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R. Pablo Arribillaga (UNSL-CONICET)

Agustín G. Bonifacio (UNSL-CONICET)

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Not obviously manipulable allotment rules*

R. Pablo Arribillaga† Agustín G. Bonifacio†

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Abstract

In the problem of allocating a single non-disposable commodity among agents whose preferences are single-peaked, we study a weakening of strategy-proofness called not obvious manipulability (NOM). If agents are cognitively limited, then NOM is sufficient to describe their strategic behavior. We characterize a large family of own-peak-only rules that satisfy efficiency, NOM, and a minimal fairness condition. We call these rules "simple". In economies with excess demand, simple rules fully satiate agents whose peak amount is less than or equal to equal division and assign, to each remaining agent, an amount between equal division and his peak. In economies with excess supply, simple rules are defined symmetrically. We also show that the single-plateaued domain is maximal for the characterizing properties of simple rules. Therefore, even though replacing strategy-proofness with NOM greatly expands the family of admissible rules, the maximal domain of preferences involved remains basically unaltered.

JEL classification: D71, D72.

Keywords: obvious manipulations, allotment rules, maximal domain, single-peaked preferences, single-plateaued preferences.

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†Instituto de Matemática Aplicada San Luis (UNSL-CONICET) and Departamento de Matemática, Universidad Nacional de San Luis, San Luis, Argentina. Emails: rarribi@unsl.edu.ar (R. P. Arribillaga) and abonifacio@unsl.edu.ar (A. G. Bonifacio).

1 Introduction

Consider the problem of allocating a single non-disposable commodity among a group of agents with single-peaked preferences: up to some critical level, called the peak, an increase in an agent's consumption raises his welfare; beyond that level, the opposite holds. In this context, an allotment rule is a systematic procedure that allows agents to select an allotment, among many, according to their preferences. An allotment rule is strategy-proof if misreporting preferences is never better than truth-telling. In this paper, we study a weakening of strategy-proofness called not obvious manipulability.

Criteria of incentive compatibility, efficiency, and fairness are considered the three perennial goals of mechanism design. When those three criteria are materialized in the properties of strategy-proofness, efficiency, and symmetry, they single out the "uniform" rule (Sprumont, 1991; Ching, 1994); whereas relaxing symmetry by permitting an asymmetric treatment of agents expands the options to the versatile family of "sequential" allotment rules (Barberà et al., 1997). Insisting on efficiency and on a minimal fairness condition, and in line with a current strand in the literature, we investigate what happens when strategy-proofness is weakened. The idea underlying this approach is that even though manipulations are pervasive, agents may not realize they can manipulate a rule because they lack information about others' behavior or they are cognitively limited. A misreport that in a specific situation (a specific preference profile of the other agents) is a profitable manipulation may actually be worse than truth-telling in another situation. Therefore, if we consider an agent with limited information on other agents' preferences, it may be very unclear whether such manipulation will be profitable in practice. Assuming that the agent knows all possible outcomes under any misreport and under truth-telling, there are several different ways to determine when a misreport is recognized or identified as a profitable manipulation. Troyan and Morrill (2020), in the context of two-sided matching, propose a simple and tractable way to do this by considering best/worst-case scenarios to formalize the notion of obvious manipulation. A manipulation is obvious if it either makes the agent better off than truth-telling in the worst case or makes the agent better off than truth-telling in the best case. An allotment rule is not obviously manipulable (NOM) if it has no obvious manipulation.

If agents are cognitively limited, then NOM is sufficient to describe their strategic behavior. Therefore, the question arises to what extent NOM rules enrich the landscape of strategy-proof rules. We will focus on own-peak-only rules. This means that the sole information collected by these rules from an agent's preference to determine his allotment is his peak amount. Because of their simplicity, own-peak-only rules are important rules in their own right and are both useful in practice and extensively studied in the literature. Furthermore, the own-peak-only property follows from efficiency and strategy-proofness (see Sprumont, 1991; Ching, 1994). However, since we do not impose strategy-proofness, we explicitly invoke this property.

Our goal is to introduce and characterize a large family of own-peak-only rules, which we call "simple". Their definition is as follows. In economies with excess demand, simple rules fully satiate agents whose peak amount is less than or equal to equal division and assign to each remaining agent an amount between equal division and his peak. Symmetrically, in economies with excess supply, simple rules fully satiate agents whose peak is greater than or equal to equal division and assign to each remaining agent an amount between his peak and equal division. We believe that the full family of simple rules has been dormant in the literature because many of the rules (and families of rules) identified in the literature belong to this large family, even though their definition needs more specifications and details. This is the case, for example, of the family of sequential rules in Barberà et al. (1997) that are defined throughout an intricated process needed to cope with strategy-proofness.

In our main result, Theorem 2, we show that an allotment rule is simple if and only if it is own-peak-only, efficient, satisfies NOM, and meets the equal division guarantee. This last property is a minimal fairness requirement that states that whenever an agent demands equal division, the rule must guarantee him that amount. If we narrow the picture to symmetric rules, i.e., rules that assign indifferent allotments to agents with the same preferences, then we also obtain a characterization of a big subfamily of simple rules that satisfies all five properties (Corollary 1).[1] As we previously said, in contrast, when we replace NOM with strategy-proofness the only efficient and symmetric rule that remains is the uniform rule (Ching, 1994). We also provide three variants of our main characterization: (i) adapting the result to economies with individual endowments (Theorem 3), (ii) invoking a peak monotonicity property that also encompasses symmetry and the equal division guarantee (Theorem 4), and (iii) getting rid of efficiency (Theorem 5).

[1]Note that although symmetry and the equal division guarantee can both be considered as fairness requirements, they are logically independent properties.

Next, we analyze the maximality of the domain of preferences (including the domain of single-peaked preferences) for which a rule satisfying own-peak-onliness, efficiency, the equal division guarantee, and NOM exists. For the properties of efficiency, strategy-proofness, and symmetry, the single-plateaued domain is maximal (Ching and Serizawa, 1998; Massó and Neme, 2001). In Theorem 6, we show that the single-plateaued domain is maximal for our properties as well. Therefore, even though replacing strategy-proofness with NOM greatly expands the family of admissible rules, the maximal domain of preferences involved remains basically unaltered.

To the best of our knowledge, our paper is the first one that applies Troyan and Morrill (2020) notion of obvious manipulations to the allocation of a non-disposable commodity among agents with single-peaked preferences. In the context of voting, Aziz and Lam (2021) and Arribillaga and Bonifacio (2022) study non-peaks-only and peaks-only rules, respectively. Other recent papers that study obvious manipulations, in other situations, are Ortega and Segal-Halevi (2022), Psomas and Verma (2022), and Arribillaga and Risma (2023).

The rest of the paper is organized as follows. The model and the concept of obvious manipulations are introduced in Section 2. In Section 3, we present the main characterization of simple rules. Section 4 analyses some related characterizations. The maximal domain result is presented in Section 5. To conclude, some final remarks are gathered in Section 6.

2 Preliminaries

2.1 Model

A social endowment $\Omega \in \mathbb{R}_{++}$ is an amount of a perfectly divisible commodity to be distributed among a set of agents $N = \{1, 2, \ldots, n\}$. Each $i \in N$ is equipped with a continuous preference relation R_i defined over $\mathbb{R}_+ \cup \{\infty\}$. Call P_i and I_i to the strict preference and indifference relations associated with R_i, respectively. Denote by $\mathcal{U}$ the domain of all such preferences. Given $R_i \in \mathcal{U}$, let $p(R_i) = \{x \in \mathbb{R}_+ \cup \{\infty\} : xR_iy \text{ for each } y \in \mathbb{R}_+ \cup \{\infty\}\}$ be the set of preferred consumptions according to R_i, called the **peak** of R_i. When $p(R_i)$ is a singleton, we slightly abuse notation and use $p(R_i)$ to denote its single element. Agents i's preference $R_i \in \mathcal{U}$ is **single-peaked** if $p(R_i)$ is a singleton and, for each pair $\{x_i, x_i'\} \subseteq \mathbb{R}_+$, we have $x_i P_i x'_i$ as long as either $x'_i < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < x'_i$ holds. Denote by $\mathcal{SP}$ the domain of all such preferences.

A (generic) domain of preferences $\mathcal{D}$ is a subset of $\mathcal{U}$. Given a domain of preferences $\mathcal{D} \subseteq \mathcal{U}$, an **economy** in $\mathcal{D}$ consists of a profile of preferences $R = (R_j)_{j \in N} \in \mathcal{D}^N$ and a social endowment $\Omega \in \mathbb{R}_{++}$ and is denoted by (R, Ω). Let $\mathcal{E}_\mathcal{D}$ be the domain of all such economies. Given a social endowment $\Omega \in \mathbb{R}_{++}$, the set of **(feasible) allotments** of Ω is $X(\Omega) = \{x \in \mathbb{R}^N_+ : \sum_{j \in N} x_j = \Omega\}$. An **(allotment) rule** on $\mathcal{E}_\mathcal{D}$ is a function $\varphi : \mathcal{E}_\mathcal{D} \longrightarrow \mathbb{R}^N_+$ such that $\varphi(R, \Omega) \in X(\Omega)$ for each $(R, \Omega) \in \mathcal{E}_\mathcal{D}$.

Given a rule φ defined on a generic domain $\mathcal{E}_\mathcal{D}$, some desirable properties we consider are listed next.

Efficiency: For each $(R, \Omega) \in \mathcal{E}_\mathcal{D}$, there is no $x \in X(\Omega)$ such that $x_i R_i \varphi_i(R, \Omega)$ for each $i \in N$ and $x_i P_i \varphi_i(R, \Omega)$ for some $i \in N$.

Efficiency is the usual Pareto optimality criterion. Under this condition, for each economy, the allocation selected by the rule should be such that there is no other allocation that all agents find at least as desirable and at least one agent (strictly) prefers.

Next, we introduce a useful property that is equivalent to efficiency on $\mathcal{E}_{\mathcal{SP}}$.

Same-sidedness: For each $(R, \Omega) \in \mathcal{E}_{\mathcal{SP}}$,

(i) $\sum_{j \in N} p(R_j) \geq \Omega$ implies $\varphi_i(R) \leq p(R_i)$ for each $i \in N$, and

(ii) $\sum_{j \in N} p(R_j) \leq \Omega$ implies $\varphi_i(R) \geq p(R_i)$ for each $i \in N$.

Remark 1 *Let φ be a rule defined on $\mathcal{E}_{\mathcal{SP}}$. Then, φ is efficient if and only if it is same-sided.*

A rule is strategy-proof if, for each agent,truth-telling is always optimal, regardless of the preferences declared by the other agents. To define it formally, let $R_i, R'_i \in \mathcal{D}$, and $\Omega \in \mathbb{R}_{++}$. Preference R'_i is a **manipulation of $\boldsymbol{\varphi}$ at $(\boldsymbol{R_i}, \boldsymbol{\Omega})$** if there is $R_{-i} \in \mathcal{D}^{n-1}$ such that $\varphi_i(R'_i, R_{-i}, \Omega) P_i \varphi_i(R_i, R_{-i}, \Omega)$.

Strategy-proofness: For each $i \in N$ and each $(R_i, \Omega) \in \mathcal{D} \times \mathbb{R}_{++}$, there is no manipulation of φ at (R_i, Ω).

The following is an informational simplicity property stating that if an agent unilaterally changes his preference for another one with the same peak, then his allotment remains unchanged.[2]

Own-peak-onliness: For each $(R, \Omega) \in \mathcal{E}_{\mathcal{D}}$, each $i \in N$, and each $R'_i \in \mathcal{D}$ such that $p(R'_i) = p(R_i)$, we have $\varphi_i(R, \Omega) = \varphi_i(R'_i, R_{-i}, \Omega)$.

Analyzing the uniform rule, Sprumont (1991) derives the own-peak-only property from efficiency and strategy-proofness (see also Ching, 1994). Since we do not impose strategy-proofness, we explicitly invoke own-peak-onliness.

Next, we present a minimal fairness condition. It states that whenever an agent has equal division as his preference's peak, the rule must assign it to him.

Equal division guarantee: For each $(R, \Omega) \in \mathcal{D}$ and each $i \in N$ such that $\frac{\Omega}{n} \in p(R_i)$, we have $\varphi_i(R) I_i \frac{\Omega}{n}$.

Another well-known fairness property states that agents with the same preferences should be assigned indifferent allocations.

Symmetry: For each $(R, \Omega) \in \mathcal{D}$ and each $\{i, j\} \subseteq N$ such that $R_i = R_j$, $\varphi_i(R, \Omega) I_i \varphi_j(R, \Omega)$.

2.2 Obvious manipulations

The notion of obvious manipulation is introduced by Troyan and Morrill (2020). They try to single out those manipulations that are easily identifiable by the agents. A manipulation is obvious if the worst possible outcome under the manipulation is strictly better than the worst possible outcome under truth-telling.[3] When the set of alternatives is infinite and preferences allow indifferences, worst possible outcomes may not be well-defined and a more general definition is necessary in this case. We say that a manipulation is obvious if each possible outcome under the manipulation is strictly better than some possible outcome under truth-telling.

Before we present the formal definition, we introduce some notation. Given a rule φ

[2] This property is weaker than the "peak-only" property, that has been imposed in a number of axiomatic studies. See Section 6 for more details.

[3] Troyan and Morrill (2020) also consider that a manipulation is obvious when the best possible outcome under the manipulation is strictly better than the best possible outcome under truth-telling. However, under EFF, the best possible outcome under truth-telling for an agent is always a peak alternative for that agent in our model. Therefore, no manipulation becomes obvious by considering best possible outcomes and so we omit this in our definition.

defined on $\mathcal{E}_\mathcal{D}$ and $(R_i, \Omega) \in \mathcal{D} \times \mathbb{R}_{++}$, the **option set** attainable with (R_i, Ω) at φ is

$$O^\varphi(R_i, \Omega) = \left\{ \varphi_i(R_i, R_{-i}) \in [0, \Omega] \text{ such that } R_{-i} \in \mathcal{D}^{n-1} \right\}.$$ [4]

Definition 1 *Let φ be a rule defined on $\mathcal{E}_\mathcal{D}$ and let $R_i, R_i' \in \mathcal{D}$ and $\Omega \in \mathbb{R}_{++}$ be such that R_i' is a manipulation of φ at (R_i, Ω). Then, R_i' is an* ***obvious manipulation*** *if for each $x' \in O^\varphi(R_i', \Omega)$ there is $x \in O^\varphi(R_i, \Omega)$ such that $x' P_i x$. The rule φ is* ***not obviously manipulable (NOM)*** *if it does not admit any obvious manipulation.*

The next proposition shows that our definition generalizes Troyan and Morrill (2020)'s definition. In fact, both definitions are equivalent whenever worst possible outcomes exist. Given R_i and $Y \subseteq [0, \Omega]$, denote by $W(R_i, Y)$ the element in Y (if any) such that $x P_i W(R_i, Y)$ for each $x \in Y \setminus \{W(R_i, Y)\}$.

Proposition 1 *Let φ be a rule defined on $\mathcal{E}_\mathcal{D}$ and let $R_i, R_i' \in \mathcal{D}$ and $\Omega \in \mathbb{R}_{++}$ be such that R_i' is a manipulation of φ at (R_i, Ω). Assume that $W(R_i, O^\varphi(R_i, \Omega))$ and $W(R_i, O^\varphi(R_i', \Omega))$ exist. Then, R_i' is an obvious manipulation if and only if $W(R_i, O^\varphi(R_i', \Omega) P_i W(R_i, O^\varphi(R_i, \Omega)$.*

Proof. ($\Longrightarrow$) Let R_i' be an obvious manipulation of φ at (R_i, Ω). As $W(R_i, O^\varphi(R_i', \Omega)) \in O^\varphi(R_i', \Omega)$, there is $x \in O^\varphi(R_i, \Omega))$ such that $W(R_i, O^\varphi(R_i, \Omega)) P_i x$. Then, by definition of $W(R_i, O^\varphi(R_i, \Omega))$,

$$W(R_i, O^\varphi(R_i', \Omega)) P_i W(R_i, O^\varphi(R_i, \Omega)) \tag{1}$$

($\Longleftarrow$) Assume that (1) holds. We will see that R_i' is an obvious manipulation of φ at (R_i, Ω). Let $x' \in O^\varphi(R_i', \Omega)$. Then, by definition of $W(R_i, O^\varphi(R_i', \Omega))$ and (1),

$$x' R_i W(R_i, O^\varphi(R_i', \Omega)) P_i W(R_i, O^\varphi(R_i, \Omega)).$$

As $W(R_i, O^\varphi(R_i, \Omega)) \in O^\varphi(R_i, \Omega)$, the proof is complete. □

3 Simple rules

In subsection 3.1, the formal definition of simple rule and our two main characterizations are presented. In subsection 3.2, the independence of the axioms involved in the characterizations is discussed.

[4] Barbera and Peleg (1990) were the first to use option sets in the context of preference aggregation.

3.1 Definition and characterizations

Given $(R,\Omega) \in \mathcal{E}_{\mathcal{SP}}$, let $z(R,\Omega) = \sum_{j \in N} p(R_j) - \Omega$. If $z(R,\Omega) \geq 0$ we say that economy (R,Ω) has **excess demand** whereas if $z(R,\Omega) < 0$ we say that economy (R,Ω) has **excess supply**.[5] Given economy $(R,\Omega) \in \mathcal{E}_{\mathcal{SP}}$, agent $i \in N$ is **simple** if either $z(R,\Omega) \geq 0$ and $p(R_i) < \frac{\Omega}{n}$ or $z(R,\Omega) \leq 0$ and $p(R_i) > \frac{\Omega}{n}$. Let $N^\star(R,\Omega)$ denote the set of simple agents of economy (R,Ω).

Given $x,y,z \in \mathbb{R}$, we say that ***x* is between *y* and *z*** if $y \leq z$ and $x \in [y,z]$ or $z \leq y$ and $x \in [z,y]$.

In economies with excess demand, simple rules fully satiate agents whose peak amount is less than or equal to equal division and assign to each remaining agent an amount between equal division and his peak. Symmetrically, in economies with excess supply, simple rules fully satiate agents whose peak is greater than or equal to equal division and assign to each remaining agent an amount between his peak and equal division. Formally,

Definition 2 *An own-peak-only rule φ defined on $\mathcal{E}_{\mathcal{SP}}$ is **simple** if for each $(R,\Omega) \in \mathcal{E}_{\mathcal{SP}}$,*

(i) $\varphi_i(R,\Omega) = p(R_i)$ if $i \in N^\star(R,\Omega)$, and

(ii) $\varphi_i(R,\Omega)$ is between $\frac{\Omega}{n}$ and $p(R_i)$ for each $i \in N \setminus N^\star(R,\Omega)$.

An algorithmic procedure to construct a simple rule is provided in Appendix A. A prominent member of the family of simple rules is the uniform rule, first characterized by Sprumont (1991).

Uniform rule, *u*: For each $(R,\Omega) \in \mathcal{E}^N$ and each $i \in N$,

$$u_i(R,\Omega) = \begin{cases} \min\{p(R_i),\lambda\} & \text{if } z(R,\Omega) \geq 0 \\ \max\{p(R_i),\lambda\} & \text{if } z(R,\Omega) < 0 \end{cases}$$

where $\lambda \geq 0$ and solves $\sum_{j \in N} u_j(R,\Omega) = \Omega$.

The following result shows that simple rules cannot be obviously manipulated.

Theorem 1 *A simple rule defined on $\mathcal{E}_{\mathcal{SP}}$ satisfies NOM.*

[5]We deliberately include the **balanced** case where $z(R,\Omega) = 0$ in the excess demand case. No confusion will arise.

Proof. Let φ be a simple rule defined on $\mathcal{E}_{\mathcal{SP}}$, and let $i \in N$ and $(R_i, \Omega) \in \mathcal{SP} \times \mathbb{R}_{++}$.

Claim: $\boldsymbol{O^{\varphi}(R_i, \Omega) = \begin{cases} \left[\frac{\Omega}{n}, p(R_i)\right] & \text{if } p(R_i) > \frac{\Omega}{n} \\ \left[p(R_i), \frac{\Omega}{n}\right] & \text{if } p(R_i) \leq \frac{\Omega}{n} \end{cases}}$

Assume that $\frac{\Omega}{n} < p(R_i)$ (the other case is symmetric). That $O^{\varphi}(R_i, \Omega) \subseteq [\frac{\Omega}{n}, p(R_i)]$ is clear by the definition of the rule. Now, let $x \in [\frac{\Omega}{n}, p(R_i)]$. We will prove that $x \in O^{\varphi}(R_i, \Omega)$. Consider $R_{-i} \in \mathcal{SP}^{n-1}$ such that $p(R_j) = \frac{\Omega}{n} - \frac{x - \frac{\Omega}{n}}{n-1}$ for each $j \in N \setminus \{i\}$. Then, as $\frac{\Omega}{n} \leq x$, $p(R_j) \leq \frac{\Omega}{n} \leq x$ for all $j \in N \setminus \{i\}$ and $\sum_{j \in N \setminus \{i\}} p(R_j) + x = \Omega$. By definition of the rule, $\varphi_j(R, \Omega) = p(R_j)$ for all $j \in N \setminus \{i\}$. Then, by feasibility, $\varphi_i(R, \Omega) = x$. Therefore, $[\frac{\Omega}{n}, p(R_i)] \subseteq O^{\varphi}(R_i, \Omega)$. This completes the proof of the Claim.

To see that φ satisfies NOM, assume that R_i' is a manipulation of φ at (R_i, Ω). By the Claim, $\frac{\Omega}{n} \in O^{\varphi}(R_i')$. Furthermore, by the Claim and single-peakedness, $x R_i \frac{\Omega}{n}$ for each $x \in O^{\varphi}(R_i)$. Then, R_i' is not an obvious manipulation and so φ is NOM. □

Next, we present our first characterization result.

Theorem 2 *An own-peak-only rule defined on $\mathcal{E}_{\mathcal{SP}}$ satisfies efficiency, the equal division guarantee, and NOM if and only if it is a simple rule.*

Proof. ($\Longleftarrow$) Let φ be a simple rule on $\mathcal{E}_{\mathcal{SP}}$. Then, φ is trivially own-peak-only and satisfies the equal division guarantee. By definition, φ satisfies same-sidedness. Therefore, by Remark 1, φ is efficient. Lastly, by Theorem 1, φ satisfies NOM.

($\Longrightarrow$) Let φ be an own-peak-only rule defined on $\mathcal{E}_{\mathcal{SP}}$ that satisfies efficiency, the equal division guarantee, and NOM. We now prove that φ is simple. Let $(R, \Omega) \in \mathcal{E}_{\mathcal{SP}}$ and assume that $z(R, \Omega) \geq 0$. Let $i \in N$. By efficiency, $\varphi_i(R, \Omega) \leq p(R_i)$. There are two cases to consider:

1. $\boldsymbol{p(R_i) \leq \frac{\Omega}{n}}$. We need to show that $\varphi_i(R, \Omega) \geq p(R_i)$ also holds. Assume, by way of contradiction, that $\varphi_i(R, \Omega) < p(R_i)$. By own-peak-onliness, we can assume that R_i is such that $\frac{\Omega}{n} \; P_i \; \varphi_i(R, \Omega)$. Let $R_i' \in \mathcal{SP}$ be such that $p(R_i') = \frac{\Omega}{n}$. By the equal division guarantee, $\varphi_i(R_i', R_{-i}, \Omega) = \frac{\Omega}{n}$. Hence,

$$\varphi_i(R_i', R_{-i}, \Omega) = \frac{\Omega}{n} \; P_i \; \varphi_i(R, \Omega) \tag{2}$$

and R_i' is a manipulation of φ at (R_i, Ω). Furthermore, by the equal division guarantee, $O^{\varphi}(R_i', \Omega) = \left\{\frac{\Omega}{n}\right\}$. As $\varphi_i(R, \Omega) \in O^{\varphi}(R_i, \Omega)$, by (2), R_i' is an obvious manipulation of φ at (R_i, Ω), contradicting that φ is NOM. Thus, $\varphi_i(R, \Omega) \geq p(R_i)$ and, since by hypothesis $\varphi_i(R, \Omega) \leq p(R_i)$, we have $\varphi_i(R, \Omega) = p(R_i)$.

2. $\boldsymbol{p(R_i) > \frac{\Omega}{n}}$. We need to show that $\varphi_i(R,\Omega) \geq \frac{\Omega}{n}$. Assume, by way of contradiction, that $\varphi_i(R,\Omega) < \frac{\Omega}{n}$. By single-peakedness and own-peak-onliness, $\frac{\Omega}{n} \; P_i \; \varphi_i(R,\Omega)$ also holds here, and the proof follows an argument similar to that of the previous case.

We have proved that if $p(R_i) \leq \frac{\Omega}{n}$, $\varphi_i(R,\Omega) = p(R_i)$; whereas if $p(R_i) > \frac{\Omega}{n}$, $\varphi_i(R,\Omega) \in [\frac{\Omega}{n}, p(R_i)]$. To complete the proof that φ is a simple rule, a symmetric argument can be used for the case $z(R,\Omega) < 0$. □

Simple rules only meet a minimal fairness requirement: whenever an agent has equal division as his peak consumption, the rule must assign it to him. Despite this requirement, simple rules can fail symmetry. If we want to avoid this adding symmetry to the picture, the following corollary follows.[6]

Corollary 1 *An own-peak-only rule defined on $\mathcal{E}_{\mathcal{SP}}$ satisfies efficiency, the equal division guarantee, symmetry, and NOM if and only if it is a symmetric and simple rule.*

Ching (1994) shows that if we require strategy-proofness instead of NOM in Corollary 1, the unique rule that prevails is the uniform rule. Therefore, Corollary 1 evidences how the family of characterizing rules enriches when NOM is sufficient to describe the strategic behavior of the agents. An interesting member of this family is the simple rule that applies the constrained equal distance criterion to non-simple agents. If we use the distance between assignments and peak amounts as a measure of the sacrifice made by the agents at an allocation, this criterion tries to equate that sacrifice among agents as much as possible. It is easy to see that this criterion is incompatible with the definition of simple rule when all agents are considered. However, it is still relevant when applied only to non-simple agents.

CED simple rule, $\boldsymbol{\varphi^{CED}}$: For each $(R,\Omega) \in \mathcal{E}^N$ and each $i \in N$,

(i) if $i \in N^\star(R,\Omega)$, $\varphi_i^{CED}(R,\Omega) = p(R_i)$,

(ii) if $i \in N \setminus N^\star(R,\Omega)$,

$$\varphi_i^{CED}(R,\Omega) = \begin{cases} \max\{\frac{\Omega}{n}, p(R_i) - d\} & \text{if } z(R,\Omega) \geq 0 \\ \min\{\frac{\Omega}{n}, p(R_i) + d\} & \text{if } z(R,\Omega) < 0 \end{cases}$$

where $d \geq 0$ and solves $\sum_{j \in N} \varphi_j^{CED}(R,\Omega) = \Omega$.

[6] As we will see in Subsection 3.2, the equal division guarantee and symmetry are independent axioms.

This rule is a compelling example of a rule that can be relevant in some situations but is completely overlooked when the strong incentive compatibility requirement of strategy-proofness is imposed. We can recover this rule, however, if we only demand the NOM property.

3.2 Independence of axioms

Next, we present some examples that prove the independence of the axioms in Theorem 2 and Corollary 1.

- Let $\widetilde{\varphi} : \mathcal{E}_{\mathcal{SP}} \longrightarrow \mathbb{R}^N_+$ be such that, for each $(R, \Omega) \in \mathcal{E}_{\mathcal{SP}}$ and each $i \in N$, $\varphi_i(R, \Omega) = \frac{\Omega}{n}$. Then $\widetilde{\varphi}$ satisfies all properties but efficiency.

- Let $\varphi^\star : \mathcal{E}_{\mathcal{SP}} \longrightarrow \mathbb{R}^N_+$ be such that, for each $(R, \Omega) \in \mathcal{E}_{\mathcal{SP}}$ and each $i \in N$,

$$\varphi^\star(R,\Omega) = \begin{cases} (p(R_1), p(R_2), 0, \ldots, 0)) & \text{if } p(R_1) + p(R_2) = \Omega \text{ and} \\ & p(R_j) \notin \{p(R_1), p(R_2)\} \text{ for each } j \in N \setminus \{1,2\} \\ u(R,\Omega) & \text{otherwise} \end{cases}$$

  Observe that this rule does meet the equal division guarantee. It is clear that $\varphi^\star$ is own-peak-only, efficient, and symmetric. Now, we prove that $\varphi^\star$ also satisfies NOM. Let $i \in N$ and $(R_i, \Omega) \in \mathcal{SP} \times \mathbb{R}_+$. Notice that $O^{\varphi^\star}(R_i, \Omega) = O^u(R_i, \Omega) \cup \{0\}$ if $i \in N \setminus \{1,2\}$ and $O^{\varphi^\star}(R_i, \Omega) = O^u(R_i, \Omega)$ if $i \in \{1,2\}$. Then, the fact that $\varphi^\star$ does not have obvious manipulations for agents in $N \setminus \{1,2\}$ follows a similar argument as in Theorem 1 using the fact that u is a simple rule. Then, $\varphi^\star$ satisfies all properties but the equal division guarantee.

- Let $\overline{\varphi} : \mathcal{E}_{\mathcal{SP}} \longrightarrow \mathbb{R}^N_+$ be such that, for each $(R, \Omega) \in \mathcal{E}_{\mathcal{SP}}$,

$$\overline{\varphi}(R,\Omega) = \begin{cases} (\frac{1}{3}\Omega, \frac{2}{3}\Omega, 0, \ldots, 0) & \text{if } p(R_1) = p(R_2) = \Omega \text{ and} \\ & p(R_j) = 0 \text{ for each } j \in N \setminus \{1,2\} \\ u(R,\Omega) & \text{otherwise} \end{cases}$$

  Observe that $\overline{\varphi}$ is simple. Thus, it satisfies own-peak-onliness, efficiency, the equal division guarantee, and NOM. However, it does not satisfy symmetry.

- Let $\widehat{\varphi} : \mathcal{E}_{\mathcal{SP}} \longrightarrow \mathbb{R}^N_+$ be such that, for each $(R, \Omega) \in \mathcal{E}_{\mathcal{SP}}$,

(i) if $z(R,\Omega) \geq 0, \widehat{\varphi}(R,\Omega) = u(R,\Omega)$.

(ii) if $z(R,\Omega) < 0$, let $\widehat{N} = \{i \in N : p(R_i) = \min_{j \in N}\{p(R_j)\}\}$. Then, for each $i \in N$,

$$\widehat{\varphi}_i(R,\Omega) = \begin{cases} p(R_i) & \text{if } i \in N \setminus \widehat{N} \\ \lambda & \text{if } i \in \widehat{N} \end{cases}$$

where $\lambda \geq 0$ and solves $\sum_{j \in N} \widehat{\varphi}_j(R,\Omega) = \Omega$.

Observe that $\widehat{\varphi}$ is not simple because it could be that $p(R_i) < \frac{\Omega}{n} < \widehat{\varphi}_i(R)$ for some $i \in \widehat{N}$. It is easy to see that $\widehat{\varphi}$ is own-peak-only, efficient, and symmetric. If $z(R,\Omega) < 0$ and $p(R_i) = \frac{\Omega}{n}$, then $i \in N \setminus \widehat{N}$. Therefore, $\widehat{\varphi}$ satisfies the equal division guarantee. This implies, by Theorem 2, that $\widehat{\varphi}$ satisfies all properties but NOM.

- Let $\underline{\varphi} : \mathcal{E}_{\mathcal{SP}} \longrightarrow \mathbb{R}^N_+$ be such that, for each $(R,\Omega) \in \mathcal{E}_{\mathcal{SP}}$ and each $i \in N$,

$$\underline{\varphi}_i(R,\Omega) = \begin{cases} u_i(R_1^0, R_{-1}, \Omega) & \text{if } z(R_{-1},\Omega) \geq 0,\ 0P_1\frac{\Omega}{n}, \text{ and} \\ & p(R_1) < p(R_j) \text{ for each } j \in N \setminus \{1\} \\ u_i(R,\Omega) & \text{otherwise} \end{cases}$$

where $R_1^0 \in \mathcal{SP}$ is such that $p(R_1^0) = 0$.

Observe that $\underline{\varphi}$ is not simple because it is not own-peak-only and it could be that $\varphi_1(R,\Omega) < p(R_1) < \frac{\Omega}{n}$. It is clear that $\underline{\varphi}$ satisfies the equal division guarantee and symmetry. To see that $\underline{\varphi}$ is efficient, let $(R,\Omega) \in \mathcal{E}_{\mathcal{SP}}$ be such that $z(R_{-1},\Omega) \geq 0$, $0P_1\frac{\Omega}{n}$, and $p(R_1) < p(R_j)$ for each $j \in N \setminus \{1\}$. Then, by the definition of u and the fact that u satisfies same-sideness, $\underline{\varphi}_1(R,\Omega) = 0 \leq p(R_1)$ and $\varphi_i(R,\Omega) \leq p(R_i)$ for each $i \in N \setminus \{1\}$. Then, $\underline{\varphi}_i(R,\Omega) \leq p(R_i)$ for each $i \in N$. Since in any other case $\underline{\varphi}$ operates as u and u satisfies same-sideness, $\underline{\varphi}$ also satisfies this property. Thus, by Remark 1, $\underline{\varphi}$ is efficient. Now, we prove that $\underline{\varphi}$ satisfies NOM. For each $i \in N \setminus \{1\}$ and each $(R_i,\Omega) \in \mathcal{SP} \times \mathbb{R}_+$, $O^{\underline{\varphi}}(R_i,\Omega) = O^u(R_i,\Omega)$. Then, the fact that $\underline{\varphi}$ does not have obvious manipulations for agents in $N \setminus \{1\}$ follows the same arguments used in the proof of Theorem 1 and the fact that u is a simple rule. Let us next consider agent 1. First, let $(R_1,\Omega) \in \mathcal{SP} \times \mathbb{R}_+$ be such that $\frac{\Omega}{n}R_1 0$. Then, $O^{\underline{\varphi}}(R_1,\Omega) = O^u(R_1,\Omega)$. Again, the fact that $\underline{\varphi}$ does not have obvious manipulations for agent 1 at (R_1,Ω) follows the same arguments used in the proof of Theorem 1 together with the facts that u is a simple rule and that $\frac{\Omega}{n} \in O^{\varphi}(R_1',\Omega)$ for each $R'_1 \in \mathcal{SP}$. Second, let $(R_1, \Omega) \in \mathcal{SP} \times \mathbb{R}_+$ be such that $0 P_1 \frac{\omega}{n}$. Then, by single-peakedness of R_1, $p(R_1) < \frac{\Omega}{n}$. By definition of $\underline{\varphi}$, $O^{\underline{\varphi}}(R_1, \Omega) = O^u(R_1^0, \Omega) = [0, \frac{\Omega}{n}]$. Using single-peakedness again and the fact that $0 P_1 \frac{\Omega}{n}$, it follows that $x R_1 \frac{\Omega}{n}$ for each $x \in O^{\varphi}(R_1, \Omega)$. Furthermore, as $\frac{\Omega}{n} \in O^{\varphi}(R'_1, \Omega)$ for each $R'_1 \in \mathcal{SP}$, agent 1 does not have obvious manipulations of $\underline{\varphi}$ at (R_1, Ω). Therefore, $\underline{\varphi}$ is NOM. We conclude that $\underline{\varphi}$ satisfies all properties but own-peak-onliness.

4 Further characterizations involving NOM

Insisting on the NOM property, in this section we provide new characterizations of rules in which NOM together with other relevant properties are invoked. These considerations lead to three interesting variants of Theorem 2. Insisting on efficiency as well, the first one is an adaptation of Theorem 2 to economies with individual endowments and the second one involves a monotonicity condition concerning the peaks of the preference profile. Lastly, the third one gets rid of efficiency.

4.1 Endowments guarantee

In some situations, it is more appropriate to assume that instead of a social endowment $\Omega \in \mathbb{R}_+$, there is a profile $\omega = (\omega_i)_{i \in N} \in \mathbb{R}_+^N$ where, for each $i \in N$, ω_i denotes agent i's individual endowment of the non-disposable commodity and $\sum_{j \in N} \omega_j = \Omega$. In such cases, it is natural to replace the equal division guarantee with the following property.

Endowments guarantee: For each $(R, \omega) \in \mathcal{E}_{\mathcal{D}}$ and each $i \in N$ such that $\omega_i \in p(R_i)$, we have $\varphi_i(R) I_i \omega_i$.

If, for each $i \in N$, we replace $\frac{\Omega}{n}$ with ω_i in the definition of simple agent, we can define a **simple reallocation rule** as one that assigns: (i) to each simple agent, his peak amount and, (ii) to each remaining agent, an amount between his endowment and his peak. Then, for economies with individual endowments, the following variant of Theorem 2 is obtained.

Theorem 3 *An own-peak-only rule defined on $\mathcal{E}_{\mathcal{SP}}$ satisfies efficiency, the endowments guarantee, and NOM if and only if it is a simple reallocation rule.*

The analysis of the independence of axioms involved is similar to that of Theorem 2 (see Subsection 3.2).

4.2 Peak monotonicity

The following monotonicity property says that if the peak of one agent is greater than or equal to the peak of another agent, then the first agent must be assigned an amount greater than or equal to that of the second agent.

Peak monotonicity: For each $(R, \Omega) \in \mathcal{E}_{\mathcal{SP}}$ and each $\{i, j\} \subseteq N$ with $i \neq j$, $p(R_i) \leq p(R_j)$ implies $\varphi_i(R, \Omega) \leq \varphi_j(R, \Omega)$.

It is clear that peak monotonicity implies symmetry. Furthermore, we next prove that peak monotonicity together with NOM imply the equal division guarantee for an own-peak-only rule defined on $\mathcal{E}_{\mathcal{SP}}$.

Lemma 1 *Any own-peak-only rule defined on $\mathcal{E}_{\mathcal{SP}}$ that satisfies peak monotonicity and NOM meets the equal division guarantee.*

Proof. Let φ be an own-peak-only rule defined on $\mathcal{E}_{\mathcal{SP}}$ that satisfies peak monotonicity and NOM. Assume, on contradiction, that there are $(R, \Omega) \in \mathcal{E}_{\mathcal{SP}}$ and $i \in N$ such that $p(R_i) = \frac{\Omega}{n}$ and $\varphi_i(R, \Omega) \neq \frac{\Omega}{n}$. Consider the case $\varphi_i(R, \Omega) < \frac{\Omega}{n}$ (the other one is symmetric). Let $R'_i \in \mathcal{SP}$ be such that $p(R'_i) = \infty$. By peak monotonicity, $\varphi_i(R'_i, R_{-i}, \Omega) \geq \varphi_j(R'_i, R_{-i}, \Omega)$ for each $j \in N \setminus \{i\}$. Then, by feasibility, $\varphi_i(R'_i, R_{-i}, \Omega) \geq \frac{\Omega}{n}$. By own-peak-onliness, we can assume that R_i is such that $\varphi_i(R'_i, R_{-i}, \Omega) P_i \varphi_i(R, \Omega)$. Then, R'_i is a manipulation of φ at (R_i, Ω). Furthermore, if $x \in O^{\varphi}(R'_i, \Omega)$, then $x = \varphi_i(R'_i, R'_{-i})$ for some $R'_{-i} \in \mathcal{SP}^{n-1}$. Then, by peak monotonicity and feasibility, $x \geq \frac{\Omega}{n}$. Therefore, by own-peak-onliness, we can assume that $x R_i \varphi_i(R)$. Therefore, R'_i is an obvious manipulation of φ at (R, Ω), which is a contradiction. □

Hence, under own-peak-onliness and NOM, peak monotonicity implies the fairness properties of symmetry and the equal division guarantee.[7] By using Lemma 1 and a proof similar to that of Theorem 2, the following characterization is obtained.

Theorem 4 *An own-peak-only rule defined on $\mathcal{E}_{\mathcal{SP}}$ satisfies efficiency, peak monotonicity, and NOM if and only if it is a peak monotonic and simple rule.*

Let us check the independence of axioms involved in Theorem 4. Considering the rules defined in Subsection 3.2, the equal division rule $\widetilde{\varphi}$ satisfies all properties but efficiency;

[7] Remember that, as we saw in Section 3, symmetry and the equal division guarantee are independent properties.

by Lemma 1, rule $\varphi^\star$ satisfies all properties but peak monotonicity; and rule $\underline{\varphi}$ satisfies all properties but own-peak-onliness. Lastly, consider the following rule:

Constrained equal-distance rule, *c*: For each $(R, \Omega) \in \mathcal{E}_{\mathcal{SP}}$ and each $i \in N$,

$$c_i(R, \Omega) = \begin{cases} \max\{p(R_i) - d, 0\} & \text{if } z(R, \Omega) \geq 0 \\ p(R_i) + d & \text{if } z(R, \Omega) < 0 \end{cases}$$

where $d \geq 0$ and solves $\sum_{j \in N} c_j(R, \Omega) = \Omega$.

This rule is own-peak-only by definition; it is same-sided, and therefore efficient; and it is also peak monotonic. However, since it may not assign the peak amount to some simple agent, it is not NOM.

4.3 Dropping efficiency

Getting rid of efficiency leads us to a straightforward relaxation of simple rules. A **generalized simple rule** is an own-peak-only rule that assigns to each agent an amount between his peak and equal division. It turns out that such rules are characterized by the equal division guarantee and the NOM property.

Theorem 5 *An own-peak-only rule defined on $\mathcal{E}_{\mathcal{SP}}$ satisfies the equal division guarantee and NOM if and only if it is a generalized simple rule.*

To see that a generalized simple rule satisfies the equal division guarantee and NOM just note that, by the definition of the rule, option sets of an agent are always included in the interval between the peak of that agent and equal division. This fact immediately implies the equal division guarantee, whereas the NOM property follows as in the proof of Theorem 1. Conversely, assume that an own-peak-only rule satisfies the equal division guarantee and NOM and that one agent does not get an allotment between his peak and equal division. Further, by own-peak-onliness, assume w.l.o.g. that whatever he gets is worse than equal division. Then, reporting a preference with equal division as its peak amount entails an obvious manipulation by the equal division guarantee, contradicting NOM. Therefore, this rule must be generalized simple.

Finally, Table 1 summarizes the different characterizations involving NOM presented in this study.

	Th. 2	Cor. 1	Th. 3	Th. 4	Th. 5
Own-peak-onliness	+	+	+	+	+
Efficiency	+	+	+	+	
Equal division guarantee	+	+			+
NOM	+	+	+	+	+
Symmetry		+			
Endowments guarantee			+		
Peak monotonicity				+	

Table 1: *Characterizations of NOM rules.*

5 Maximal domain for NOM

Now, a relevant question that arises is whether the domain of preferences can be enlarged maintaining the compatibility of the properties. Ching and Serizawa (1998) show that the single-plateaued domain is maximal for efficiency, symmetry, and strategy-proofness. As we have seen, when we weaken strategy-proofness to NOM (and explicitly invoke own-peak-onliness), the class of rules that also meet the equal division guarantee enlarges considerably. One may suspect that the maximal domain of preferences for these new properties enlarges as well. However, as we will see in this section, the single-plateaued domain is still maximal for these properties.[8]

We start with the definition of maximality for a list of properties.

Definition 3 *A domain $\mathcal{D}^\star \subseteq \mathcal{U}$ is* ***maximal for a list of properties*** *if (i) $\mathcal{SP} \subseteq \mathcal{D}^\star$, (ii) there is a rule defined on $\mathcal{E}_{\mathcal{D}^\star}$ satisfying the properties, and (iii) for each $\mathcal{E}_{\overline{\mathcal{D}}}$ with $\mathcal{D}^\star \subsetneq \overline{\mathcal{D}} \subseteq \mathcal{U}$ there is no rule defined on $\mathcal{E}_{\overline{\mathcal{D}}}$ satisfying the same properties.*

Given $i \in N$ and $R_i \in \mathcal{U}$, let $\underline{p}(R_i) = \inf p(R_i)$ and $\overline{p}(R_i) = \sup p(R_i)$. Agent i's preference $R_i \in \mathcal{U}$ is **single-plateaued** if $p(R_i)$ is the interval $[\underline{p}(R_i), \overline{p}(R_i)]$ and, for each pair $\{x_i, x_i'\} \subseteq \mathbb{R}_+$, we have $x_i P_i x_i'$ as long as either $x_i' < x_i \leq \underline{p}(R_i)$ or $\overline{p}(R_i) \leq x_i < x_i'$ holds. Denote by $\mathcal{SPL}$ the domain of all such preferences. Agent i's preference $R_i \in \mathcal{U}$

[8] Ching and Serizawa (1998) show that the single-plateaued domain is *the* unique maximal domain for their properties. We show that this domain is *one* maximal domain for our properties. There could exist "pathological" maximal domains containing just a portion of the single-plateaued domain. Nevertheless, all of them have to be contained in the domain of convex preferences.

is **convex** if $p(R_i)$ is the interval $[\underline{p}(R_i), \overline{p}(R_i)]$ and, for each pair $\{x_i, x_i'\} \subseteq \mathbb{R}_+$, we have $x_i R_i x_i'$ as long as either $x_i' < x_i \leq \underline{p}(R_i)$ or $\overline{p}(R_i) \leq x_i < x_i'$ holds. Denote by $\mathcal{C}$ the domain of all such preferences.

Lemma 2 *Let $\mathcal{D}$ be such that $\mathcal{SP} \subseteq \mathcal{D} \subseteq \mathcal{U}$ and let φ be an own-peak-only rule defined on $\mathcal{E}_\mathcal{D}$ that satisfies the equal division guarantee and NOM. Let $i \in N$ and $(R, \Omega) \in \mathcal{E}_\mathcal{D}$. If $p(R_i)$ is a singleton, then $\varphi_i(R, \Omega)$ is between $p(R_i)$ and $\frac{\Omega}{n}$.*

Proof. Let $\varphi : \mathcal{E}_\mathcal{D} \longrightarrow \mathbb{R}_+$ be an own-peak-only rule that satisfies the equal division guarantee and NOM. Let $(R, \Omega) \in \mathcal{E}_\mathcal{D}$ and let $i \in N$ be such that $p(R_i) \leq \frac{\Omega}{n}$. We will prove that $\varphi_i(R, \Omega) \in [p(R_i), \frac{\Omega}{n}]$. By own-peak-onliness and the fact that $\mathcal{SP} \subseteq \mathcal{D}$, the proof that $\varphi_i(R, \Omega) \geq p(R_i)$ follows the same lines as Case 1 in part ($\Longrightarrow$) of the proof of Theorem 2. Next, we need to show that $\varphi_i(R, \Omega) \leq \frac{\Omega}{n}$. Assume, by way of contradiction, that $\frac{\Omega}{n} < \varphi_i(R, \Omega)$. By own-peak-onliness and the fact that $\mathcal{SP} \subseteq \mathcal{D}$, we can assume w.l.o.g. that $\frac{\Omega}{n} \; P_i \; \varphi_i(R, \Omega)$. Let $R_i' \in \mathcal{SP}$ be such that $p(R_i') = \frac{\Omega}{n}$. By the equal division guarantee, R_i' is an obvious manipulation of φ at (R_i, Ω). This contradicts the fact that φ satisfies NOM. Therefore, $\varphi_i(R, \Omega) \in [p(R_i), \frac{\Omega}{n}]$.

The case $\frac{\Omega}{n} > p(R_i)$ is symmetric, so we omit it. □

Lemma 3 *Let $\mathcal{D}$ be such that $\mathcal{SP} \subseteq \mathcal{D} \subseteq \mathcal{U}$. If there is an own peak-only rule defined on $\mathcal{E}_\mathcal{D}$ that satisfies efficiency, the equal division guarantee, and NOM, then $\mathcal{D} \subseteq \mathcal{C}$.*

Proof. Let $\mathcal{D}$ be such that $\mathcal{SP} \subseteq \mathcal{D} \subseteq \mathcal{U}$ and let φ be an own-peak-only rule defined on $\mathcal{E}_\mathcal{D}$ that is efficient, NOM, and meets the equal division guarantee. Assume, by way of contradiction, that there is $R_0 \in \mathcal{D} \setminus \mathcal{C}$. Then, there are $x, y, z \in \mathbb{R}_+$ such that $x < y < z$, xP_0y and zP_0y. Without loss of generality, we can assume that zR_0x (the case xR_0z is symmetric).

Let $x' = \max\{w \in [x, y] \text{ such that } wI_0x\}$ and $y' = \min\{w \in [y, z] \text{ such that } wI_0x\}$. As R_0 is continuous, both x' and y' are well defined, $x' < y < y'$, and $y'P_0w$ for each $w \in (x', y')$. Let $(R, \Omega) \in \mathcal{E}_\mathcal{D}$ be such that $\Omega = ny'$, $R_1 = R_0$, and $R_{-1} \in \mathcal{SP}^{n-1}$ is such that $p(R_j) \in (y', \frac{\Omega - x'}{n-1})$ and $\Omega R_j y'$ for each $j \in N \setminus \{1\}$. By Lemma 2, $\varphi_j(R, \Omega) \in [y', p(R_j)]$ for each $j \in N \setminus \{1\}$. Then, by feasibility and the fact that $p(R_j) < \frac{\Omega - x'}{n-1}$, it follows that $\varphi_1(R, \Omega) \in (x', y']$. If $\varphi_1(R, \Omega) \in (x', y')$, by the definitions of x' and y', $y'P_1\varphi_i(R, \Omega)$. If we consider $R_1' \in \mathcal{SP}$ be such that $p(R_1') = \frac{\Omega}{n} = y'$, by the equal division guarantee R_1'

is an obvious manipulation of φ at (R_1, Ω), contradicting that φ satisfies NOM. Therefore, $\varphi_1(R, \Omega) = y'$. Hence, again by Lemma 2 and feasibility, $\varphi_j(R, \Omega) = y'$ for each $j \in N$. Now, consider the feasible allocation in which agent 1 gets x' and the remaining agents get $\frac{\Omega - x'}{n-1}$. Since $x' I_1 y'$ and $\frac{\Omega - x'}{n-1} P_j y'$ for each $j \in N \setminus \{1\}$, this new allocation Pareto improves upon $\varphi(R, \Omega)$, contradicting efficiency. We conclude that $\mathcal{D} \subseteq \mathcal{C}$. □

Theorem 6 *Domain $\mathcal{SPL}$ is a maximal domain for own-peak-onliness, efficiency, the equal division guarantee, and NOM.*

Proof. Let $\mathcal{D}$ be a domain for the properties listed in the theorem such that $\mathcal{SPL} \subseteq \mathcal{D}$. By Lemma 3, $\mathcal{D} \subseteq \mathcal{C}$. Assume, by way of contradiction, that there is $R_0 \in \mathcal{D} \setminus \mathcal{SPL}$. Without loss of generality, we can assume that there are $x, y \in \mathbb{R}_+$ such that $x < y < \underline{p}(R_0)$ and $x I_0 y$.

Assume further that there is a rule φ defined on $\mathcal{E}_{\mathcal{D}}$ that satisfies the properties listed in the Theorem. Consider $(R, \Omega) \in \mathcal{E}_{\mathcal{D}}$ with $\Omega = ny$, $R_1 = R_0$, and $R_{-1} \in \mathcal{SP}^{n-1}$ such that $p(R_j) = y + \frac{y-x}{n-1}$ for each $j \in N \setminus \{1\}$. By Lemma 2, $\varphi_j(R, \Omega) \in [y, p(R_j)]$ for each $j \in N \setminus \{1\}$. Then, by feasibility, $\varphi_1(R, \Omega) \leq y$. If $\varphi_1(R, \Omega) < y$ we can assume, by own-peak-onliness and the fact that $\mathcal{SPL} \subseteq \mathcal{D}$, that $y P_1 \varphi_1(R, \Omega)$. Next, let us consider $R'_1 \in \mathcal{SP}$ such that $p(R'_1) = \frac{\Omega}{n} = y$. By the equal division guarantee, R'_1 is an obvious manipulation of φ at R_i, contradicting the fact that φ satisfies NOM. Therefore, $\varphi_1(R, \Omega) = y$. Hence, again by Lemma 2 and feasibility, $\varphi_j(R, \Omega) = y$ for each $j \in N$. Now, consider the feasible allocation in which agent 1 gets x and the remaining agents get $y + \frac{y-x}{n-1}$. Since $x I_1 y$ and $y + \frac{y-x}{n-1} P_j y$ for each $j \in N \setminus \{1\}$, this new allocation Pareto improves upon $\varphi(R, \Omega)$, contradicting efficiency. We conclude that $\mathcal{D} = \mathcal{SPL}$.

The proof is completed by considering the extension of the uniform rule to domain $\mathcal{SPL}$ defined in the proof of the Theorem in Ching and Serizawa (1998). As they observe, this rule satisfies efficiency and strategy-proofness on $\mathcal{SPL}$. As NOM is a weakening of strategy-proofness, this rule satisfies NOM as well. Furthermore, it is easy to see that this rule is own-peak-only and meets the equal division guarantee.[9] □

[9]We can also consider extensions of simple rules defined on $\mathcal{E}_{\mathcal{SP}}$ to domain $\mathcal{E}_{\mathcal{SPL}}$. Before defining them, we need some notation. Given $(R, \Omega) \in \mathcal{E}_{\mathcal{SPL}}$, let $\underline{z}(R, \Omega) = \sum_{j \in N} \underline{p}(R_j) - \Omega$ and $\overline{z}(R, \Omega) = \sum_{j \in N} \overline{p}(R_j) - \Omega$. For $i \in N$ and $R_i \in \mathcal{SPL}$, let $\underline{R}_i \in \mathcal{SP}$ be such that $p(\underline{R}_i) = \underline{p}(R_i)$ and let $\overline{R}_i \in \mathcal{SP}$ be such that $p(\overline{R}_i) = \overline{p}(R_i)$. An own-peak-only rule $\varphi^\star$ defined on $\mathcal{E}_{\mathcal{SPL}}$ is an **extension** of simple rule φ defined on $\mathcal{E}_{\mathcal{SP}}$ if, for each $(R, \Omega) \in \mathcal{E}_{\mathcal{SPL}}$,

Finally, it is worth mentioning that the single-plateaued domain remains maximal for the properties involved in the characterization of Corollary 1, i.e., when symmetry is also considered.

6 Final Remarks

Some final remarks are in order. The well-known property of peaks-onliness is a strong form of the own-peak-only property and requires that if two profiles of preferences have the same peaks, then the allocations recommended by the rule in both profiles are the same. Although own-peak-onliness follows from efficiency and strategy-proofness, peak-onliness does not. If we replace own-peak-onliness with peak-onliness in the definition of essential rules, in Section 3, then all our results remain valid replacing own-peak-onliness by peak-onliness.

In this paper, we introduced the equal division guarantee as a minimal fairness requirement. We showed that it is not implied by symmetry. However, two well-known fairness properties, envy-freeness and the equal division lower bound, imply it for own-peak-only rules. Remember that a rule is envy-free if no agent prefers some other agent's allotment to his own; and that a rule meets the equal division lower bound if, for each agent, the allotment recommended by the rule is at least as good as equal division. Formally,

Envy-free: For each $(R, \Omega) \in \mathcal{E}_{\mathcal{SP}}$ and each $\{i, j\} \subseteq N$ with $i \neq j$, $\varphi_i(R, \Omega) R_i \varphi_j(R, \Omega)$.

Equal division lower bound:[10] For each $(R, \Omega) \in \mathcal{E}_{\mathcal{SP}}$ and each $i \in N$, $\varphi_i(R, \Omega) R_i \frac{\Omega}{n}$.

Lemma 4 *For any own-peak-only rule defined on $\mathcal{E}_{\mathcal{SP}}$, envy-freeness or the equal division lower bound imply the equal division guarantee.*

(i) $\varphi^\star_i(R, \Omega) = \varphi_i(\underline{R}, \Omega)$ if $\underline{z}(R, \Omega) \geq 0$,

(ii) $\varphi^\star_i(R, \Omega) = \varphi_i(\overline{R}, \Omega)$ if $\overline{z}(R, \Omega) \leq 0$,

(iii) $\varphi^\star_i(R, \Omega) \in [\underline{p}(R_i), \overline{p}(R_i)]$ if $\underline{z}(R, \Omega) < 0$ and $\overline{z}(R, \Omega) > 0$.

With similar arguments to those used in the proof of Theorem 2, it follows that these rules also are efficient, meet the equal division guarantee, and are NOM.

[10] This property is also known as *individual rationality from equal division.*

Proof. Let φ be an own-peak-only rule defined on $\mathcal{E}_{\mathcal{SP}}$. First, assume that φ meets the equal division lower bound. Let $(R, \Omega) \in \mathcal{E}_{\mathcal{SP}}$ and $i \in N$ be such that $p(R_i) = \frac{\Omega}{n}$. Then, $\varphi_i(R, \Omega) R_i \frac{\Omega}{n} = p(R_i)$ and thus $\varphi_i(R, \Omega) = p(R_i)$, implying that φ meets the equal division guarantee. Second, assume that φ is envy-free but does not meet the equal division guarantee. Then, there are $(R, \Omega) \in \mathcal{E}_{\mathcal{SP}}$ and $i \in N$ with $p(R_i) = \frac{\Omega}{n}$ such that $\varphi_i(R) \neq \frac{\Omega}{n}$. Consider the case $\varphi_i(R) > \frac{\Omega}{n}$ (the other one is symmetric). By feasibility, there is $j \in N \setminus \{i\}$ such that $\varphi_j(R) < \frac{\Omega}{n}$. As $p(R_i) = \frac{\Omega}{n}$, by own-peak-onliness, we can assume that R_i is such that $\varphi_j(R) P_i \varphi_i(R)$. Then, φ does not satisfy envy-freeness. This contradiction shows that φ meets the equal division guarantee. □

References

ARRIBILLAGA, R. P. AND A. G. BONIFACIO (2022): "Obvious manipulations of tops-only voting rules," *arXiv preprint arXiv:2210.11627*.

ARRIBILLAGA, R. P. AND E. RISMA (2023): "Obvious manipulations in matching with and without contracts," *arXiv preprint arXiv:2306.17773*.

AZIZ, H. AND A. LAM (2021): "Obvious manipulability of voting rules," in *International Conference on Algorithmic Decision Theory*, Springer, 179–193.

BARBERÀ, S., M. O. JACKSON, AND A. NEME (1997): "Strategy-proof allotment rules," *Games and Economic Behavior*, 18, 1–21.

BARBERA, S. AND B. PELEG (1990): "Strategy-proof voting schemes with continuous preferences," *Social Choice and Welfare*, 7, 31–38.

CHING, S. (1994): "An alternative characterization of the uniform rule," *Social Choice and Welfare*, 11, 131–136.

CHING, S. AND S. SERIZAWA (1998): "A maximal domain for the existence of strategy-proof rules," *Journal of Economic Theory*, 78, 157–166.

MASSÓ, J. AND A. NEME (2001): "Maximal domain of preferences in the division problem," *Games and Economic Behavior*, 37, 367–387.

ORTEGA, J. AND E. SEGAL-HALEVI (2022): "Obvious manipulations in cake-cutting," *Social Choice and Welfare*, 1–20.

PSOMAS, A. AND P. VERMA (2022): "Fair and efficient allocations without obvious manipulations," *arXiv preprint arXiv:2206.11143*.

SPRUMONT, Y. (1991): "The division problem with single-peaked preferences: a characterization of the uniform allocation rule," *Econometrica*, 509–519.

TROYAN, P. AND T. MORRILL (2020): "Obvious manipulations," *Journal of Economic Theory*, 185, 104970.

A Constructing a simple rule

Even though simple rules are easy to define and we can think of several different ways to construct them, in this appendix we provide a particular algorithmic approach.[11]

Once we specify an economy $(R, \Omega) \in \mathcal{E}_{\mathcal{SP}}$, the procedure in Figure 1 shows how we can construct the outcome of a simple rule in such an economy.

Assume that $z(R, \Omega) \geq 0$ (the other case is similar). To begin, an initial allotment is performed. Each simple agent is assigned his peak amount and each non-simple agent is assigned equal division. This frees up an amount $\overline{\Omega}_1 \geq 0$ that still has to be divided among non-simple agents. To do this, we consider a sequential adjustment process. First, we order non-simple agents arbitrarily. Following this order, in step t of the adjustment process non-simple agent i_t's surplus, λ_t, is determined fulfilling three requirements: (i) agent i_t's final allotment $\frac{\Omega}{n} + \lambda_t$ should be between $\frac{\Omega}{n}$ and $p(R_{i_t})$ to comply with the definition of simple rule; (ii) surplus λ_t should be small enough to achieve feasibility, i.e., $\lambda_t \leq \overline{\Omega}_t$; and (iii) surplus λ_t should be large enough to maintain the economy in excess demand for not yet adjusted agents, i.e., $\Omega - [\sum_{j \in N^\star(R,\Omega) \cup \{i_1, \cdots, i_{t-1}\}} \alpha_j + (\frac{\Omega}{n} + \lambda_t)] \leq \sum_{j=t+1}^{k} p(R_{i_j})$ or, equivalently, $\underline{\Omega}_t + p(R_{i_t}) - \frac{\Omega}{n} \leq \lambda_t$. To finish step t, amounts $\overline{\Omega}_t$ and $\underline{\Omega}_t$ are updated accordingly. Finally, each simple agent gets as final allotment his peak, each non-simple agent i_t different from i_k gets as final allotment $\frac{\Omega}{n} + \lambda_t$ and the allotment of agent i_k is determined by feasibility.

[11]Of course, this process will be nothing more than a way to resolve the system of inequalities posed by the definition of simple rule once an economy is fixed.

Algorithm:

Input An economy $(R, \Omega) \in \mathcal{E}_{\mathcal{SP}}$

Initialization

$$\alpha_i := \begin{cases} p(R_i) & \text{if } i \in N^\star(R,\Omega) \\ \frac{\Omega}{n} & \text{if } i \in N \setminus N^\star(R,\Omega) \end{cases} \qquad \text{(initial allotment)}$$

Adjustment

$N \setminus N^\star(R,\Omega) := \{i_1, i_2, \ldots, i_k\}$ (order $N \setminus N^\star(R,\Omega)$)

$t := 1$

$\overline{\Omega}_1 := \Omega - \sum_{j \in N} \alpha_j$ (amount still left to allocate)

$\underline{\Omega}_1 := \Omega - \sum_{j \in N} p(R_j)$ (amount to control for type of case)

`IF` $z(R,\Omega) \geq 0$ (excess demand case)

`WHILE` $t < k$ `DO`

Choose λ_t such that

$$\max\{0, \underline{\Omega}_t + p(R_{i_t}) - \tfrac{\Omega}{n}\} \leq \lambda_t \leq \min\{p(R_{i_t}) - \tfrac{\Omega}{n}, \overline{\Omega}_t\}$$

$\alpha_{i_t} = \alpha_{i_t} + \lambda_t$

$\overline{\Omega}_{t+1} = \overline{\Omega}_t - \lambda_t$

$\underline{\Omega}_{t+1} = \underline{\Omega}_t - \alpha_{i_t} + p(R_{i_t})$

$t = t + 1$

`ELSE` (excess supply case: $z(R,\Omega) < 0$)

`WHILE` $t < k$ `DO`

Choose λ_t such that

$$\max\{0, \tfrac{\Omega}{n} - p(R_{i_t}) - \underline{\Omega}_t\} \leq \lambda_t \leq \min\{\tfrac{\Omega}{n} - p(R_{i_t}), -\overline{\Omega}_t\}$$

$\alpha_{i_t} = \alpha_{i_t} - \lambda_t$

$\underline{\Omega}_{t+1} = \underline{\Omega}_t - \alpha_{i_t} + p(R_{i_t})$

$t = t + 1$

Output

$$\varphi_i(R,\Omega) = \begin{cases} \alpha_i & \text{if } i \in N \setminus \{i_k\} \\ \Omega - \sum_{j \in N \setminus \{i_k\}} \alpha_j & \text{if } i = i_k \end{cases} \qquad \text{(final allotment)}$$

Figure 1: *Algorithm to construct a simple rule.*

Notice that the algorithm is correct. To see this, observe that in each step t of the algorithm $\alpha_i \leq p(R_i)$ for all $i \in N \setminus \{i_t\}$ and at the beginning of each step t, $\alpha_{i_t} = \frac{\Omega}{n} \leq p(R_{i_t})$. Therefore, $\underline{\Omega}_t + p(R_{i_t}) - \frac{\Omega}{n} \leq \overline{\Omega}_t$. Furthermore, $\underline{\Omega}_1 \leq 0$ in the initialization and $\underline{\Omega}_2 = \underline{\Omega}_1 - \alpha_{i_1} + p(R_{i_1}) = \underline{\Omega}_1 - (\frac{\Omega}{n} + \lambda_1) + p(R_{i_1})$. Therefore, by the choice of λ_1, $\underline{\Omega}_2 \leq 0$. Following a similar argument we get that $\underline{\Omega}_t \leq 0$ at each step of the algorithm. Therefore, $\underline{\Omega}_t + p(R_{i_t}) - \frac{\Omega}{n} \leq p(R_{i_t}) - \frac{\Omega}{n}$ holds. Then, the election of λ_t is always possible.