)$. Therefore, $f' = f$. By (15) and definition of the choice set, $C^{P^\star}_f(h_F(Q)(f) \cup \{w\})$ is a subset of $h_F(Q)(f) \cup \{w\}$ containing worker w and is an acceptable set for f under $P^\star$. This contradicts the definition of $P^\star$ and the claim is proven.

Now, since $h_F(Q)$ and $h(P^\star)$ are stable under $P^\star$, using the Rural Hospital Theorem,[10] $|h(P^\star)(f)| = |h_F(Q)(f)|$. Since the firm f in $P^\star$, only list all subsets of $h_F(Q)(f)$, we have that $h(P^\star)(f) \subseteq h_F(Q)(f)$. Therefore, $h(P^\star)(f) = h_F(Q)(f)$. This last together with (14) imply that

$$h(P^\star)(f) = C^P_f\left(h(P)^\star(f) \cup h(Q)(f)\right) \neq h(Q)(f).$$

This contradicts that Q is a Nash equilibrium of the game (h, P). □

In the Appendix 4 we present an example showing that the requirement of LAD is necessary for Lemma 2. Now, we are in a position to prove Theorem 2.

Proof of Theorem 2. Let P be a market and h a stable rule. In order to show that the game (h, P) implements in Nash equilibrium $S(P)$ we need to prove the following items:

***i*)** Let $Q = (P_f, Q_w)$ be a Nash equilibrium of the game (h, P). By Lemma 2, $h(Q) = h_F(Q)$. Now, Lemma 1 implies that $h_F(Q)$ is stable under P. Therefore, $h(Q)$ is stable under P.

***ii*)** We need to prove that each stable matching under P can be supported by a Nash equilibrium. In order to do so, given $\mu \in S(P)$ we consider the strategic profile $Q^\star = (P_f, Q^\star_w)$ in which each firm declares the true preference P_f and each worker w choose $Q^\star_w = \mu(w)$. Note that by definition, $Q^\star_f = P_f$ is substitutable for each $f \in F$. Now, we need to prove that 1) $h(Q^\star) = \mu$, and 2) $Q^\star$ is a Nash equilibrium of the game.

[10] The *Rural Hospital Theorem* is proven in different contexts by many authors (see McVitie and Wilson, 1971; Roth, 1984, 1985; Martínez et al., 2000; Alkan, 2002; Kojima, 2012, among others). The version of this theorem for a many-to-many matching market where all agents have substitutable choice functions satisfying *LAD*, that also applies in our setting, is presented in Alkan (2002) and states that each agent is matched with the same number of partners in every stable matching.

1) $\boldsymbol{h(\mathrm{Q}^{\star}) = \mu}$. Since h is a stable rule under $\mathrm{Q}^{\star}$, to prove 1) is suffices to show that $S(\mathrm{Q}^{\star}) = \{\mu\}$. First, we show that $\mu \in S(\mathrm{Q}^{\star})$. Since $\mu \in S(P)$, the definition of the preference profile $\mathrm{Q}^{\star}$ implies that $\mu \in I(\mathrm{Q}^{\star})$. Assume that $\mu \notin S(\mathrm{Q}^{\star})$. Then, there is a blocking pair (f, w) of μ under $\mathrm{Q}^{\star}$. Thus, $\mu(w) \neq f$, $w \in C_f^{\mathrm{Q}^{\star}}(\mu(f) \cup \{w\})$ and $f\mathrm{Q}^{\star}_w\mu(w)$. But, $f\mathrm{Q}^{\star}_w\mu(w)$ contradicts the definition of $\mathrm{Q}^{\star}_w$. Second, we claim that μ is the unique stable matching under $\mathrm{Q}^{\star}$. Otherwise, there is $\mu' \in S(\mathrm{Q}^{\star})$ and $\mu' \neq \mu$. Thus, there is $w' \in W$ such that $\mu(w') \neq \mu'(w')$. There are two cases to consider:

Case 1.1: $\boldsymbol{\mu(w') = \emptyset}$ and there is $\boldsymbol{f \in F}$ such that $\boldsymbol{\mu'(w') = f}$. Definition of $\mathrm{Q}^{\star}_{w'}$ implies that $\emptyset = \mu(w')\mathrm{Q}^{\star}_{w'}\mu'(w') = f$, contradicting that $\mu' \in I(\mathrm{Q}^{\star})$.

Case 1.2: There is $\boldsymbol{f \in F}$ such that $\boldsymbol{\mu(w') = f}$ and $\boldsymbol{\mu'(w') = \emptyset}$. We claim that $\mu'(f) \subseteq \mu(f)$. Otherwise, there is $w'' \in W$ such that $w'' \in \mu'(f)$ and $w'' \notin \mu(f)$. The definition of $\mathrm{Q}^{\star}_{w''}$ implies that

$$\mu(w'')\mathrm{Q}^{\star}_{w''}\emptyset\mathrm{Q}^{\star}_{w''}\ \mu'(w'') = f$$

contradicting that $\mu' \in I(\mathrm{Q}^{\star})$. Thus, the claim is proved. Since $\mu'(f) \subseteq \mu(f)$ and $w' \in \mu'(f)$ imply that,

$$w' \in \mu(f) = C_f^{\mathrm{Q}^{\star}}\left(\mu(f) \cup \mu'(f)\right).$$

By substitutability,

$$w' \in C_f^{\mathrm{Q}^{\star}}\left(\mu'(f) \cup \{w'\}\right). \tag{16}$$

By the case's hypothesis,

$$f = \mu(w')\mathrm{Q}^{\star}_{w'}\mu'(w') = \emptyset. \tag{17}$$

By (16) and (17), (f, w') blocks μ' under $\mathrm{Q}^{\star}$. This contradicts that $\mu' \in S(\mathrm{Q}^{\star})$. Therefore, by Cases 1.1 and 1.2, μ is the unique stable matching under $\mathrm{Q}^{\star}$.

2) $\boldsymbol{\mathrm{Q}^{\star} = (P_f, \mathrm{Q}^{\star}_{\mathbf{w}})}$ **is a Nash equilibrium of the game.** Otherwise, there are two cases to consider:

Case 2.1: There is a firm $\boldsymbol{f \in F}$ and a strategy $\boldsymbol{\widehat{\mathrm{Q}}_f}$ such that $\boldsymbol{h(\mathrm{Q}^{\star})\, R_f^B h(\mathrm{Q}^{\star}_{-f}, \widehat{\mathrm{Q}}_f)}$ does not hold. The strategic profile $(\mathrm{Q}^{\star}_{-f}, \widehat{\mathrm{Q}}_f)$ is denote by Q'. Thus,

$$h(\mathrm{Q}^{\star})(f) \neq C_f^P\left(h(\mathrm{Q}^{\star})(f) \cup h(\mathrm{Q}')(f)\right).$$

Then, there is $w' \in C_f^P\left(h(\mathrm{Q}^{\star})(f) \cup h(\mathrm{Q}')(f)\right)$ such that $w' \notin h(\mathrm{Q}^{\star})(f)$ and $w' \in h(\mathrm{Q}')(f)$. By 1), $h(\mathrm{Q}^{\star})(f) = \mu(f)$ hence, $f \neq \mu(w')$. Recall that, $\mathrm{Q}'_{w'} = \mathrm{Q}^{\star}_{w'}$ then, $\mu(w')$ is the unique acceptable firm under Q' for w'. Hence, $\emptyset\mathrm{Q}'_{w'}f$ and $f = h(\mathrm{Q}')(w')$, contradicting that $h(\mathrm{Q}') \in I(\mathrm{Q}')$.

Case 2.2: There is a worker $w \in W$ and a strategy $\widehat{\mathsf{Q}}_w$ such that $h(\mathsf{Q}^\star)R_w h(\mathsf{Q}^\star_{-w}, \widehat{\mathsf{Q}}_w)$ does not hold. We consider the strategic profile $\mathsf{Q}'' = (\mathsf{Q}^\star_{-w}, \widehat{\mathsf{Q}}_w)$. Thus,

$$h(\mathsf{Q}''(w))P_w h(\mathsf{Q}^\star)(w). \tag{18}$$

Since $h(\mathsf{Q}^\star)(w) = \mu(w)$ and $\mu \in I(P)$, there is $f' \in F$ such that $f' = h(\mathsf{Q}'')(w)$. Thus,

$$w \in h(\mathsf{Q}'')(f') \text{ and } w \notin h(\mathsf{Q}^\star)(f'). \tag{19}$$

We claim that

$$h(\mathsf{Q}'')(f') \subsetneq h(\mathsf{Q}^\star)(f') \cup \{w\}. \tag{20}$$

First, we prove that $h(\mathsf{Q}'')(f') \subseteq h(\mathsf{Q}^\star)(f') \cup \{w\}$. Notice that $w \in h(\mathsf{Q}^\star)(f') \cup \{w\}$. Let $\widehat{w} \neq w$ such that $\widehat{w} \in h(\mathsf{Q}'')(f')$. Definition of Q'' and 1) imply that $\mathsf{Q}''_{\widehat{w}} = \mathsf{Q}^\star_{\widehat{w}} = \mu(\widehat{w}) = h(\mathsf{Q}^\star)(\widehat{w})$. This last, together with $h(\mathsf{Q}'')(\widehat{w}) = f'$ and the fact that $h(\mathsf{Q}'') \in I(\mathsf{Q}'')$ imply that $h(\mathsf{Q}'')(\widehat{w}) = f' = h(\mathsf{Q}^\star)(\widehat{w})$. Then $\widehat{w} \in h(\mathsf{Q}^\star)(f')$, and the inclusion is proven. Second, we prove that $h(\mathsf{Q}'')(f')$ is a proper subset of $h(\mathsf{Q}^\star)(f') \cup \{w\}$. Otherwise,

$$h(\mathsf{Q}'')(f') = h(\mathsf{Q}^\star)(f') \cup \{w\}. \tag{21}$$

Now, assume that

$$w \in C^P_{f'}\left(h(\mathsf{Q}^\star)(f') \cup \{w\}\right). \tag{22}$$

By 1), $h(\mathsf{Q}^\star) = \mu$, together with (18) and (22), imply that (f', w) blocks μ under P. This contradicts that $\mu \in S(P)$. Therefore,

$$w \notin C^P_{f'}\left(h(\mathsf{Q}^\star)(f') \cup \{w\}\right).$$

The fact that $h(\mathsf{Q}'')(f') \subseteq h(\mathsf{Q}^\star)(f') \cup \{w\}$ together with the substitutability of P implies that,

$$w \notin C^P_{f'}\left(h(\mathsf{Q}^\star)(f') \cup h(\mathsf{Q}'')(f')\right) = C^P_{f'}\left(h(\mathsf{Q}^\star)(f') \cup \{w\}\right). \tag{23}$$

By (23), the definition of $\mathsf{Q}^\star$ and the fact that $h(\mathsf{Q}^\star) \in S(\mathsf{Q}^\star)$ we have,

$$w \notin C^{\mathsf{Q}^\star}_{f'}\left(h(\mathsf{Q}^\star)(f') \cup \{w\}\right) = h(\mathsf{Q}^\star)(f'). \tag{24}$$

Using (21), (24), and (19) we get,

$$C^{\mathsf{Q}^\star}_{f'}\left(h(\mathsf{Q}'')(f')\right) = C^{\mathsf{Q}^\star}_{f'}\left(h(\mathsf{Q}^\star)(f') \cup \{w\}\right) = h(\mathsf{Q}^\star)(f') \neq h(\mathsf{Q}'')(f'). \tag{25}$$

Furthermore, by definition of Q'', we have that $C^{\mathsf{Q}^\star}_{f'}\left(h(\mathsf{Q}'')(f')\right) = C^{\mathsf{Q}''}_{f'}\left(h(\mathsf{Q}'')(f')\right)$. Now, using (25) it follows that,

$$C_{f'}^{Q''}\left(h(Q'')(f')\right) \neq h(Q'')(f'),$$

contradicting $h(Q'') \in I(Q'')$. Therefore, (20) holds. Thus, there is $w^\star \in h(Q^\star)(f') \setminus h(Q'')(f')$. Assume that $w^\star \notin C_{f'}^{Q''}\left(h(Q'')(f') \cup \{w^\star\}\right)$. Using (20) and substitutability,

$$w^\star \notin C_{f'}^{Q''}\left(h(Q^\star)(f') \cup \{w\} \cup \{w^\star\}\right) = C_{f'}^{Q''}\left(h(Q^\star)(f') \cup \{w\}\right)$$

By definition of Q'',

$$w^\star \notin C_{f'}^{Q^\star}\left(h(Q^\star)(f') \cup \{w\}\right). \tag{26}$$

Since $h(Q^\star) \in I(Q^\star)$,

$$w^\star \in h(Q^\star)(f') = C_{f'}^{Q^\star}\left(h(Q^\star)(f')\right). \tag{27}$$

Conditions (26) and (27) imply that

$$C_{f'}^{Q^\star}\left(h(Q^\star)(f')\right) \neq C_{f'}^{Q^\star}\left(h(Q^\star)(f') \cup \{w\}\right).$$

Thus,

$$w \in C_{f'}^{Q^\star}\left(h(Q^\star)(f') \cup \{w\}\right). \tag{28}$$

By (18) and (28), (f', w) blocks $h(Q^\star)$, contradicting that $h(Q^\star) \in S(Q^\star)$. Therefore

$$w^\star \in C_{f'}^{Q''}\left(h(Q'')(f') \cup \{w^\star\}\right). \tag{29}$$

Since $w^\star \neq w$ then $Q''_{w^\star} = Q^\star_{w^\star} = h(Q^\star)(w^\star) = f'$. This means that,

$$f' = h(Q^\star)(w^\star) P_{w^\star} h(Q'')(w^\star) \tag{30}$$

Using (29) and (30) , $(f', w^\star)$ blocks $h(Q'')$ under Q'', contradicting the stability of $h(Q'')$ under Q''. By Cases 2.1 and 2.2, $Q^\star$ is a Nash equilibrium.

Therefore by ***i)*** and ***ii)***, h implements in Nash equilibrium the stable matchings. □

4 Concluding Remarks

The main motivation of this paper is to provide a framework to study the Nash equilibrium solutions of the game induced by stable rules. In a many-to-one matching market with substitutable preferences, we show that any stable rule implements, in Nash equilibrium, the individually rational matchings. In this market, by requiring LAD on the preferences of firms and assuming that only workers play strategically, we show any stable rule implements, in Nash equilibrium, the stable matchings.

It is usual in the literature to study many-to-one markets assuming that firms' preferences are responsive. This is due to the close relationship between this market with responsive preferences and the marriage market (for a thorough survey on this fact, see Roth and Sotomayor (1990)). However, when firms are endowed with substitutable preferences (a much less restrictive requirement), this relation with the marriage market no longer holds. Thus, extending the results of a many-to-one market with responsive preferences to substitutable preferences is not straightforward.

Although the result of the implementation of individually rational matchings can be generalized to a many-to-many matching market under substitutability, the results of the implementation of stable matchings are not. This fact has already been noted in Sotomayor (2012).

The study of the implementability of several solution concepts under other equilibria notions is an interesting topic for future research.

Appendix

The following example shows that without LAD the Lemma 2 is not valid.

Example 1 *Let P be a market where $F=\{f_1,f_2,f_3\}$, $W=\{w_1,w_2,w_3,w_4\}$ and the preference profile is given by:*

$$\begin{array}{ll} P_{f_1}: w_1w_2, w_1, w_2, w_3w_4, w_3, w_4, \emptyset & P_{w_i}: f_2, f_3, f_1, \emptyset \quad \textit{for } i=1,2 \\ P_{f_2}: w_3, w_1w_4, w_4, w_1w_2, w_1, w_2, \emptyset & P_{w_3}: f_1, f_3, f_2, \emptyset \\ P_{f_3}: w_4, w_2w_3, w_1w_2, w_3, w_1, w_2, \emptyset & P_{w_4}: f_1, f_2, f_3, \emptyset \end{array}$$

We compute the two optimal stable matchings under P:

$$\mu_F = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1w_2 & w_3 & w_4 \end{pmatrix}, \textit{ and } \mu_W = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_3w_4 & w_1w_2 & \emptyset \end{pmatrix}.$$

Note that the preferences of the firms are substitutable but do not satisfy LAD. Consider the following sets $\{w_2,w_3,w_4\}$ and $\{w_3,w_4\}$. We can observe that

$$|C^P_{f_1}(\{w_2,w_3,w_4\})| = |\{w_2\}| < |C^P_{f_1}(\{w_3,w_4\})| = |\{w_3,w_4)|,$$

contradicting the definition of LAD. Now, consider the worker-optimal stable rule h_W. We claim that P is a Nash equilibrium of the game (h_W,P). To see this, observe that if worker true-telling, then under the worker-optimal stable rule h_W, the worker gets her top. Therefore, no worker can achieve a better solution by deviating from P assuming that the other agents *maintain their strategy P. The firms to manipulate could declare the subset assigned to them by h_W as not acceptable and the corresponding single sets too. For instance, assume that the firm f_1 misrepresents her preferences,*

$$Q_{f_1} : w_1w_2, w_1, w_2, \emptyset$$

Let $Q = (P_{-f_1}, Q_{f_1})$, it can be check that $h_W(Q) = \begin{pmatrix} f_1 & f_2 & f_3 \\ \emptyset & w_1w_4 & w_2w_3 \end{pmatrix}$. So firm f_1 does not obtain a preferred solution. It can be verified that we obtain the same solution if the firm f_1 only declares as acceptable to w_1 or w_2. Now assume that f_2 misrepresents her preferences,

$$Q_{f_2} : w_3, w_1w_4, w_4, w_1, \emptyset$$

Let $Q = (P_{-f_2}, Q_{f_2})$, it can be check that $h_W(Q) = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_3w_4 & w_1 & w_2 \end{pmatrix}$. So firm f_2 does not obtain a preferred solution. It can be verified that we obtain the same solution if firm f_2 declares, $Q_{f_2} : w_1w_4, w_4, w_1, \emptyset$.

Assume that f_2 misrepresents her preferences,

$$Q_{f_2} : w_3, w_4, \emptyset$$

Let $Q = (P_{-f_2}, Q_{f_2})$, it can be check that $h_W(Q) = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_3w_4 & \emptyset & w_1w_2 \end{pmatrix}$. So firm f_2 does not obtain a preferred solution. It can be verified that we obtain the same solution if the firm f_2 only declares as acceptable to w_3 or w_4.

Since f_3 is not at the top of any worker's preference in this example, even if the f_3 does not true-telling then the stable matching rule h_W will not assign any subsets of workers if the other agents true-telling. Therefore, f_3 not will obtain a preferred solution deviating from P. Thus, preference profile P is a Nash equilibrium of the game. Since, $\mu_W \neq \mu_F$ then $h_W(P) \neq h_F(P)$ showing that Lemma 2 does not hold without LAD. ♢

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