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# Obvious Strategy-proofness with Respect to a Partition*

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Abstract: We define and study obvious strategy-proofness with respect to a partition of the set of agents. It has as special cases strategy-proofness, when the partition is the coarsest one, and obvious strategy-proofness, when the partition is the finest one. For any partition, it lies between these two extreme implementation notions. We give two general properties of the new implementation notion and apply it to the simple voting problem with two alternatives and strict references. We also propose the notion of strong obvious strategy-proofness and show that it coincides with obvious strategy-proofness.

Keywords: Obvious strategy-proofness; Extended majority voting.
JEL Classification Number: D71.

# 1 Introduction

We propose and study a new implementation concept to which we refer to as *obvious strategy-proofness with respect to a partition*. For any given partition of the set of agents, it is stronger than strategy-proofness and weaker than obvious strategy-proofness (as defined in

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Li (2017)). It coincides with strategy-proofness for the coarsest partition and with obvious strategy-proofness for the finest partition.

A social choice function is strategy-proof if the direct revelation mechanism induces the social choice function and truth-telling is a dominant strategy.[1] Li (2017) argues that strategy-proofness requires that agents are able to perform complex contingent reasoning: For each of the potentially declared preference profiles of the other agents (the contingencies that any of the agents face when deciding what preference to declare), the agent is able to identify that truth-telling is one of the optimal choices.

To relieve the burden of agents' reasoning, Li (2017) suggests that the hypothetical contingencies of the direct revelation mechanism may be replaced in a sequential mechanism (*i.e.*, an extensive game form) with reliable facts that can be observed by the agent at any moment in which it has to make a choice along the extensive game form. In addition, to evaluate the consequence of truth-telling compared to the consequence of making any other choice, a behavioral hypothesis is used about the future behavior of all other agents playing thereafter: It is pessimistic in evaluating truth-telling (the worst of all possible future results will occur) and it is optimistic in evaluating any of the deviations (the best of all possible future results will occur). If the worst result attached to truth-telling is at least as good as the best result attached to deviating, then truth-telling appears as being an obviously optimal choice (that is, obviously dominant). There are already many papers that study obvious strategy-proofness. For a general setting, see for instance, Bade and Gonczarowski (2017), Mackenzie (2020), Mackenzie and Zhou (2022), and Pycia and Troyan (2023). For particular settings studying specific obviously strategy-proof social choice functions, see for instance Arribillaga, MassÃ³ and Neme (2020 and 2023), Ashlagi and Gonczarowski (2018) and Troyan (2019).

For a given partition of the set of agents, our notion is a hybrid of the two extreme notions, maintaining the sequential interpretation of the direct revelation mechanism. Given a partition of the set of agents, each agent, at any moment in which it has to make a choice, considers that the strategy of the other agents that belong to the same subset of the partition as itself is fixed and taken as given (*i.e.*, it is one of the possible hypothetical contingencies) while, on the other hand, it uses the two most extreme behavioral hypothesis to evaluate future choices of agents that do not belong to the same subset of the partition and who

[1] The direct revelation mechanism is a normal form game that can also be described as an extensive form game with imperfect information where agents only play once with no information about the other agents' choices.

have to play from then on. To perform contingent reasoning about the choices of agents that belong to the same subset of the partition can be considered easier than the reasoning about the choices of agents that belong to the other subset of the partition. For instance, agents in the same subset may carry out pre-play communication and make a joint and common hypothesis about the choices that the members of the subset will make throughout the game; hence, it is reasonable to consider, when evaluating one's choice, the contingency of the behavior of agents in the same subset of the partition as hypothetical but at the same time as given. In contrast, information about agents that do not belong to the same subset may be scarce and/or pre-play communication may not be possible; therefore, when comparing truth-telling with deviating at the moment of making the choice, the agent may not be able to elucidate what agents outside the own subset will do thereafter and so it may leave their choices as not fixed and use instead extreme guesses about their consequences.

Our two general results are the following. First, for any partition of the set of agents, we identify in Theorem 1 a large and simple class of extensive game forms with the property that if a social choice function is implementable in dominant strategies by a game in the class, then the social choice function is implementable in obviously dominant strategies with respect to the partition by the same game. Second, in Proposition 1 we show that if a social choice function is implementable in obviously dominant strategies with respect to a partition, then the social choice function is obviously implementable with respect to any coarser partition as well.

The paper proceeds with an application of the new implementation concept of obvious strategy-proofness with respect to a partition to the simplest social choice problem in which there are only two alternatives, $x$ and $y$, and agents' preferences are strict. This simple setting admits a large family of strategy-proof social choice functions, called *extended majority voting rules.* Each member of the class can be described as a committee: A monotonic family of winning coalitions, those subsets of agents that can enforce $x$ by voting for $x$, regardless of the other agents' votes. We identify the key necessary and sufficient condition that a committee must satisfy for the obviously dominant implementability with respect to a partition of the corresponding extended majority voting rule (Theorem 2). We refer to this condition as the IUP, for *Iterated Union Property.* We finish the paper with the characterization of two nested families of extended majority voting rules that are obviously strategy-proof with respect to a partition, each family corresponding to one of the two anonymous subclasses related to two different notions of anonymity. *Anonymity relative to* *a partition*, where the allowed permutations of agents are only those that map each subset of the partition into itself (and so, the partition is not altered by the permutation), and *Strong anonymity*, where agents can be permuted in any way (and so, a partitioned set of agents can be mapped into potentially different partitions).

We finish the paper with two final remarks. In the first one, we relate our results with a class of extensive game forms that play a crucial role in the literature on obvious strategy-proofness: round table mechanisms. In the second one, we propose a natural definition of *group obvious strategy-proofness* and show that this apparently stronger notion coincides with obvious strategy-proofness.

The paper is organized as follows. Section 2 presents the basic notation and definitions. and the description of extensive game forms, required to define obvious strategy-proofness with respect to a partition which is presented in Section 3. Section 4 applies this new notion to the case of two alternatives and strict preferences. Section 5 finishes with two final remarks.

# 2 Preliminaries

## 2.1 Basic notation and definitions

We consider collective decision problems where a set of agents $N = \{1, \ldots, n\}$ has to choose an alternative from a given set $A$. Each agent $i \in N$ has a (weak) preference $R_i$ over $A$, which is a complete and transitive binary relation on $A$. Given $R_i$, we denote by $P_i$ its induced strict preference and by $t(R_i)$ the most-preferred alternative according to $R_i$, if it exists; that is, for any distinct pair $x, y \in A$, $x \, P_i \, y$ if and only if $x \, R_i \, y$ and not $y \, R_i \, x$, and $t(R_i) \, P_i \, y$ for all $y \in A \setminus \{t(R_i)\}$. Let $\mathcal{R}$ and $\mathcal{P}$ be respectively the sets of all weak and strict preferences over $A$. A (preference) *profile* is a $n$-tuple $R = (R_1, \ldots, R_n) \in \mathcal{R}^N$, an ordered list of $n$ preferences, one for each agent. Given a profile $R$, an agent $i$, and a subset of agents $S$, $R_{-i}$ and $R_{-S}$ denote the sub-profiles in $\mathcal{R}^{N \setminus \{i\}}$ and $\mathcal{R}^{N \setminus S}$ obtained by deleting $R_i$ and $R_S := (R_j)_{j \in S}$ from $R$, respectively; hence, $R$ can be written as $(R_i, R_{-i})$ or as $(R_S, R_{-S})$.

A *social choice function* $f : \mathcal{D} \to A$ on a Cartesian product domain of preference profiles $\mathcal{D} := \mathcal{D}_1 \times \cdots \times \mathcal{D}_n \subseteq \mathcal{R}^N$ selects, for each profile $R \in \mathcal{D}$, an alternative $f(R) \in A$.

Let $f : \mathcal{D} \to A$ be a social choice function. Construct its associated normal game form $(N, \mathcal{D}, f)$, where $N$ is the set of players, $\mathcal{D}$ is the Cartesian product set of strategy profiles and $f$ is the outcome function mapping strategy profiles into alternatives. Then, $f$

is implementable in dominant strategies (or $f$ is SP-implementable) if the normal game form $(N, \mathcal{D}, f)$ has the property that, for all $R \in \mathcal{D}$ and $i \in N$, $R_i$ is a weakly dominant strategy for $i$ in the game in normal form $(N, \mathcal{D}, f, R)$, where each $i \in N$ uses $R_i$ to evaluate the consequences of strategy profiles: Namely, a social choice function $f : \mathcal{D} \to A$ is *strategy-proof* (SP) if, for all $R \in \mathcal{D}$, $i \in N$, and $R_i' \in \mathcal{D}_i$,

$$f(R_i, R_{-i}) \, R_i \, f(R_i', R_{-i}).$$

The literature refers to $(N, \mathcal{D}, f)$ as the direct revelation mechanism that SP-implements $f$.

Strategy-proofness requires that agents are able to perform contingent reasoning that might be complex, even for simple social choice functions. To deal with agents that may have limited this ability, Li (2017) proposes the stronger incentive notion of obvious strategy-proofness (OSP) for general settings where agents' types (that coincide with agents' preferences in our setting) are private information. A social choice function $f : \mathcal{D} \to A$ is *obviously strategy-proof* (OSP) if two conditions hold. First, there exist an extensive game form $\Gamma$, played by the agents in $N$ and whose outcomes are the alternatives in $A$, and a type-strategy profile $(\sigma_i^{R_i})_{R_i \in \mathcal{D}_i, i \in N}$, a behavioral strategy in $\Gamma$ for each agent and for each of its types (to be defined formally in Subsection 2.2), that induce the rule; namely, for every profile of types $R = (R_1, \ldots, R_n) \in \mathcal{D}$, when each agent $i$ plays the strategy $\sigma_i^{R_i}$ that corresponds to its type $R_i$, the outcome of the game $x$ is the alternative that the social choice function would have chosen at this profile (*i.e.*, $f(R) = x$). Second, for each agent $i$ and for each of its types $R_i \in \mathcal{D}_i$, the strategy $\sigma_i^{R_i}$ that corresponds to its type $R_i$ is obviously dominant; namely, whenever $i$ has to make a choice in $\Gamma$ it evaluates the consequence of playing according to $\sigma_i^{R_i}$ in a pessimistic way (thinking that the worst possible outcome will follow) and the consequence of deviating to any other strategy $\sigma_i'$ in an optimistic way (thinking that the best possible outcome will follow) and, moreover, the pessimistic outcome associated to $\sigma_i^{R_i}$ is at least as good as the optimistic outcome associated to the deviation $\sigma_i'$, according to $R_i$. Hence, whenever an agent has to play, the choice prescribed by the strategy that corresponds to its type appears as unmistakably optimal; *i.e.*, obviously dominant. In this case, we say that the extensive game form $\Gamma$ and the type-strategy profile $(\sigma_i^{R_i})_{R_i \in \mathcal{D}_i, i \in N}$ OSP-implement $f$.

The difficulty of establishing whether a social choice function $f$ is obviously strategy-proof lies in the fact that its implementation in obviously dominant strategies must be through an extensive game form. But now the extensive game form is not given by a general revelation principle as it is for strategy-proofness in the form of the direct revelation mechanism. The main difficulty lies then in identifying, for each social choice function, the extensive game form $\Gamma$ used to OSP-implement $f$.

To propose intermediate OSP-implementability notions that require different levels of contingent reasoning we have to deal with extensive game forms, which are presented in the next section.

## 2.2 Extensive game forms

Table 1 provides basic notation for extensive game forms.

TABLE 1: NOTATION FOR EXTENSIVE GAME FORMS

| Name | Notation | Generic element |
|---|---|---|
| Players (or agents) | $N$ | $i$ |
| Outcomes (or alternatives) | $A$ | $x$ |
| Histories | $H$ | $h$ |
| Initial history | $h^0$ | |
| Nodes | $Z$ | $z$ |
| Partial order on $Z$ | $\prec$ | |
| Initial node | $z_0$ | |
| Terminal nodes | $Z_T$ | |
| Non-terminal nodes | $Z_{NT}$ | |
| Nodes where $i$ plays | $Z_i$ | $z_i$ |
| Information sets of player $i$ | $\mathcal{I}_i$ | $I_i$ |
| Choices at $z_i \in Z_{NT}$ | $Ch(z_i)$ | |
| Outcome at $z \in Z_T$ | $g(z)$ | |

An extensive game form with set of players $N$ and outcomes in $A$ (or simply, a *game*) is a seven-tuple $\Gamma = (N, A, (Z, \prec), \mathcal{Z}, \mathcal{I}, Ch, g)$, where $(Z, \prec)$ is a rooted tree. Namely, a graph with the properties that any two nodes in $Z$ are connected through a unique path and with a distinguished node (called a root) $z_0 \in Z_{NT}$ such that $z_0 \prec z$ for all $z \in Z \setminus \{z\}$. Or equivalently, every $z \in Z \setminus \{z_0\}$ has a *unique* node $z'$ with the property that $z' \prec z$ and there is no $z'' \in Z_{NT}$ for which $z' \prec z'' \prec z$; this node $z'$ is named the immediate predecessor of $z$ and it is denoted by $IP(z)$. In addition to the notation of Table 1, $\mathcal{Z} = \{Z_1, \ldots, Z_n\}$ represents the partition of $Z_{NT}$, where $z \in Z_i$ means that $i$ plays at $z$, $\mathcal{I} = \{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$ represents the partition of information sets, where $z, z' \in I_i \in \mathcal{I}_i$ means that $i$ has to play at $I_i$ (*i.e.*, $I_i \subseteq Z_i$) and $i$ does not know whether the game has reached node $z$ or $z'$, and $Ch = \bigcup_{z \in Z_{NT}} Ch(z)$ is the collection of all available choices. Of course, for each $z \in Z_{NT}$, there should be a one-to-one identification between $Ch(z)$ and the set of immediate followers of $z$, defined as $IF(z) = \{z' \in Z \mid IP(z') = z\}$. For this reason we often identify the choice made by agent $i$ at node $z \in Z_i$ with the node that follows $z$. Moreover, for each $I_i \in \mathcal{I}_i$ and any pair $z, z' \in I_i$, $Ch(z) = Ch(z')$ holds; namely, player $i$ at $I_i$ can not distinguish between $z$ and $z'$ by observing the set of their respective available actions. We write $I_i' \prec I_i$ if for each $z' \in I_i'$ there is $z \in I_i$ for which $z' \prec z$. A history $h$ (of length $t$) is a sequence $z_0, z_1, \ldots, z_t$ of $t+1$ nodes, starting at $z_0$ and finishing at $z_t$, such that for all $m = 1, \ldots, t$, $z_{m-1} = IP(z_m)$. Each history $h = z_0, \ldots, z_t$ can be uniquely identified with the node $z_t$ and each node $z$ can be uniquely identified with the history $h = z_0, \ldots, z$. Note that $\Gamma$ is not yet a game in extensive form because agents' preferences over alternatives (associated to terminal nodes) are not specified. But given a game $\Gamma$ and a profile of preferences $R \in \mathcal{D}$ over $A$, the pair $(\Gamma, R)$ defines a game in extensive form where each agent $i$ uses $R_i$ to evaluate pairs of alternatives, associated to pairs of terminal nodes. Since $N$ and $A$ will be fixed throughout the paper, let $\mathcal{G}$ be the class of all games with set of players $N$ and outcomes in $A$. From now on we shall refer to $N$ as the set of agents and to $A$ as the set of alternatives.

Fix a game $\Gamma \in \mathcal{G}$ and an agent $i \in N$. A (behavioral and pure) *strategy* of $i$ in $\Gamma$ is a function $\sigma_i : Z_i \to Ch$ such that, for each $z \in Z_i$, $\sigma_i(z) \in Ch(z)$; namely, $\sigma_i$ selects at each node where $i$ has to play one of $i$'s available choices. Moreover, $\sigma_i$ is $\mathcal{I}_i$-measurable: For any $I_i \in \mathcal{I}_i$ and any pair $z, z' \in I_i$, $\sigma_i(z) = \sigma_i(z')$. Hence, we often write $\sigma_i(I_i)$ to denote the action taken by $\sigma_i$ at all nodes in $I_i$. Let $\Sigma_i$ be the set of $i$'s strategies in $\Gamma$. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma := \Sigma_1 \times \cdots \times \Sigma_n$ is an ordered list of strategies, one for each agent. Let $z^\Gamma(z, \sigma)$ be the terminal node that results in $\Gamma$ when agents start playing at $z \in Z_{NT}$ according to $\sigma \in \Sigma$. Given $\sigma \in \Sigma$ and $S \subseteq N$, denote by $\sigma_S = (\sigma_i)_{i \in S}$ the strategy profile of agents in $S$.

Let a game $\Gamma$ and a domain $\mathcal{D}$ be given. A *type-strategy* profile $(\sigma_i^{R_i})_{R_i \in \mathcal{D}_i, i \in N}$ specifies, for each agent $i \in N$ and preference $R_i \in \mathcal{D}_i$, a behavioral strategy $\sigma_i^{R_i} \in \Sigma_i$ of $i$ in $\Gamma$. We denote by $\sigma^R$ the strategy profile $(\sigma_1^{R_1}, \ldots, \sigma_n^{R_n}) \in \Sigma$.

We say that the extensive game form $\Gamma$ and the type-strategy profile $(\sigma_i^{R_i})_{R_i \in \mathcal{D}_i, i \in N}$ SP-*implement* the social choice function $f : \mathcal{D} \to A$ if, for all $R \in \mathcal{D}$, (i) $f(R) = g(z^\Gamma(z_0, \sigma^R))$ and (ii) for all $i \in N$, $\sigma_i^{R_i}$ is a weakly dominant strategy in $\Gamma$; namely, for all $\sigma_{-i} \in \Sigma_{-i}$ and

$\sigma'_i \in \Sigma_i$,

$$g(z^{\Gamma}(z_0,(\sigma_i^{R_i},\sigma_{-i})))\,R_i\,g(z^{\Gamma}(z_0,(\sigma'_i,\sigma_{-i}))).$$

We often omit the explicit reference to the type-strategy profile and simply say that $\Gamma$ SP-implements $f$.

# 3 Obvious strategy-proofness with respect to a partition

## 3.1 Definition and example

We present several notions required to define obvious strategy-proofness with respect to a partition of agents $\mathcal{S}=\{S_1,\ldots,S_K\}$, where $1\leq K\leq n$.

Fix a game $\Gamma\in\mathcal{G}$, a strategy profile $\sigma\in\Sigma$, and a subset of agents $S\subseteq N$.

We say that a history $h=z_0,\ldots,z_t$ (or node $z_t$) is *compatible with* $\sigma_S$ if, for all $z_{t'}\in Z_i$ such that $0\leq t'<t$, $\sigma_i(z_{t'})=z_{t'+1}$ holds; namely, a history $h=z_0,\ldots,z_t$ is compatible with $\sigma_S$ if, whenever an agent $i$ has to play at a node $z_{t'}$ on the path from $z_0$ to $z_t$, $i$'s choice prescribed by $\sigma_i$ induces the node $z_{t'+1}$. Note that the compatibility of $h=z_0,\ldots,z_t$ with $\sigma_S$ does not exclude the possibility that an agent not in $S$ plays along the history towards $z_t$; namely, $z_{t'}\in Z_i$ for some $0\leq t'<t$ and $i\notin S$. Given $\sigma_S$, $i\in S$ and $\sigma'_i\in\Sigma_i\setminus\{\sigma_i\}$, an earliest point of departure for $\sigma_S$ and $\sigma'_i$ is the set of all nodes compatible with $\sigma_S$ in an information set $I_i$, with the properties that $\sigma_i$ and $\sigma'_i$ prescribe different actions at each of them but identical ones at all its previous information sets that come across to each of their paths.

**Definition 1** Let $\sigma_S$, $i\in S$, $\sigma'_i\in\Sigma_i\setminus\{\sigma_i\}$ and $I_i\in\mathcal{I}_i$ be given. We say that the set formed by of all nodes $z\in I_i$ that are compatible with $\sigma_S$, denoted by $I_i(\sigma_S,\sigma'_i)$, is an *earliest point of departure* for $\sigma_S$ and $\sigma'_i$ if
(i) $\sigma_i(I_i)\neq\sigma'_i(I_i)$,
(ii) $\sigma_i(I'_i)=\sigma'_i(I'_i)$ for all $I'_i\in\mathcal{I}_i$ such that $I'_i\prec I_i$.

Observe two things. First, an earliest point of departure is a subset of an information set of an agent. Second, it is relative to a join strategy $\sigma_S$ of agents in $S$, a subset to which $i$ belongs to, and to an alternative strategy $\sigma'_i$ different from the strategy $\sigma_i$ specified in $\sigma_S$. To illustrate the notion, consider the game $\Gamma$ depicted in Figure 1 below, which will be fully described later on. Let $S=\{1,2\}$, $(\sigma_1,\sigma_2)$ and $\sigma'_2$ be such that $\sigma_1(z_0)=y$,

$\sigma_2(I_2) = y$ and $\sigma'_2(I_2) = x$. Then, the earliest point of departure for $(\sigma_1, \sigma_2)$ and $\sigma'_2$ is $I_2((\sigma_1, \sigma_2), \sigma'_2) = \{z_1\} \subsetneq I_2$. Again, earliest points of departure may be strict subsets of information sets because the strategies of all agents in $S$ except $i$ have been fixed, excluding therefore nodes of the same information set.[2]

Given $\sigma_S$ and $\sigma'_i$, denote the set of earliest points of departures for $\sigma_S$ and $\sigma'_i$ by $\alpha(\sigma_S, \sigma'_i)$.

Given the partition $\mathcal{S}$ of $N$ and agent $i \in N$, denote by $S^i \in \mathcal{S}$ the element in $\mathcal{S}$ with the property that $i \in S^i$. Given $\sigma_{S^i}$ and $\sigma'_i$, let $o(\sigma_{S^i}, \sigma'_i)$ and $o'(\sigma_{S^i}, \sigma'_i)$ be the two sets of options left respectively by $\sigma_i$ and $\sigma'_i$ at the earliest point of departure $I_i(\sigma_{S^i}, \sigma'_i)$; namely,[3]

$$o(\sigma^i_S, \sigma'_i) = \{x \in A \mid \exists \overline{\sigma}_{-S^i} \in \Sigma_{-S^i} \text{ and } z \in I_i(\sigma_{S^i}, \sigma'_i) \text{ s.t. } x = g(z^\Gamma(z, (\sigma_i, \sigma_{S^i \setminus \{i\}}, \overline{\sigma}_{-S^i})))\}$$

and

$$o'(\sigma^i_S, \sigma'_i) = \{y \in A \mid \exists \overline{\sigma}_{-S^i} \in \Sigma_{-S^i} \text{ and } z \in I_i(\sigma_{S^i}, \sigma'_i) \text{ s.t. } y = g(z^\Gamma(z, (\sigma'_i, \sigma_{S^i \setminus \{i\}}, \overline{\sigma}_{-S^i})))\}.$$

We are now ready to define the notion of obviously dominant strategy with respect to a partition of agents $\mathcal{S}$, given a game $\Gamma$ and a domain of preferences $\mathcal{D}$.

**Definition 2** We say that $\sigma_i$ is *obviously dominant with respect to* $\mathcal{S}$ in $\Gamma$ for $i$ with $R_i \in \mathcal{D}_i$ if for all $\sigma_{S^i \setminus \{i\}} \in \Sigma_{S^i \setminus \{i\}}$, all $\sigma'_i \neq \sigma_i$ and all $I_i(\sigma_{S^i}, \sigma'_i) \in \alpha(\sigma_{S^i}, \sigma'_i)$,

$$x \, R_i \, y$$

holds, for all $x \in o(\sigma_{S^i}, \sigma'_i)$ and all $y \in o'(\sigma_{S^i}, \sigma'_i)$.[4]

**Definition 3** A social choice function $f : \mathcal{D} \to A$ is *obviously strategy-proof* (OSP) *with respect to* $\mathcal{S}$ if there exist an extensive game form $\Gamma \in \mathcal{G}$ and a type-strategy profile $(\sigma_i^{R_i})_{R_i \in \mathcal{D}_i, i \in N}$ for $\Gamma$ such that, for each $R \in \mathcal{D}$, (i) $f(R) = g(z^\Gamma(z_0, \sigma^R))$ and (ii) for all $i \in N$, $\sigma_i^{R_i}$ is obviously dominant with respect to $\mathcal{S}$ in $\Gamma$ for $i$ with $R_i$.

When (i) holds we say that $\Gamma$ and $(\sigma_i^{R_i})_{R_i \in \mathcal{D}_i, i \in N}$ induce $f$. When (i) and (ii) hold we say that $\Gamma$ OSP-*implements* $f$ with respect to $\mathcal{S}$.

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[2] Observe that if we consider $S = \{2\}, I_2(\sigma_2, \sigma'_2) = I_2$

[3] Observe that $o(\sigma^i_S, \sigma'_i)$ and $o'(\sigma^i_S, \sigma'_i)$ depend on the choice of $I_i(\sigma_{S^i}, \sigma'_i)$ although this fact is not reflected in the notation. This will not cause any confusion as the earliest point of departure in question will be clear from the context.

[4] Namely, given $\sigma_{S^i}$, the worst alternative that can be reached by $i$ playing $\sigma_i$ is at least as preferred according to $R_i$ as the best alternative that can be reach by $i$ playing $\sigma'_i$; in this sense, $\sigma_i$ is undoubtedly better.

**Remark 1** *Let $f : \mathcal{D} \to A$ be a social choice function. Then,*

- *$f$ is OSP with respect to $\mathcal{S} = \{\{1\}, \ldots, \{n\}\}$ if and only if $f$ is OSP and*
- *$f$ is OSP with respect to $\mathcal{S} = \{N\}$ if and only if $f$ is SP.*

Example 1 illustrates the notion of obvious strategy-proofness with respect to a partition.

**Example 1** Let $N = \{1, 2, 3, 4, 5\}$ be the set of agents, let $\mathcal{S}^* = \{\{1, 2\}, \{3\}, \{4, 5\}\}$ be the partition, and let $A = \{x, y\}$ be the set of alternatives. For each $i \in N$, let $\mathcal{D}_i = \mathcal{P} = \{P_i^x, P_i^y\}$ be the domain of the two strict preferences over $A$, where $x\, P_i^x\, y$ and $y\, P_i^y\, x$ (*i.e.*, $x = t(P_i^x)$ and $y = t(P_i^y)$). When it does not lead to any confusion, we will refer to $P_i^x$ and $P_i^y$ only by their preferred alternatives $x$ and $y$, respectively. Define the social choice function $f : \mathcal{P}^N \to \{x, y\}$ as follows: For each $P \in \mathcal{P}^N$, $f(P) = x$ if (i) $t(P_1) = t(P_2) = x$, or (ii) $t(P_1) = t(P_3) = x$ or (iii) $t(P_2) = t(P_4) = t(P_5) = x$ hold; otherwise, $f(P) = y$.[5]

Consider the extensive game form $\Gamma$ depicted in Figure 1, where agents play only once, information sets of agents 1, 3 and 4 contain a unique node ($z_0$, $z_3$ and $z_4$, respectively), and agents 2 and 5 have an information set with two nodes ($I_2 = \{z_1, z_2\}$ and $I_5 = \{z_5, z_6\}$, respectively) and, at each $z \in Z_{NT}$, $Ch(z) = \{x, y\}$.

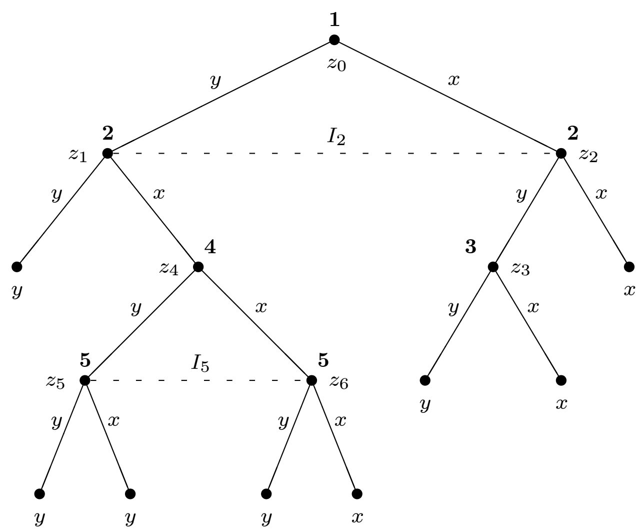


Figure 1: An extensive game form $\Gamma$ that illustrates Definition 3

[5]This is a particular instance of an extended majority voting rule that we shall define later through a family of minimal winning coalitions for $x$, $\mathcal{C}_m^x$. The family contains those subsets of agents that can impose $x$ whenever all their members declare $x$ as their top alternative; in this case, $\mathcal{C}_m^x = \{\{1, 2\}, \{1, 3\}, \{2, 4, 5\}\}$. By Arribillaga, MassÃ³ and Neme (2020), this voting rule is not obviously strategy-proof.

For agent $i \in N$ with preference $P_i \in \mathcal{P}$, define the truth-telling strategy $\sigma_i^{P_i}$ by setting, for $z \in Z_i$, $\sigma_i^{P_i}(z) = t(P_i)$.

It is easy to check that this particular social choice function $f$ is induced by $\Gamma$ and $(\sigma_i^{P_i})_{P_i \in \mathcal{P}, i \in N}$. To complete the verification that $f$ is OSP with respect to $\mathcal{S}^*$, we check that, for each $i \in N$ and each $P_i \in \mathcal{P}$, $\sigma_i^{P_i}$ is obviously dominant with respect to $\mathcal{S}^* = \{\{1,2\},\{3\},\{4,5\}\}$ in $\Gamma$ for $i$ with $P_i$.

Consider coalition $S_1^* = \{1,2\}$ and agent 1.

Assume $x \, P_1 \, y$ (*i.e.*, $P_1 = P_1^x$). Then, agent 1's truth-telling strategy is $\sigma_1^{P_1}(z_0) = x$ and let $\sigma_1'(z_0) = y$ be agent 1's deviating strategy. For any $\sigma_2 \in \Sigma_2$, write $\sigma_{S_1^*} = (\sigma_1^{P_1}, \sigma_2)$. Fix $\sigma_2(I_2) = x$. Hence, $\alpha(\sigma_{S_1}^*, \sigma_1') = \{I_1(\sigma_{S_1^*}, \sigma_1')\}$ and $I_1(\sigma_{S_1^*}, \sigma_1') = \{z_0\}$, and so $o(\sigma_{S_1^*}, \sigma_1') = \{x\}$ and $o'(\sigma_{S_1^*}, \sigma_1') = \{x, y\}$. Then, $x$ is the worst (and unique) alternative of playing according to the truth-telling strategy $\sigma_1^{P_1}(z_0) = x$, which is weakly preferred to $x$, the best possible alternative of playing according to the deviating strategy $\sigma_1'(z_0) = y$. Fix $\sigma_2(I_2) = y$. Hence, $\alpha(\sigma_{S_1}^*, \sigma_1') = \{I_1(\sigma_{S_1^*}, \sigma_1')\}$ and $I_1(\sigma_{S_1^*}, \sigma_1') = \{z_0\}$ and so $o(\sigma_{S_1^*}, \sigma_1') = \{x, y\}$ and $o'(\sigma_{S_1^*}, \sigma_1') = \{y\}$. Then, $y$ is the worst possible alternative of playing according to the truth-telling strategy $\sigma_1^{P_1}(z_0) = x$, which is weakly preferred to $y$, the best (and unique) alternative of playing according to the deviating strategy $\sigma_1'(z_0) = y$.

Assume $y \, P_1 \, x$ (*i.e.*, $P_1 = P_1^y$). Then, agent 1's truth-telling strategy is $\sigma_1^{P_1}(z_0) = y$ and let $\sigma_1'(z_0) = x$ be agent 1's deviating strategy. For any $\sigma_2 \in \Sigma_2$, write $\sigma_{S_1^*} = (\sigma_1^{P_1}, \sigma_2)$. Fix $\sigma_2(I_2) = x$. Hence, $\alpha(\sigma_{S_1}^*, \sigma_1') = \{I_1(\sigma_{S_1^*}, \sigma_1')\}$ and $I_1(\sigma_{S_1^*}, \sigma_1') = \{z_0\}$, and so $o(\sigma_{S_1^*}, \sigma_1') = \{x, y\}$ and $o'(\sigma_{S_1^*}, \sigma_1') = \{x\}$. Then, $x$ is the worst possible alternative of playing according to the truth-telling strategy $\sigma_1^{P_1}(z_0) = y$, which is weakly preferred to $x$, the best (and unique) alternative of playing according to the deviating strategy $\sigma_1'(z_0) = x$. Fix $\sigma_2(I_2) = y$. Hence, $\alpha(\sigma_{S_1}^*, \sigma_1') = \{I_1(\sigma_{S_1^*}, \sigma_1')\}$ and $I_1(\sigma_{S_1^*}, \sigma_1') = \{z_0\}$ and so $o(\sigma_{S_1^*}, \sigma_1') = \{y\}$ and $o'(\sigma_{S_1^*}, \sigma_1') = \{x, y\}$. Then, $y$ is the worst (and unique) alternative of playing according to the truth-telling strategy $\sigma_1^{P_1}(z_0) = y$, which is weakly preferred to $y$, the best possible alternative of playing according to the deviating strategy $\sigma_1'(z_0) = x$.

Consider now agent 2.

Assume $x \, P_2 \, y$ (*i.e.*, $P_2 = P_2^x$). Then, agent 2's truth-telling strategy is $\sigma_2^{P_2}(I_2) = x$ and let $\sigma_2'(I_2) = y$ be agent 2's deviating strategy. For any $\sigma_1 \in \Sigma_1$, write $\sigma_{S_1^*} = (\sigma_1, \sigma_2^{P_2})$. Fix $\sigma_1(z_0) = x$. Hence, $\alpha(\sigma_{S_1}^*, \sigma_2') = \{I_2(\sigma_{S_1^*}, \sigma_2')\}$ and $I_2(\sigma_{S_1^*}, \sigma_2') = \{z_2\}$, and so $o(\sigma_{S_1^*}, \sigma_2') = \{x\}$ and $o'(\sigma_{S_1^*}, \sigma_2') = \{x, y\}$. Then, $x$ is the worse (and unique) alternative of playing according to the truth-telling strategy $\sigma_2^{P_2}(I_2) = x$, which is weakly preferred to $x$, the best possible alternative of playing according to the deviating strategy $\sigma'_2(I_2) = y$. Fix $\sigma_1(z_0) = y$. Hence, $\alpha(\sigma^*_{S_1}, \sigma'_2) = \{I_2(\sigma_{S^*_1}, \sigma'_2)\}$ and $I_2(\sigma_{S^*_1}, \sigma'_2) = \{z_1\}$, and so $o(\sigma_{S^*_1}, \sigma'_2) = \{x, y\}$ and $o'(\sigma_{S^*_1}, \sigma'_2) = \{y\}$. Then, $y$ is the worse possible alternative of playing according the truth-telling strategy $\sigma_2^{P_2}(I_2) = x$, which is weakly preferred to $y$, the best (and unique) alternative of playing according to the deviating strategy $\sigma'_2(I_2) = y$.

Assume $y\, P_2\, x$ (*i.e.*, $P_2 = P_2^y$). Then, agent 2's truth-telling strategy is $\sigma_2^{P_2}(I_2) = y$ and let $\sigma'_2(I_2) = x$ be agent 2's deviating strategy. For any $\sigma_1 \in \Sigma_1$, write $\sigma_{S^*_1} = (\sigma_1, \sigma_2^{P_2})$. Fix $\sigma_1(z_0) = x$. Hence, $\alpha(\sigma^*_{S_1}, \sigma'_2) = \{I_2(\sigma_{S^*_1}, \sigma'_2)\}$ and $I_2(\sigma_{S^*_1}, \sigma'_2) = \{z_2\}$, and so $o(\sigma_{S^*_1}, \sigma'_2) = \{x, y\}$ and $o'(\sigma_{S^*_1}, \sigma'_2) = \{x\}$. Then, $x$ is the worse possible alternative of playing according to the truth-telling strategy $\sigma_2^{P_2}(I_2) = y$, which is weakly preferred to $x$, the best (and unique) alternative of playing according to the deviating strategy $\sigma'_2(I_2) = x$. Fix $\sigma_1(z_0) = y$. Hence, $\alpha(\sigma^*_{S_1}, \sigma'_2) = \{I_2(\sigma_{S^*_1}, \sigma'_2)\}$ and $I_2(\sigma_{S^*_1}, \sigma'_2) = \{z_1\}$, and so $o(\sigma_{S^*_1}, \sigma'_2) = \{y\}$ and $o'(\sigma_{S^*_1}, \sigma'_2) = \{x, y\}$. Then, $y$ is the worse (and unique) possible alternative of playing according to the truth-telling strategy $\sigma_2^{P_2}(I_2) = y$, which is weakly preferred to $y$, the best possible alternative of playing according to the deviating strategy $\sigma'_2(I_2) = x$.

Therefore, truth-telling is obviously dominant with respect to $\mathcal{S}^*$ in $\Gamma$ for agents 1 and 2 with each of the two preferences.

Consider coalition $S^*_2 = \{3\}$. For any $P_3 \in \mathcal{P}$ and deviating strategy $\sigma'_3$, $\alpha(\sigma_3^{P_3}, \sigma'_3) = \{I_3(\sigma_3^{P_3}, \sigma'_3)\}$ and $I_3(\sigma_3^{P_3}, \sigma'_3) = \{z_3\}$ hold, and so $o(\sigma_{S^*_3}, \sigma'_3) = t(P_3)$, and $o'(\sigma_{S^*_3}, \sigma'_3) \neq t(P_3)$ hold. Then, $t(P_3)$ is the worse (and unique) possible alternative of playing according to the truth-telling strategy, which is strictly preferred to $\sigma'_3(I_3) \neq t(P_3)$, the best possible alternative of playing according to the deviating strategy.

Therefore, truth-telling is obviously dominant with respect to $\mathcal{S}^*$ in $\Gamma$ for agent 3 with each of the two preferences.

Consider coalition $S^*_3 = \{4, 5\}$ and agent 4.

Assume $x\, P_4\, y$ (*i.e.*, $P_4 = P_4^x$). Then, agent 4's truth-telling strategy is $\sigma_4^{P_4}(z_4) = x$ and let $\sigma'_4(z_4) = y$ be agent 4's deviating strategy. For any $\sigma_5 \in \Sigma_5$, write $\sigma_{S^*_3} = (\sigma_4^{P_4}, \sigma_5)$. Fix $\sigma_5(I_5) = x$. Hence, $\alpha(\sigma^*_{S_3}, \sigma'_4) = \{I_4(\sigma_{S^*_3}, \sigma'_4)\}$ and $I_4(\sigma_{S^*_3}, \sigma'_4) = \{z_4\}$, and so $o(\sigma_{S^*_3}, \sigma'_4) = \{x\}$ and $o'(\sigma_{S^*_3}, \sigma'_4) = \{y\}$. Then, $x$ is the worst (and unique) alternative of playing according to the truth-telling strategy $\sigma_4^{P_4}(z_4) = x$, which is strictly preferred to $y$, the best (and unique) alternative of playing according to the deviating strategy $\sigma'_4(z_4) = y$. Fix $\sigma_5(I_2) = y$. Hence, $\alpha(\sigma^*_{S_3}, \sigma'_4) = \{I_4(\sigma_{S^*_3}, \sigma'_4)\}$ and $I_4(\sigma_{S^*_3}, \sigma'_4) = \{z_4\}$ and so $o(\sigma_{S^*_3}, \sigma'_4) = \{y\}$ and $o'(\sigma_{S^*_3}, \sigma'_4) = \{y\}$. Then, $y$ is the worst (and unique) alternative of playing according to the truth-telling strategy $\sigma_4^{P_4}(z_4) = y$, which is weakly preferred to $y$, the best (and unique) alternative of playing according to the deviating strategy $\sigma_4'(z_4) = x$.

Assume $y\, P_4\, x$ (*i.e.*, $P_4 = P_4^y$). Then, agent 4's truth-telling strategy is $\sigma_4^{P_4}(z_4) = y$ and let $\sigma_4'(z_4) = x$ be agent 4's deviating strategy. For any $\sigma_5 \in \Sigma_5$, write $\sigma_{S_3^*} = (\sigma_4^{P_4}, \sigma_5)$. Fix $\sigma_5(I_5) = x$. Hence, $\alpha(\sigma_{S_3}^*, \sigma_4') = \{I_4(\sigma_{S_3^*}, \sigma_4')\}$ and $I_4(\sigma_{S_3^*}, \sigma_4') = \{z_4\}$, and so $o(\sigma_{S_3^*}, \sigma_4') = \{y\}$ and $o'(\sigma_{S_3^*}, \sigma_4') = \{x\}$. Then, $y$ is the worst (and unique) alternative of playing according to the truth-telling strategy $\sigma_4^{P_4}(z_4) = y$, which is strictly preferred to $y$, the best (and unique) alternative of playing according to the deviating strategy $\sigma_4'(z_4) = x$. Fix $\sigma_5(I_2) = y$. Hence, $\alpha(\sigma_{S_3}^*, \sigma_4') = \{I_4(\sigma_{S_3^*}, \sigma_4')\}$ and $I_4(\sigma_{S_3^*}, \sigma_4') = \{z_4\}$ and so $o(\sigma_{S_3^*}, \sigma_4') = \{y\}$ and $o'(\sigma_{S_3^*}, \sigma_4') = \{y\}$. Then, $y$ is the worst (and unique) alternative of playing according to the truth-telling strategy $\sigma_4^{P_4}(z_4) = y$, which is weakly preferred to $y$, the best (and unique) alternative of playing according to the deviating strategy $\sigma_4'(z_4) = x$.

Consider now agent 5.

Assume $x\, P_5\, y$ (*i.e.*, $P_5 = P_5^x$). Then, agent 5's truth-telling strategy is $\sigma_5^{P_5}(I_5) = x$ and let $\sigma_5'(I_5) = y$ be agent 5's deviating strategy. For any $\sigma_4 \in \Sigma_4$, write $\sigma_{S_3^*} = (\sigma_4, \sigma_5^{P_5})$. Fix $\sigma_4(x_4) = x$. Hence, $\alpha(\sigma_{S_3}^*, \sigma_5') = \{I_5(\sigma_{S_3^*}, \sigma_5')\}$ and $I_5(\sigma_{S_3^*}, \sigma_5') = \{z_6\}$, and so $o(\sigma_{S_3^*}, \sigma_5') = \{x\}$ and $o'(\sigma_{S_3^*}, \sigma_5') = \{y\}$. Then, $x$ is the worst (and unique) alternative of playing according to the truth-telling strategy $\sigma_5^{P_5}(I_5) = x$, which is strictly preferred to $y$, the best (and unique) alternative of playing according to the deviating strategy $\sigma_5'(I_5) = y$. Fix $\sigma_4(z_4) = y$. Hence, $\alpha(\sigma_{S_3}^*, \sigma_5') = \{I_5(\sigma_{S_3^*}, \sigma_5')\}$ and $I_5(\sigma_{S_3^*}, \sigma_5') = \{z_5\}$ and so $o(\sigma_{S_3^*}, \sigma_5') = \{y\}$ and $o'(\sigma_{S_3^*}, \sigma_5') = \{y\}$. Then, $y$ is the worst (and unique) alternative of playing according to the truth-telling strategy $\sigma_5^{P_5}(I_5) = y$, which is weakly preferred to $y$, the best (and unique) alternative of playing according to the deviating strategy $\sigma_5'(I_5) = y$.

Assume $y\, P_5\, x$ (*i.e.*, $P_5 = P_5^y$). Then, agent 5's truth-telling strategy is $\sigma_5^{P_5}(I_5) = y$ and let $\sigma_5'(I_5) = x$ be agent 5's deviating strategy. For any $\sigma_4 \in \Sigma_4$, write $\sigma_{S_3^*} = (\sigma_4, \sigma_5^{P_5})$. Fix $\sigma_4(x_4) = x$. Fix $\sigma_4(z_4) = x$. Hence, $\alpha(\sigma_{S_3}^*, \sigma_5') = \{I_5(\sigma_{S_3^*}, \sigma_5')\}$ and $I_5(\sigma_{S_3^*}, \sigma_5') = \{z_6\}$, and so $o(\sigma_{S_3^*}, \sigma_5') = \{y\}$ and $o'(\sigma_{S_3^*}, \sigma_5') = \{x\}$. Then, $y$ is the worst (and unique) alternative of playing according to the truth-telling strategy $\sigma_5^{P_5}(I_5) = y$, which is strictly preferred to $x$, the best (and unique) alternative of playing according to the deviating strategy $\sigma_5'(I_5) = x$. Fix $\sigma_4(z_4) = y$. Hence, $\alpha(\sigma_{S_3}^*, \sigma_5') = \{I_5(\sigma_{S_3^*}, \sigma_5')\}$ and $I_5(\sigma_{S_3^*}, \sigma_5') = \{z_5\}$ and so $o(\sigma_{S_3^*}, \sigma_5') = \{y\}$ and $o'(\sigma_{S_3^*}, \sigma_5') = \{y\}$. Then, $y$ is the worst (and unique) alternative of playing according to the truth-telling strategy $\sigma_5^{P_5}(I_5) = y$, which is weakly preferred to $y$, the best (and unique) alternative of playing according to the deviating strategy $\sigma_5'(I_5) = y$.

Therefore, truth-telling is obviously dominant with respect to $\mathcal{S}^*$ in $\Gamma$ for agents 4 and 5 with each of the two preferences. Thus, $\Gamma$ and $(\sigma_i^{P_i})_{P_i\in\mathcal{P},\, i\in N}$ OSP-implement $f$ with respect to $\mathcal{S}^*$. $\square$

## 3.2 Two general results

Proposition 1 establishes that for any social choice function $f$ the property of being OSP with respect to a given partition is inherited by all of its coarser partitions. Thus, in Example 1 above, $f$ is also OSP with respect to the coarser partition $\mathcal{S} = \{\{1,2,3\},\{4,5\}\}$ of $\mathcal{S}^* = \{\{1,2\},\{3\},\{4,5\}\}$. We now state and prove Proposition 1.

**Proposition 1** *Let $\mathcal{S}$ be a coarser partition of $\mathcal{S}^*$ and let $f : \mathcal{D} \to A$ be OSP with respect to $\mathcal{S}^*$. Then, $f : \mathcal{D} \to A$ is OSP with respect to $\mathcal{S}$.*

**Proof.** Let $\Gamma$ and $(\sigma_i^{R_i})_{R_i\in\mathcal{D}_i,\, i\in N}$ be the game in extensive form and the type-strategy profile that OSP-implement $f$ with respect to $\mathcal{S}^*$. Hence, they induce $f$. Thus, it only remains to be shown that $(\sigma_i^{R_i})_{R_i\in\mathcal{D}_i,\, i\in N}$ is obviously dominant with respect to $\mathcal{S}$ in $\Gamma$.

Fix $i \in N$ and $R_i \in \mathcal{D}_i$. To lighten the notation is this proof, we will write $\sigma_i$ instead of $\sigma_i^{R_i}$. Let $S \in \mathcal{S}$ and $S^* \in \mathcal{S}^*$ be such that $i \in S^* \subseteq S$. Fix a strategy $\sigma_j$ for all $j \in S \setminus \{i\}$ and let $\sigma_i' \neq \sigma_i$.

**Claim** Let $I_i \in \mathcal{I}_i$ be such that $\sigma_i(I_i) \neq \sigma_i'(I_i)$ and $\sigma_i(I_i') = \sigma_i'(I_i')$ for all $I_i' \prec I_i$. Then,
(i) if $I_i(\sigma_S, \sigma_i') \in \alpha(\sigma_S, \sigma_i')$, then $I_i(\sigma_S, \sigma_i') \subseteq I_i(\sigma_{S^*}, \sigma_i')$, and
(ii) if $\overline{\sigma}_{-S} \in \Sigma_{-S}$, then $(\overline{\sigma}_{-S}, \sigma_{S\setminus S^*}) \in \Sigma_{-S^*}$.

**Proof of the Claim.** To prove (i), let $I_i(\sigma_S, \sigma_i') \in \alpha(\sigma_S, \sigma_i')$ and $z_t \in I_i(\sigma_S, \sigma_i')$ be arbitrary. Then, the history $h = z_0, \ldots, z_t$ is compatible with $\sigma_S$. Hence, if $z_{t'} \in Z_j$, with $t' < t$ and $j \in S$, then $\sigma_j(z_{t'}) = z_{t'+1}$. Therefore, as $S^* \subseteq S$, if $z_{t'} \in Z_j$, with $t' < t$ and $j \in S^*$, then $\sigma_j(z_{t'}) = z_{t'+1}$. Therefore, $h = z_0, \ldots, z_t$ is compatible with $\sigma_{S^*}$. Hence $z_t \in I_i(\sigma_{S^*}, \sigma_i')$.

The proof of (ii) follows immediately from the observation that $S^* \subseteq S$. $\square$

To proceed with the proof of Proposition 1, let $I_i(\sigma_S, \sigma_i') \in \alpha(\sigma_S, \sigma_i')$ be given. By the claim above,

$$\begin{aligned}
&\min_{R_i}\{x \in X \mid \exists \overline{\sigma}_{-S} \in \Sigma_{-S} \text{ and } z \in I_i(\sigma_S, \sigma_i') \text{ such that } x = g(z^\Gamma(z, (\sigma_i, \sigma_{S\setminus\{i\}}, \overline{\sigma}_{-S})))\} \\
R_i &\min_{R_i}\{x \in X \mid \exists \overline{\sigma}_{-S^*} \in \Sigma_{-S^*} \text{ and } z \in I_i(\sigma_{S^*}, \sigma_i') \text{ such that } x = g(z^\Gamma(z, (\sigma_i, \sigma_{S^*\setminus\{i\}}, \overline{\sigma}_{-S^*})))\},
\end{aligned}$$

because the first set of options, where the minimum is taken, is a subset of the second one, and

$$\max_{R_i}\{x \in X \mid \exists\overline{\sigma}_{-S^*} \in \Sigma^*_{-S} \text{ and } z \in I_i(\sigma_{S^*}, \sigma'_i) \text{ such that } x = g(z^\Gamma(z, (\sigma'_i, \sigma_{S^*\setminus\{i\}}, \overline{\sigma}_{-S^*})))\}$$
$$R_i \max_{R_i}\{x \in X \mid \exists\overline{\sigma}_{-S} \in \Sigma_{-S} \text{ and } z \in I_i(\sigma_S, \sigma'_i) \text{ such that } x = g(z^\Gamma(z, (\sigma'_i, \sigma_{S\setminus\{i\}}, \overline{\sigma}_{-S})))\}$$

because the first set of options, where the maximum is taken, contains the second one. Therefore, as $f$ is OSP with respect to $\mathcal{S}^*$,

$$\min_{R_i}\{x \in X \mid \exists\overline{\sigma}_{-S^*} \in \Sigma_{-S^*} \text{ and } z \in I_i(\sigma_{S^*}, \sigma'_i) \text{ such that } x = g(z^\Gamma(z, (\sigma_i, \sigma_{S^*\setminus\{i\}}, \overline{\sigma}_{-S^*})))\}$$
$$R_i \max_{R_i}\{x \in X \mid \exists\widehat{\sigma}_{-S^*} \in \Sigma_{-S^*} \text{ and } z \in I_i(\sigma_{S^*}, \sigma'_i) \text{ such that } x = g(z^\Gamma(z, (\sigma'_i, \sigma_{S^*\setminus\{i\}}, \widehat{\sigma}_{-S^*})))\}.$$

Applying the transitivity of $R_i$, we obtain that

$$\min_{R_i}\{x \in X \mid \exists\overline{\sigma}_{-S} \in \Sigma_{-S} \text{ and } z \in I_i(\sigma_S, \sigma'_i) \text{ such that } x = g(z^\Gamma(z, (\sigma_i, \sigma_{S\setminus\{i\}}, \overline{\sigma}_{-S})))\}$$
$$R_i \max_{P_i}\{x \in X \mid \exists\widehat{\sigma}_{-S} \in \Sigma_{-S} \text{ and } z \in I_i(\sigma_S, \sigma'_i) \text{ such that } x = g(z^\Gamma(z, (\sigma'_i, \sigma_{S\setminus\{i\}}, \widehat{\sigma}_{-S})))\}.$$

Thus, for all $x \in o(\sigma_S, \sigma'_i)$ and $y \in o'(\sigma_S, \sigma'_i)$,

$$x \, R_i \, y.$$

Then, $\sigma_i^{R_i}$ is obviously dominant with respect to $\mathcal{S}$ in $\Gamma$ for $i$ with $R_i$. Therefore, $f$ is OSP with respect to $\mathcal{S}$. ■

Given a partition $\mathcal{S}$ of the set of agents and a domain $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n \subseteq \mathcal{R}^N$ of preferences, define the class of finite extensive game forms $\mathcal{G}^{\mathcal{S}}$ through the following finite sequence of steps, Namely, $\Gamma \in \mathcal{G}^{\mathcal{S}}$ if the following conditions hold.

- Step 1: There exists $S_1 \in \mathcal{S}$ such that agents in $S_1$ play only once and simultaneously, and the set of available choices of each $i \in S_1$ is a partition of $\mathcal{D}_i$.

- Step 2: For each non-terminal and commonly known history $h^1$ of Step 1, there exists $S_2 \in \mathcal{S}$ such that agents in $S_2$ play only once and simultaneously, and the set of available choices for each agent $i \in S_2$ is a partition of $\mathcal{D}_i$, if $i$ has not played yet along $h^1$, or a partition of the subset of preferences chosen by $i$ in Step 1, otherwise. Moreover, if agent $i \in S_2$ had only one available action in Step 1 (which would imply that $S_1 = S_2$), then $i$ has the same singleton set of available actions in this Step 2.

- ...

  Given $S_1, \ldots, S_{k-1}$ identified in steps from 1 to $k-1$.

- Step $k$: For each non-terminal and commonly known history $h^{k-1}$ of Step $k-1$, there exists $S_k \in \mathcal{S}$ such that agents in $S_k$ play only once and simultaneously, and the set of available choices for each $i \in S_k$ is a partition of $\mathcal{D}_i$, if $i$ has not played yet along $h^{k-1}$, or a partition of the subset of preferences chosen by $i$ last step $i$ has played along $h^{k-1}$, otherwise. Moreover, if agent $i \in S_k$ had only one available action last step $k' < k$ where $i$ has played (which would imply that $S'_k = S_k$), then $i$ has the same singleton set of available actions in this Step $k$.

Observe that $S_k$ and $S_{k'}$ may coincide for some pair of steps $k \neq k'$. However, to be in $\mathcal{G}^{\mathcal{S}}$ the game $\Gamma$ has to finish after a finite number of steps.

The game $\Gamma$ depicted in Figure 1 belongs to $\mathcal{G}^{\mathcal{S}^*}$ for $\mathcal{S}^* = \{\{1,2\},\{3\},\{4,5\}\}$.

We say that $(\sigma_i^{R_i})_{R_i \in \mathcal{D}_i}$ is the *truth-telling type-strategy* of $i$ in $\Gamma \in \mathcal{G}^{\mathcal{S}}$ if, for each $R_i \in \mathcal{D}_i$ and each information set $I_i \in \mathcal{I}_i$, such that there exits $a_i \in Ch(I_i)$ with $R_i \in a_i$, $\sigma_i^{R_i}(I_i) = a_i$; namely, $i$ always chooses the set in the available partition of preferences that contains $R_i$, if any.[6]

**Theorem 1** *Let $f : \mathcal{D} \to A$ be a social choice function and let $\mathcal{S}$ be a partition of $N$. Assume that $\Gamma \in \mathcal{G}^{\mathcal{S}}$ and the truth-telling type-strategy profile $(\sigma_i^{R_i})_{R_i \in \mathcal{D}_i, i \in N}$ SP-implement $f$. Then, $\Gamma$ and $(\sigma_i^{R_i})_{R_i \in \mathcal{D}_i, i \in N}$ OSP-implement $f$ with respect to $\mathcal{S}$.*

**Proof.** Let $\Gamma \in \mathcal{G}^{\mathcal{S}}$ and $(\sigma_i^{R_i})_{R_i \in \mathcal{D}_i, i \in N}$ be the game and the truth-telling type-strategy profile that SP-implement $f$. Hence, for each $R \in \mathcal{D}$, (i) $f(R) = g(z^\Gamma(z_0, \sigma^R))$ and (ii) for all $i \in N$, $\sigma_i^{R_i}$ is weakly dominant in $\Gamma$ for $i$ with $R_i$. Let $\sigma'_i \in \Sigma_i \setminus \{\sigma_i^{R_i}\}$ be any deviating strategy of agent $i$. Fix an strategy, $\sigma_{S^i \setminus \{i\}}$, for agents in $S^i \setminus \{i\}$ and let $\sigma_{S^i} = (\sigma_{S^i \setminus \{i\}}, \sigma_i^{R_i})$. Let $I_i(\sigma_{S^i}, \sigma'_i) \in \alpha(\sigma_{S^i}, \sigma'_i)$

Select any $\theta_{-S^i}, \theta'_{-S^i} \in \Sigma_{-S^i}$, $z, z' \in I_i(\sigma_{S^i}, \sigma'_i)$ and $y, y' \in A$ for which

$$x \, R_i \, y = g(z^\Gamma(z, (\sigma_S^i, \theta_{-S^i}))),$$

for all $x \in o(\sigma_{S^i}, \sigma'_i)$ and

$$y' = g(z^\Gamma(z''_i, \sigma_{S^i \setminus \{i\}}, \theta'_{-S^i}))) \, R_i \, x',$$

for all $x' \in o'(\sigma_{S^i}, \sigma'_i)$.

[6] Observe that this definition does not specify the choice of the strategy in an information set $I_i$ such that there is no $a_i \in Ch(I_i)$ with $R_i \in a_i$. In such information sets the strategy can chose any available choice.

Namely, given $\sigma_{S^i}$ and $\sigma'_i$, $\theta_{-S^i}$ and $\theta'_{-S^i}$ are two profiles of strategies of the agents not in $S^i$ that induce respectively alternatives $y$ and $y'$, who are one of the least or most preferred alternatives respectively in the sets of options left by $\sigma_{S^i}$ together with $\sigma'_i$ at the earliest point of departure $I_i(\sigma_{S^i}, \sigma'_i)$. Without lost of generality, by definition of information sets in the game, we can modify $\theta_{-S^i}$ and $\theta'_{-S^i}$ and obtain that $z$ and $z'$ are compatible with $\theta_{-S^i}$ and $\theta'_{-S^i}$, respectively. Then we can assume that

$$y = g(z^\Gamma(z_0, (\sigma^i_S, \theta_{-S^i}))),$$

and

$$y' = g(z^\Gamma(z_0, (\sigma'_i, \sigma_{S^i \setminus \{i\}}, \theta'_{-S^i}))),$$

Define, for each $j \notin S^i$, the behavioral strategy $\widehat{\sigma}_j$ such that, for each $z \in Z_j$,

$$\widehat{\sigma}_j(z) = \begin{cases} \theta_j & \text{if agents in } S^i \text{ play in the history towards } z \text{ according to } (\sigma_i^{R_i}, \sigma_{S^i\setminus\{i\}}^{R_{S^i\setminus\{i\}}}) \\ \theta'_j & \text{if agents in } S^i \text{ play in the history towards } z \text{ according to } (\sigma'_i, \sigma_{S^i\setminus\{i\}}^{R_{S^i\setminus\{i\}}}). \end{cases}$$

Then, for all $x \in o(\sigma_S^{R_S}, \sigma'_i)$ and $x' \in o'(\sigma_S^{R_S}, \sigma'_i)$,

$$\begin{aligned} x\, R_i\, y &= g(z^\Gamma(z_0, (\sigma_{S^i}, \theta_{-S^i}))) && \text{by definitions of } \theta_{-S^i} \text{ and } y \\ &= g(z^\Gamma(z_0, (\sigma_{S^i}, \widehat{\sigma}_{-S^i}))) && \text{by definition of } \widehat{\sigma}_{-S^i} \\ &R_i \\ &g(z^\Gamma(z_0, (\sigma'_i, \sigma_{S^i\setminus\{i\}}, \widehat{\sigma}_{-S^i}))) && \text{because } \sigma_i^{R_i} \text{ is a dominant strategy in } \Gamma \\ &= g(z^\Gamma(z_0, (\sigma'_i, \sigma_{S^i\setminus\{i\}}, \theta'_{-S^i}))) && \text{by definition of } \widehat{\sigma}_{-S^i} \\ &= y'\, R_i\, x' && \text{by definitions of } \theta'_{-S^i} \text{ and } y'. \end{aligned}$$

Therefore, $\sigma_i^{R_i}$ is obviously dominant with respect to $\mathcal{S}$ in $\Gamma$ for $i$ with $R_i$ and $\Gamma$ OSP-implements $f$ with respect to $\mathcal{S}$. ■

# 4 An application to extended majority voting

In this section, we apply the notion of OSP with respect to a partition to the simplest social choice problem where there are only two alternatives and agents' preferences are strict.

In the first subsection we identify the class of all obviously strategy-proof social choice functions with respect to any partition. In the second one we identify, among them, two anonymous subclasses.

## 4.1 The general case

Let $A = \{x, y\}$ be the set of alternatives and $\mathcal{P}$ be the set of the two strict preferences on $A$; namely, $\mathcal{P} = \{P^x, P^y\}$, where $x\, P^x\, y$ and $y\, P^y\, x$.

Since obvious strategy-proofness with respect to a partition is stronger than strategy-proofness, the first class in this simple case will be contained in the set of all strategy-proof social choice functions $f : \mathcal{P}^N \to \{x, y\}$, which we now describe using the notion of a committee.

Let $2^N$ denote the family of all subsets of $N$ (we call them coalitions). A family $\mathcal{C} \subset 2^N$ of coalitions is a *committee* if it is (coalition) monotonic in the sense that, for each pair $T, T' \subseteq N$ such that $T \in \mathcal{C}$ and $T \subsetneq T'$, we have $T' \in \mathcal{C}$. Coalitions in $\mathcal{C}$ are called *winning.* Given $\mathcal{C}$, denote by $\mathcal{C}_m$ the family of *minimal winning coalitions* of $\mathcal{C}$; namely,

$$\mathcal{C}_m = \{T \in \mathcal{C} \mid \text{there is no } T' \in \mathcal{C} \text{ such that } T' \subsetneq T\}.$$

Observe that by the monotonicity property of a committee, specifying $\mathcal{C}_m$ is enough to completely determine $\mathcal{C}$.

**Definition 3** A social choice function $f : \mathcal{P}^N \to \{x, y\}$ is an *extended majority voting rule* (EMVR) if there exists a committee $\mathcal{C}^x$ with the property that, for all $P \in \mathcal{P}^N$,

$$f(P) = x \text{ if and only if } \{i \in N \mid P_i = P^x\} \in \mathcal{C}^x. \tag{1}$$

Before proceeding, two remarks about the definition of an EMVR are in order.

First, the above definition is relative to a committee for $x$ (this is reflected in the use of the notation $\mathcal{C}^x$). It is possible to define the symmetric condition of (1) relative to a committee for $y$, denoted by $\mathcal{C}^y$, by replacing $x$ by $y$ everywhere in (1). Then, it is easy to show that $\mathcal{C}^x$ and $\mathcal{C}^y$ define the same $f$ if and only if

$$T \in \mathcal{C}_y \text{ if and only if } T \cap T' \neq \emptyset \text{ for all } T' \in \mathcal{C}^x. \tag{2}$$

We say that agent $i$ is *dummy* in $\mathcal{C}$ if there does not exist $M \in \mathcal{C}_m$ such that $i \in M$; otherwise, $i$ is *non-dummy.*

Second, if the EMVR is onto then its associated committee $\mathcal{C}$ is not trivial (*i.e.*, $\emptyset \notin \mathcal{C} \neq \{\emptyset\}$). However, if the EMVR is not onto, and so it is constant, then $\emptyset \in \mathcal{C}$ if it is the constant EMVR that always elects $x$ and $\mathcal{C} = \emptyset$ if it is the constant EMVR that always selects $y$. Since constant social choice functions are obviously strategy-proof with respect to any partition, from now on we will assume that all committees under consideration are not trivial and, accordingly, their associated EMVRs are onto.

We denote the extended majority voting rule whose associated committee is $\mathcal{C}$ by $f_{\mathcal{C}}$.

We state as a remark the characterization of the class of all EMVR in this simple context (it follows from a more general result in Barberá , Sonnenschein and Zhou (1991)).

**Remark 2** *A social choice function $f : \mathcal{P}^N \to \{x, y\}$ is strategy-proof if and only if $f$ is an EMVR; namely, there exists a committee $\mathcal{C}^x$ such that $f = f_{\mathcal{C}^x}$.*

We now define recursively a critical property of a committee that will play an important role in our results. Fix a partition $\mathcal{S} = \{S_1, \ldots, S_K\}$, with $K$ subsets of $N$, and a committee $\mathcal{C}^x$.

For $k = 1$, and given $S_1 \in \mathcal{S}$, define the following three families of sets.

$\mathcal{C}_m^{x,1} = \mathcal{C}_m^x$,

$\mathcal{ND}^1 = \{i \in S_1 \mid \text{there exists } M^1 \in \mathcal{C}_m^{x,1} \text{ with } i \in M^1\}$ and

$\mathcal{X}^1 = \{X = S_1 \cap M^1 \mid M^1 \in \mathcal{C}_m^{x,1} \text{ and } X \notin \mathcal{C}_m^{x,1}\}$.

For $1 < k \leq K$, given $X_1, \ldots, X_{k-1}$, where for each $t = 1, \ldots, k-1$, $X_t \in \mathcal{X}^t$, and given $S_k \notin \{S_1, \ldots, S_{k-1}\}$, define the following three families of sets.

$\mathcal{C}_m^{x,k} = \{M \setminus \cup_{t=1}^{k-1} S_t \mid M \in \mathcal{C}_m^x \text{ and } \ X_t = S_t \cap M \text{ for each } t = 1, \ldots, k-1\}$,

$\mathcal{ND}^k = \{i \in S_k \mid \text{there exits } M^k \in \mathcal{C}_m^{x,k} \text{ with } i \in M^k\}$ and

$\mathcal{X}^k = \{X = S_k \cap M^k \mid M^k \in \mathcal{C}_m^{x,k} \text{ and } X \notin \mathcal{C}_m^{x,k}\}$.

**Iterated Union Property (IUP)** A committee $\mathcal{C}^x$ satisfies the *Iterated Union Property* with respect to the partition $\mathcal{S}$ if, for each $1 \leq k \leq K-1$ and each $X_1, \ldots, X_{k-1}$, where $X_t \in \mathcal{X}^t$ for all $t = 1, \ldots, k-1$, there exists $S_k \in \mathcal{S} \setminus \{S_1, \ldots, S_{k-1}\}$ such that, for each $X \in \mathcal{X}^k$ and $i \in \mathcal{ND}^k \setminus X$,

$$X \cup \{i\} \in \mathcal{C}_m^{x,k}. \tag{3}$$

**Remark 3** *Condition (3) implies that*

$$(S_k \setminus X) \cup \{j\} \in \mathcal{C}_m^{y,k} \tag{4}$$

*holds for all $j \in X$.*

To see that Remark 3 holds, assume otherwise. Then, by (2), there exist $M' \in \mathcal{C}_m^{x,k}$ and $j \in X \in \mathcal{X}^k$ such that $[(S_k \setminus X) \cup \{j\}] \cap M' = \emptyset$. This implies

$$(S_k \setminus X) \cap M' = \emptyset \text{ and } j \notin M'. \tag{5}$$

Denote $X' = (S_k \cap M') \in \mathcal{X}^k$. By (5), $X' \subsetneq X$. Because $\mathcal{C}^x$ satisfies the IUP, $X' \cup \{j\} \in \mathcal{C}^{x,k}$, and $X' \cup \{j\} \subseteq X$. This implies that $X \in \mathcal{C}^{x,k}$, contradicting that $X \in \mathcal{X}^k$.

It is immediate to check that, by just applying the definitions of OSP-implementability and of the IUP with respect to the finer partition, the following remark holds.

**Remark 4** *A social choice function $f : \mathcal{P}^N \to \{x, y\}$ is obviously strategy-proof if and only if $f$ is an extended majority voting rule whose associated committee $\mathcal{C}^x$ satisfies the IUP with respect to the partition $\mathcal{S} = \{\{1\}, \ldots, \{n\}\}$.*

Example 2 illustrates the IUP with respect to a partition $\mathcal{S}$.

**Example 2** Let $N = \{1, 2, 3, 4, 5, 6\}$ be the set of agents, $\mathcal{S} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$ be the partition of $N$ and $\mathcal{C}^x_m = \{\{1, 2\}, \{1, 3, 5, 6\}, \{1, 4, 5, 6\}, \{2, 3, 4, 6\}\}$ be the committee. We argue that $S_1 = \{1, 2\}$ is the subset whose existence is required by the IUP with respect to $\mathcal{S}$; we later shall show that (3) would not be satisfied by neither of the other two subsets. Then, $\mathcal{C}^{x,1}_m = \mathcal{C}^x_m$, $\mathcal{ND}^1 = \{1, 2\}$ and $\mathcal{X}^1 = \{\{1\}, \{2\}\}$.

1. For $X = \{1\} \in \mathcal{X}^1$ and $2 \in \mathcal{ND}^1 \setminus X = \{2\}$, $\{1, 2\} \in \mathcal{C}^{x,1}$. Hence, (3) holds.

   1.1. We argue that, given $X = \{1\}$, $S_2 = \{5, 6\}$ is the subset whose existence is required by the IUP with respect to $\mathcal{S}$; we later shall show that (3) would not be satisfied by the subset $\{3, 4\}$. Then,
   $\mathcal{C}^{x,2}_m = \{\{3, 5, 6\}, \{4, 5, 6\}\}$, $\mathcal{ND}^2 = \{5, 6\}$ and $\mathcal{X}^2 = \{\{5, 6\}\}$.
   Since for $X = \{5, 6\}$, $\mathcal{ND}^2 \setminus X = \emptyset$, (3) does not impose any restriction

2. For $X = \{2\} \in \mathcal{X}^1$ and $1 \in \mathcal{ND}^1 \setminus X = \{1\}$, $\{1, 2\} \in \mathcal{C}^{x,1}$. Hence, (3) holds.

   2.1. Assume $S_2 = \{3, 4\}$. Then,
   $\mathcal{C}^{x,2} = \{\{3, 4, 6\}\}$, $\mathcal{ND}^2 = \{3, 4\}$ and $\mathcal{X}^2 = \{\{3, 4\}\}$.
   Since $X = \{3, 4\}$ and $\mathcal{ND}^2 \setminus X = \{\emptyset\}$, (3) does not impose any restriction.

   2.2. Assume $S_2 = \{5, 6\}$. Then,
   $\mathcal{C}^{x,2} = \{\{3, 4, 6\}\}$, $\mathcal{ND}^2 = \{6\}$ and $\mathcal{X}^2 = \{\{6\}\}$.
   Since for $X = \{6\}$, $\mathcal{ND}^2 \setminus X = \emptyset$, (3) does not impose any restriction.

Therefore, the committee $\mathcal{C}^x$ satisfies the IUP with respect to the partition $\mathcal{S}$.

Now we see that (3) does not hold at $k = 1$, given $X = \{1\}$, for neither $S_1 = \{3, 4\}$ nor $S_1 = \{5, 6\}$. Suppose $S_1 = \{3, 4\}$. Then for $M^1 = \{1, 2\} \in \mathcal{C}^x$, we have that $X = \{3, 4\} \cap$

$\{1,2\} = \emptyset \in \mathcal{X}^1$ and, since $\mathcal{ND}^1 = \{3,4\}$, it follows that for any $i \in \mathcal{ND}^1 \setminus X = \{3,4\}$, $\{i\} \notin \mathcal{C}^{x,1}$, which implies that (3) does not hold. Similarly, if $S_1 = \{5,6\}$. Hence, for the IUP with respect to $\mathcal{S}$ to be satisfied at $k=1$, it must occur that $S_1 = \{1,2\}$.

Now, we see that (3) does not hold at $k=2$, given $X = \{1\}$, for $S_2 = \{3,4\}$. Assume otherwise. Then, $\mathcal{C}_m^{x,2} = \{\{3,5,6\},\{4,5,6\}\}$, $\mathcal{ND}^2 = \{3,4\}$ and $\mathcal{X}^2 = \{\{3\},\{4\}\}$. For $X = \{3\} \in \mathcal{X}^2$ and $4 \in \mathcal{ND}^2 \setminus X = \{4\}$, $\{3,4\} \notin \mathcal{C}_m^{x,2}$. Hence, (3) does not hold. Thus, $S_2 \neq \{3,4\}$ is not the subset whose existence is required by the IUP at $k=2$, after $S_1 = \{1,2\}$ at $X = \{1\}$ and $k=1$. □

Example 3 illustrates, given an arbitrary partition $\mathcal{S}$, different ways of constructing committees that satisfy the IUP with respect to $\mathcal{S}$. It shows that, although the IUP with respect to $\mathcal{S}$ is restrictive, there are many committees satisfying it with respect to any arbitrary partition. For brevity, we shall omit some details required to check that the committees in Example 3 satisfy the IUP with respect to $\mathcal{S}$.

**Example 3** Let $\mathcal{S} = \{S_1, \ldots, S_K\}$ be given. Define the following three committees that satisfy the IUP with respect to $\mathcal{S}$.

1. From each subset $S_k \in \mathcal{S}$, select an arbitrary agent $i_k \in S_k$. Then, define the committee as follows.

   $\mathcal{C}_m^x = \{S_1, (S_1 \setminus \{i_1\}) \cup S_2, (S_1 \setminus \{i_1\}) \cup (S_2 \setminus \{i_2\}) \cup S_3, \ldots, \cup_{k=1}^{K-1} (S_k \setminus \{i_k\}) \cup S_K\}.$

   To check that $\mathcal{C}^x$ satisfies the IUP with respect to $\mathcal{S}$, observe that for $k=1$, $\mathcal{ND}^1 = S_1$, $\mathcal{X}^1 = \{S_1 \setminus \{i_1\}\}$, and $\{i_1\} = \mathcal{ND}^1 \setminus \{S_1 \setminus \{i_1\}\}$; accordingly, since $X \cup \{i_1\} = S_1 \in \mathcal{C}_m^{x,1}$, (3) is satisfied. For $k=2$, and given $X = S_1 \setminus \{i_1\}$, observe that
   $\mathcal{C}_m^{x,2} = \{S_2, (S_2 \setminus \{i_2\}) \cup S_3, \ldots, \cup_{k=2}^{K-1} (S_k \setminus \{i_k\}) \cup S_K\}$. Then, $\mathcal{ND}^2 = \{S_2\}$ and $\mathcal{X}^2 = \{S_2 \setminus \{i_2\}\}$ and $\{i_2\} = \mathcal{ND}^2 \setminus \{S_2 \setminus \{i_2\}\}$; accordingly, since $X \cup \{i_2\} = S_2 \in \mathcal{C}_m^{x,2}$, (3) is satisfied. For any $k > 2$, the verification proceeds similarly.

2. Select two arbitrary agents $i_1, i_1' \in S_1$ and, for each $k = 2, \ldots, K$, select and arbitrary agent $i_k \in S_k$. Then, define the committee as follows.

$$\begin{aligned} \mathcal{C}_m^x &= \{S_1, (S_1 \setminus \{i_1\}) \cup S_2), (S_1 \setminus \{i_1'\}) \cup S_3, (S_1 \setminus \{i_1\}) \cup (S_2 \setminus \{i_2\}) \cup S_4, \\ &\quad (S_1 \setminus \{i_1'\}) \cup (S_3 \setminus \{i_3\}) \cup S_5, \ldots, \} \end{aligned}$$

   To check that $\mathcal{C}^x$ satisfies the IUP with respect to $\mathcal{S}$, observe that for $k=1$, $\mathcal{ND}^1 = S_1$, $\mathcal{X}^1 = \{S_1 \setminus \{i_1\}, S_1 \setminus \{i_1'\}\}$, $\{i_1\} = \mathcal{ND}^1 \setminus \{S_1 \setminus \{i_1\}\}$ and $\{i_1'\} = \mathcal{ND}^1 \setminus \{S_1 \setminus \{i_1'\}\}$;

accordingly, since $(S_1 \setminus \{i_1\}) \cup \{i_1\} = S_1 \in \mathcal{C}^{x,1}$ and $(S_1 \setminus \{i_1'\}) \cup \{i_1'\} = S_1 \in \mathcal{C}^{x,1}$, (3) is satisfied. First, fix $X_1 = S_1 \setminus \{i_1\} \in \mathcal{X}^1$. For $k = 2$, observe that $\mathcal{C}_m^{x,2} = \{S_2, (S_2 \setminus \{i_2\}) \cup S_4, \dots\}$ Then, $\mathcal{ND}^2 = S_2$, $\mathcal{X}^2 = \{S_2 \setminus \{i_2\}\}$ and $\{i_2\} = S_2 \setminus (S_2 \setminus \{i_2\})$; accordingly, since $(S_2 \setminus \{i_2\}) \cup \{i_2\} = S_2 \in \mathcal{C}^{x,2}$, (3) is satisfied. Now, fix $X_1 = S_1 \setminus \{i_1\} \in \mathcal{X}^1$ and $X_2 = S_2 \setminus \{i_2\} \in \mathcal{X}^2$, and proceed similarly for $k \geq 3$. Second, fix $X_1' = S_1 \setminus \{i_1'\} \in \mathcal{X}^1$. For $k = 2$, observe that $\mathcal{C}'^{x,2}_m = \{S_3, (S_3 \setminus \{i_3\}) \cup S_5, \dots\}$. Then, $\mathcal{ND}'^2 = S_3$, $\mathcal{X}'^2 = \{S_3 \setminus \{i_3\}\}$ and $\{i_3\} = S_3 \setminus (S_3 \setminus \{i_3\})$; accordingly, since $(S_3 \setminus \{i_3\}) \cup \{i_3\} = S_3 \in \mathcal{C}'^{x,2}$, (3) is satisfied. Now, fix $X_1' = S_1 \setminus \{i_1'\} \in \mathcal{X}^1$ and $X_2 = S_2 \setminus \{i_2\} \in \mathcal{X}^2$, and proceed similarly for $k \geq 3$.

3. Select an arbitrary subset of agents $N^* = \{j_1^*, \dots, j_r^*\} \subseteq N$, with $1 \leq r < n$. For each $k = 1, \dots, K$ define $\widehat{S}_k = S_k \setminus N^*$, and let $j_k, j_k' \in \widehat{S}_k$ be arbitrary. Define the committee as follows.

$$
\begin{aligned}
\mathcal{C}_m^x = \; & \{\{j_1^*\}, \dots, \{j_r^*\}, \widehat{S}_1, \left(\widehat{S}_1 \setminus \{j_1\}\right) \cup \widehat{S}_2, \left(\widehat{S}_1 \setminus \{j_1'\}\right) \cup \widehat{S}_2, \\
& \left(\widehat{S}_1 \setminus \{j_1\}\right) \cup \left(\widehat{S}_2 \setminus \{j_2\}\right) \cup \widehat{S}_3, \left(\widehat{S}_1 \setminus \{j_1'\}\right) \cup \left(\widehat{S}_2 \setminus \{j_2'\}\right) \cup \widehat{S}_3, \dots, \\
& \cup_{k=1}^{K-1} \left(\widehat{S}_k \setminus \{j_k\}\right) \cup \widehat{S}_K, \cup_{k=1}^{K-1} \left(\widehat{S}_k \setminus \{j_k'\}\right) \cup \widehat{S}_K\}.
\end{aligned}
$$

□

Theorem 2 below shows that the IUP is the key property to characterize obviously strategy-proof social choice functions with respect to a partition in this setting. First, for a committee $\mathcal{C}^x$ to satisfy IUP with respect to a partition $\mathcal{S}$, it is a sufficient condition guaranteeing that $f_{\mathcal{C}^*}$ is OSP with respect to $\mathcal{S}$. Second, if the extensive game form $\Gamma$ that OSP-implements $f_{\mathcal{C}^x}$ with respect to $\mathcal{S}$ belongs to the family of games $\mathcal{G}^{\mathcal{S}}$, then $\mathcal{C}^x$ satisfies the IUP with respect to $\mathcal{S}$. Example 4 below will make clear that the condition that $\Gamma \in \mathcal{G}^{\mathcal{S}}$ can not be dispensed with for the sufficiency of the IUP for OSP-implementation.

**Theorem 2** *Let $f_{\mathcal{C}^x}$ be the EMVR associated to a committee $\mathcal{C}^x$ and let $\mathcal{S}$ be a partition of $N$. Then, $\mathcal{C}^x$ satisfies the IUP with respect to $\mathcal{S}$ if and only if there exists a game $\Gamma \in \mathcal{G}^{\mathcal{S}}$ such that $(\Gamma, (\sigma_i^{P_i})_{P_i \in \mathcal{P}, i \in N}))$ OSP-implements $f_{\mathcal{C}^x} : \mathcal{P}^N \to A$ with respect to $\mathcal{S}$.*

Before moving directly to the proof of Theorem 2 we present Example 4 to show why the IUP of a committee $C^x$ with respect to a partition $\mathcal{S}$ is too strong to guarantee that $f_{\mathcal{C}^x}$ is OSP with respect to $\mathcal{S}$; in particular, the example contains a committee $\mathcal{C}^x$ and a partition $\mathcal{S}$ for which (i) $f_{\mathcal{C}^x}$ is OSP with respect to $\mathcal{S}$, (ii) the IUP is not satisfied with respect to $\mathcal{S}$, and (iii) the game used to OSP-implement $f_{\mathcal{C}^*}$ does not belong to the class $\mathcal{G}^{\mathcal{S}}$. However, there exists a finer partition $\mathcal{S}^*$ of $\mathcal{S}$ such that $f_{\mathcal{C}^x}$ satisfies the IUP with respect the finer partition $\mathcal{S}^*$ and, as Theorem 2 establishes, there exists $\Gamma \in \mathcal{G}^{\mathcal{S}^*}$ such that $\Gamma$ OSP-implements $f_{\mathcal{C}^*}$ with respect to $\mathcal{S}^*$. Hence, by Proposition 1, $f_{\mathcal{C}^x}$ is also OSP with respect to $\mathcal{S}$. Indeed, the game $\Gamma$ of Figure 1 is the one that OSP-implements $f_{\mathcal{C}^x}$ with respect $\mathcal{S}^*$ and $\Gamma \in \mathcal{G}^{\mathcal{S}^*} \setminus \mathcal{G}^{\mathcal{S}}$.

**Example 4** Let $N = \{1,2,3,4,5\}$ be the set of agents, $\mathcal{S} = \{\{1,2,3\},\{4,5\}\}$ be the partition of $N$ and $\mathcal{C}^x_m = \{\{1,2\},\{1,3\},\{2,4,5\}\}$ be the committee. We first argue that the committee $\mathcal{C}^x_m$ does not satisfy the IUP with respect to $\mathcal{S}$.

Assume first that $S_1 = \{1,2,3\}$ is the subset for which the IUP is satisfied at $k = 1$. Then, $\mathcal{ND}^1 = \{1,2,3\}$ and $\mathcal{X}^1 = \{\{2\}\}$. For $X_1 = \{2\}$, $3 \in \mathcal{ND}^1 \setminus X_1$ but $\{2,3\} \notin \mathcal{C}^x_m$. So, $S_1$ is not equal to $\{1,2,3\}$. Assume now that $S_1 = \{4,5\}$ is the subset for which the IUP is satisfied at $k = 1$. Then, $\mathcal{ND}^1 = \{4,5\}$ and $\mathcal{X}^1 = \{\{4,5\},\emptyset\}$. For $X_1 = \emptyset$, $4 \in \mathcal{ND}^1 \setminus X_1$ but $X_1 \cup \{4\} = \{4\} \notin \mathcal{C}^x$. So, (3) is not satisfied and accordingly, $S_1$ is not equal to $\{4,5\}$. Hence, $\mathcal{C}^x$ does not satisfy IUP with respect to $\mathcal{S}$.

Remember that in Example 1 we already showed that $f_{\mathcal{C}^x}$ is OSP-implementable with respect to $\mathcal{S}^* = \{\{1,2\},\{3\},\{4,5\}\}$ and that the extensive game form $\Gamma$ depicted in Figure 1 OSP-implements $f_{\mathcal{C}^x}$ with respect to $\mathcal{S}^*$. Since $\mathcal{S}^*$ is a finer partition of $\mathcal{S} = \{\{1,2,3\},\{4,5\}\}$, by Proposition 1, $f_{\mathcal{C}^x}$ is also OSP-implementable with respect to $\mathcal{S}$. Nonetheless, the extensive game form $\Gamma$ that OSP-implements $f_{\mathcal{C}^x}$ belongs to $\mathcal{G}^{\mathcal{S}^*}$ but not to $\mathcal{G}^{\mathcal{S}}$. □

**Proof of Theorem 2.** Let $f_{\mathcal{C}^x}$ be an EMVR associated to the committee $\mathcal{C}^x$ and let $\mathcal{S}$ be a partition of $N$.

($\Rightarrow$) Assume $\mathcal{C}^x$ satisfies the IUP with respect to $\mathcal{S} = \{S_1,\ldots,S_K\}$.

Define recursively the extensive game form $\Gamma \in \mathcal{G}^{\mathcal{S}}$ through the following steps.

**Step 1.** Let $S_1 \in \mathcal{S}$ be the subset whose existence is guaranteed by the IUP with respect to $\mathcal{S}$. Agents in $S_1$ play only once and simultaneously, each $i \in S_1$ at its unique information set of this Step 1, denoted as $I^1_i$, by choosing from the following set of choices:

$$Ch(I^1_i) = \begin{cases} \{\{P^x_i\},\{P^y_i\}\} & \text{if } i \in S_1 \cap \mathcal{ND}^1 \\ \{\{P^x_i, P^y_i\}\} & \text{if } i \in S_1 \setminus \mathcal{ND}^1. \end{cases}$$

Namely, the non-dummy agents of $S_1$ by choosing one of the two preferences and the dummy agents of $S_1$ by selecting necessarily the full set of preferences $\mathcal{P}$. Let $h^1$ denote a generic history of Step 1 and refer to $a^1_i \in Ch(I^1_i)$, as the choice made by agent $i \in S_1$ along $h^1$.

For each history $h^1 = (a_i^1)_{i \in S_1}$ of Step 1, and abusing notation by writing it as a vector of choices instead of a sequence, define the set

$$\widehat{X}_1 = \{i \in S_1 \cap \mathcal{ND}^1 \mid a_i^1 = \{P_i^x\}\}.$$

We refer to $\widehat{X}_1$ as the outcome of Step 1, and distinguish among three cases.

**(1.1)** If $\widehat{X}_1 \in \mathcal{C}^{x,1}$, $h^1$ is a terminal history and the outcome of the game $\Gamma$ is $x$.

**(1.2)** If $S_1 \setminus \widehat{X}_1 \in \mathcal{C}^{y,1}$, $h^1$ is a terminal history and the outcome of the game $\Gamma$ is $y$.

**(1.3)** If neither $\widehat{X}_1 \in \mathcal{C}^{x,1}$ nor $S_1 \setminus \widehat{X}_1 \in \mathcal{C}^{y,1}$ hold, go to Step 2 with $\widehat{X}_1$.

To proceed with the definition of $\Gamma$, assume **(1.3)** holds. Before moving to Step 2 we show that $\widehat{X}_1$ belongs to the family $\mathcal{X}^1 = \{X = S_1 \cap M^1 \mid M^1 \in \mathcal{C}_m^{x,1} \text{ and } X \notin \mathcal{C}_m^{x,1}\}$, defined just before the statement of the IUP.

**Claim A.1** *Let $\widehat{X}_1$ be the outcome of Step 1 and assume that neither $\widehat{X}_1 \in \mathcal{C}^{x,1}$ nor $S_1 \setminus \widehat{X}_1 \in \mathcal{C}^{y,1}$ hold. Then, $\widehat{X}_1 \in \mathcal{X}^1$.*

**Proof of Claim A.1.** Since $S_1 \setminus \widehat{X}_1 \notin \mathcal{C}^{y,1}$ holds, by (2), there exists $M \in \mathcal{C}^{x,1}$ such that $M \cap (S_1 \setminus \widehat{X}_1) = \emptyset$. Hence, $S_1 \cap M \subseteq \widehat{X}_1$. We show that $\widehat{X}_1 = S_1 \cap M$. To obtain a contradiction, suppose there exists $i \in \widehat{X}_1 \setminus (S_1 \cap M)$. By the definition of $\widehat{X}_1$, $i \in \mathcal{ND}^1$. Hence, $i \in \mathcal{ND}^1 \setminus (S_1 \cap M)$. By monotonicity of $\mathcal{C}^{x,1}$, $S_1 \cap M \subseteq \widehat{X}_1$ and $\widehat{X}_1 \notin \mathcal{C}^{x,1}$ imply

$$(S_1 \cap M) \notin \mathcal{C}^{x,1}. \tag{6}$$

By definition of $\mathcal{X}^1$, $S_1 \cap M \in \mathcal{X}^1$. By the IUP with respect to $\mathcal{S}$, $(S_1 \cap M) \cup \{i\} \in \mathcal{C}^{x,1}$. Hence, by the monotonicity of $\mathcal{C}^{x,1}$, $\widehat{X}_1 \in \mathcal{C}^{x,1}$ which is a contradiction with one of the assumptions of Claim A.1. Therefore, $\widehat{X}_1 = S_1 \cap M$ and, by (6), $\widehat{X}_1 \in \mathcal{X}^1$. □

**Step k ≥ 2**. Given $\widehat{X}_{k-1}$, outcome of Step $k-1$ that follows $\widehat{X}_1, \ldots, \widehat{X}_{k-2}$, if any. Let $S_k \in \mathcal{S}$ be the subset whose existence is guaranteed by the IUP with respect to $\mathcal{S}$. Agents in $S_k$ play only once and simultaneously, each $i \in S_k$ at its unique information set of this Step k, denoted as $I_i^k$, by choosing from the following set of choices:

$$Ch(I_i^k) = \begin{cases} \{\{P_i^x\}, \{P_i^y\}\} & \text{if } i \in S_k \cap \mathcal{ND}^k \\ \{\{P_i^x, P_i^y\}\} & \text{if } i \in S_k \setminus \mathcal{ND}^k. \end{cases}$$

Namely, the non-dummy agents of $S_k$ by choosing one of the two preferences and the dummy agents of the set $S_k$ by selecting necessarily the full set of preferences $\mathcal{P}$. Let $a_i^k \in Ch(I_i^k)$ be the choice made by agent $i \in S_k$ along the history that follows $h^{k-1}$ and let $h^k =$

$(h^{k-1}, (a_i^k)_{i\in S_k})$ be the complete history of choices made in Steps 1 to $k$. For each history $h^k = (h^{k-1}, (a_i^k)_{i\in S_k})$ of Steps 1 to $k$, define the set

$$\widehat{X}_k = \{i \in S_k \cap \mathcal{ND}^k \mid a_i^k = \{P_i^x\}\}.$$

We refer to $\widehat{X}_k$ as the outcome of Step $k$ that follows $\widehat{X}_1, \dots, \widehat{X}_{k-1}$, and distinguish among three cases.

**(k.1)** If $\widehat{X}_1 \cup \cdots \cup \widehat{X}_k \in \mathcal{C}^x$, $h^k$ is a terminal history and the outcome of the game $\Gamma$ is $x$.

**(k.2)** If $(S_1 \setminus \widehat{X}_1) \cup \cdots \cup (S_k \setminus \widehat{X}_k) \in \mathcal{C}^y$, $h^k$ is a terminal history and the outcome of the game $\Gamma$ is $y$.

**(k.3)** If neither $\widehat{X}_1, \cup \cdots \cup \widehat{X}_k \in \mathcal{C}^x$ nor $(S_1 \setminus \widehat{X}_1) \cup \cdots \cup (S_k \setminus \widehat{X}_k) \in \mathcal{C}^y$ hold, go to Step $k+1$ with $\widehat{X}_1 \cup \cdots \cup \widehat{X}_k$.

To proceed with the definition of $\Gamma$, assume **(k.3)** holds. Before moving to Step $k+1$ we show that $\widehat{X}_k$ belongs to the family $\mathcal{X}^k = \{X = S_k \cap M^k \mid M^k \in \mathcal{C}_m^{x,k} \text{ and } X \notin \mathcal{C}^{x,k}\}$, defined just before the statement of the IUP.

**Claim A.k** *Let $\widehat{X}_k$ be the outcome of Step $k-1$ that follows $\widehat{X}_1, \dots, \widehat{X}_{k-1}$, and assume that neither $\widehat{X}_1 \cup \cdots \cup \widehat{X}_k \in \mathcal{C}^x$ nor $(S_1 \setminus \widehat{X}_1) \cup \cdots \cup (S_k \setminus \widehat{X}_k) \in \mathcal{C}^y$ hold and $\widehat{X}_t \in \mathcal{X}^t$ for each $t = 1, \dots, k-1$. Then, $\widehat{X}_k \in \mathcal{X}^k$.*

**Proof of Claim A.k**. By hypothesis, we have that

$$\widehat{X}_1 \cup \cdots \cup \widehat{X}_k \notin \mathcal{C}^x \tag{7}$$

and

$$(S_1 \setminus \widehat{X}_1) \cup \cdots \cup (S_k \setminus \widehat{X}_k) \notin \mathcal{C}^y \tag{8}$$

hold and $\widehat{X}_t \in \mathcal{X}^t$ for each $t = 1, \dots, k-1$. By (8) and (2), there exists $M \in \mathcal{C}_m^x$ such that $M \cap [(S_1 \setminus \widehat{X}_1) \cup \cdots \cup (S_k \setminus \widehat{X}_k)] = \emptyset$. Hence, $S_t \cap M \subseteq \widehat{X}_t$ for all $t = 1, \dots, k$. We show that $\widehat{X}_t = S_t \cap M$ for all $t = 1, \dots, k$. To obtain a contradiction, suppose there is $t \in \{1, \dots, k\}$ such that $S_t \cap M \subsetneq \widehat{X}_t$. Let $r$ be the smallest of these indexes. Then, $\widehat{X}_t = S_t \cap M$ for all $1 \leq t < r$, if any, and there is $i \in \widehat{X}_r \setminus (S_r \cap M)$. Let $M^r = (M \setminus \cup_{t=1}^r S_t) \in \mathcal{C}_m^{x,r}$. By definition of $\widehat{X}_r$, $i \in \mathcal{ND}^r$. Hence, $i \in \mathcal{ND}^r \setminus (S_r \cap M^r)$. By the IUP with respect to $\mathcal{S}$,

$$(S_r \cap M^r) \cup \{i\} \in \mathcal{C}^{x,r}. \tag{9}$$

By the definition of $\mathcal{C}^{x,r}$, $\widehat{X}_1 \cup \cdots \cup \widehat{X}_{r-1} \cup ((S_r \cap M^r) \cup \{i\}) \in \mathcal{C}^x$. Then, since $(S_r \cap (M^r) \cup \{i\}) \subseteq \widehat{X}_r$, monotonicity of $\mathcal{C}^x$ implies that $\widehat{X}_1 \cup \cdots \cup \widehat{X}_{r-1} \cup \widehat{X}_r \in \mathcal{C}^x$, which is a contradiction with (7). Then, $\widehat{X}_t = S_t \cap M$ for all $t = 1, \dots, k$. Then $M^k = (M \setminus \cup_{t=1}^{k-1} S_t) \in \mathcal{C}^{x,k}$ and $\widehat{X}_k = S_k \cap M^k$. Furthermore, by (7), $\widehat{X}_k \notin \mathcal{C}^{x,k}$. Then, $\widehat{X}_k \in \mathcal{X}^k$. $\square$

Observe that if $k = K$, where $K$ is the number of subsets in the partition $\mathcal{S}$, and Step $K$ is reached, then either $\widehat{X}_1 \cup \dots \cup \widehat{X}_k \in \mathcal{C}^x$ or $(S_1 \setminus \widehat{X}_1) \cup \dots \cup (S_k \setminus \widehat{X}_k) \in \mathcal{C}^y$ holds. This is because all agents have already played in $\Gamma$ and either those agents $i \in N$ choosing $P_i^x$ is a winning coalition in $\mathcal{C}^x$, in which case the outcome of $\Gamma$ is $x$, or else those agents $i \in N$ choosing $P_i^y$ is a winning coalition in $\mathcal{C}^y$, in which case the outcome of $\Gamma$ is $y$. Therefore, the outcome of $\Gamma$ is either $x$ or $y$ if $h^K$ is the terminal history identified in **K.1** or in **K.2**, respectively. Thus, this construction has at most $K$ steps and the game $\Gamma \in \mathcal{G}^{\mathcal{S}}$ is well-defined.

We now proceed with the part ($\Rightarrow$) of the proof of Theorem 2.

Let $\Gamma \in \mathcal{G}^{\mathcal{S}}$ be the game defined from $\mathcal{C}^x$ according to the at most $K$ previous steps.[7] The type-strategy $(\sigma_i^{P_i})_{P_i \in \mathcal{P}}$ is *truth-telling* if, for every $z_i \in Z_i$ such that $|Ch(z_i)| = 2$, $\sigma_i^{P_i}(z_i) = \{P_i^x\}$ if $P_i = P_i^x$ and $\sigma_i^{P_i}(z_i) = \{P_i^y\}$ if $P_i = P_i^y$.

We first observe that $\Gamma$ OSP-implements $f_{\mathcal{C}^x}$ with respect to $\mathcal{S}$. This is because if agents select their choices according to their truth-telling type-strategies, $x$ is the outcome of $\Gamma$ if a winning coalition in $\mathcal{C}^x$ has chosen $x$ along the play of $\Gamma$ and $y$ is the outcome of $\Gamma$ if a winning coalition in $\mathcal{C}^y$ has chosen $y$ along the play of $\Gamma$.

We now prove that, for each $i \in N$ and $P_i \in \mathcal{P}$, the truth-telling strategy $\sigma_i^{P_i}$ is obviously dominant with respect to $\mathcal{S}$ in $\Gamma$ for $i$ and $P_i$.

Assume agent $j$ has to choose, at information set $I_j^k$ of Step $k$ after history $h^{k-1}$, one from the set $Ch(I_j^k) = \{\{P_j^x \hat{A}\}, \{P_j^y\}\}$. By definition of $\Gamma$, $j \in \mathcal{ND}^k$ and $h^{k-1}$ can be identified with (i) $X_1, \dots, X_{k-1}$, the set of agents $i \in N$ that have chosen $P_i^x$ in Steps 1 to $k-1$, which by Claims 1 to $k-1$, $X_t \in \mathcal{X}^t$ for all $1 \leq t \leq k-1$, and (ii) a set of agents $S_k \in \mathcal{S}$, those who also play together with $j$ in Step $k$. We distinguish between two general cases which, in turn, each is divided into three subcases.

*Case A.* Assume $P_j = P_j^x$. The choice consistent with $j$'s truth-telling strategy is $\widehat{a}_j = \{P_j^x\}$. Let $\sigma_i$ be a fixed strategy for each $i \in S_k \setminus \{j\}$. Denote, for each $i \in S_k \setminus \{j\}$, $\sigma_i(I_i^k) = \widehat{a}_i$, where $I_i^k$ is agent $i$'s information set that goes across the history that starts at $h^{k-1}$ and it is played by agents in $S_k$ along Step $k$. Let $\widehat{h}^k = (h^{k-1}, (\widehat{a}_i)_{i \in S_k})$ and $\widehat{X}_k = \{i \in \mathcal{ND}^k \mid \widehat{a}_i = \{P_i^x\}\}$. We distinguish among three subcases.

[7] Observe that the number of steps of $\Gamma$ may be strictly smaller than $K$. For instance, whenever $S_K \cap \mathcal{ND} = \emptyset$.

*Case A.1.* Suppose $X_1 \cup \cdots \cup X_{k-1} \cup \widehat{X}_k \in \mathcal{C}^x$ holds. Then, $\widehat{h}^k$ is a terminal history and the outcome of the game is $x$. Therefore, as $P_j = P_j^x$, the truth-telling strategy $\sigma_j^{P_j}$ is an obvious dominant strategy with respect to $\mathcal{S}$ in $\Gamma$ for $j$ and $P_j^x$.

*Case A.2.* Suppose $(S_1 \setminus X_1) \cup \cdots \cup (S_{k-1} \setminus X_{k-1}) \cup (S_k \setminus \widehat{X}_k) \in \mathcal{C}^y$ holds. Then, $\widehat{h}^k$ is a terminal history and the outcome of the game is $y$. Suppose agent $j$ deviates and plays $\overline{a}_j = \{P_j^y\}$. Let $\overline{a} = (\overline{a}_j, (\widehat{a}_i)_{i \in S_k \setminus \{j\}})$, $\overline{h}^k = (h^{k-1}, (\overline{a}_i)_{i \in S_k})$, and $\overline{X}_k = \{i \in \mathcal{ND}^k \mid \overline{a}_i = \{P_i^x\}\}$. We have that $\widehat{X}_k = \overline{X}_k \cup \{j\}$. Then, by monotonicity of $\mathcal{C}^y$, $(S_1 \setminus X_1) \cup \cdots \cup (S_{k-1} \setminus X_{k-1}) \cup (S_k \setminus \overline{X}_k) \in \mathcal{C}^y$. Therefore, $\overline{h}^k$ is a terminal history and the outcome of the game is $y$. Thus, as $P_j = P_j^x$, agent $j$'s deviation is not profitable. Hence, the truth-telling strategy $\sigma_j^{P_j}$ is obviously dominant with respect to $\mathcal{S}$ in $\Gamma$ for $j$ and $P_j^x$.

*Case A.3.* Suppose neither Subcase A.1 nor Subcase A.2 hold. Then, by Claim A.$k$ above, $\widehat{X}_k \in \mathcal{X}^k$. Suppose agent $j$ deviates and plays $\overline{a}_j = P_j^y$. Let $\overline{a} = (\overline{a}_j, (\widehat{a}_i)_{i \in S_k \setminus \{j\}})$, $\overline{h}^k = (h^{k-1}, (\overline{a}_i)_{i \in S_k})$, and $\overline{X}_k = \{i \in \mathcal{ND}^k \mid \overline{a}_i = \{P_i^x\}\}$. We have that $\widehat{X}_k = \overline{X}_k \cup \{j\}$. Then, by (4), $(S_k \setminus \widehat{X}_k) \cup \{j\} \in \mathcal{C}_m^{y,k}$. Then, by the monotonicity of $\mathcal{C}^y$, $(S_1 \setminus X_1) \cup \cdots \cup (S_{k-1} \setminus X_{k-1}) \cup (S_k \setminus \widehat{X}_k) = (S_1 \setminus X_1) \cup \cdots \cup (S_{k-1} \setminus X_{k-1}) \cup (S_k \setminus \overline{X}_k) \cup \{j\} \in \mathcal{C}_m^y$. Therefore, $\overline{h}^k$ is a terminal history and the outcome of the game is $y$. Thus, as $P_j = P_j^x$, agent $j$'s deviation is not profitable. Hence, the truth-telling strategy $\sigma_j^{P_j}$ is obviously dominant with respect to $\mathcal{S}$ in $\Gamma$ for $j$ and $P_j^x$.

*Case B.* Assume $P_j = P_j^y$. The choice consistent with $j$'s truth-telling strategy is $\widehat{a}_j = \{P_j^y\}$. Let $\sigma_i$ be a fixed strategy for each $i \in S_k \setminus \{j\}$. Denote, for each $i \in S_k \setminus \{j\}$, $\sigma_i(I_i^k) = \widehat{a}_i$, where $I_i^k$ is agent $i$'s information set that goes across the history that starts at $h^{k-1}$ and it is played by agents in $S_k$ along Step $k$. Let $\widehat{h}^k = (h^{k-1}, (\widehat{a}_i)_{i \in S_k})$ and $\widehat{X}_k = \{i \in \mathcal{ND}^k \mid \widehat{a}_i = \{P_i^x\}\}$. We distinguish among three subcases.

*Case B.1.* Suppose $X_1 \cup \cdots \cup X_{k-1} \cup \widehat{X}_k \in \mathcal{C}^x$ holds. Then, $\widehat{h}^k$ is a terminal history and the outcome of the game is $x$. Suppose agent $j$ deviates and plays $\overline{a}_j = \{P_j^x\}$. Let $\overline{a} = (\overline{a}_j, (\widehat{a}_i)_{i \in S_k \setminus \{j\}})$, $\widehat{h}^k = (h^{k-1}, (\overline{a}_i)_{i \in S_k})$, and $\overline{X}_k = \{i \in \mathcal{ND}^k \mid \overline{a}_i = \{P_j^x\}\}$. We have that $\widehat{X}_k = \overline{X}_k \setminus \{j\}$. Then, by monotonicity of $\mathcal{C}^x$, $X_1 \cup \cdots \cup X_{k-1} \cup \overline{X}_k \in \mathcal{C}^x$. Therefore, $\overline{h}^k$ is a terminal history and the outcome of the game is $x$. Thus, as $P_j = P_j^y$, agent $j$'s deviation is not profitable. Hence, the truth-telling strategy $\sigma_j^{P_j}$ is obviously dominant with respect to $\mathcal{S}$ in $\Gamma$ for $j$ and $P_j^y$.

*Case B.2.* Suppose $(S_1 \setminus X_1) \cup \cdots \cup (S_{k-1} \setminus X_{k-1}) \cup (S_k \setminus \widehat{X}_k) \in \mathcal{C}^y$ holds. Then, $\widehat{h}^k$ is a terminal history and the outcome of the game is $y$. Therefore, as $P_j = P_j^y$, the truth-telling strategy $\sigma_j^{P_j}$ is obviously dominant with respect to $\mathcal{S}$ in $\Gamma$ for $j$ and $P_j^y$.

*Case B.3* Suppose neither Subcase B.1 nor Subcase B.2 hold. Then, by Claim $k$ above, $\widehat{X}_k \in \mathcal{X}^k$. Suppose agent $j$ deviates and plays $\overline{a}_j = \{P_j^x\}$. Let $\overline{a} = (\overline{a}_j, (\overline{a}_i)_{i \in S_k \setminus \{j\}})$, $\overline{h}^k = (h^{k-1}, (\overline{a}_i)_{i \in S_k})$ and $\overline{X}_k = \{i \in \mathcal{ND}^k \mid \overline{a}_i = \{P_i^x\}\}$. We have that $\widehat{X}_k = \overline{X}_k \cup \{j\}$. Then, by the IUP with respect to $\mathcal{S}$, $\widehat{X} \cup \{j\} \in \mathcal{C}_m^{x,k}$ and, by the monotonicity of $\mathcal{C}^x$, $\widehat{X}_1 \cup \cdots \cup \widehat{X}_{k-1} \cup \overline{X}_k = \widehat{X}_1 \cup \cdots \cup \cup \widehat{X}_{k-1} \cup \widehat{X}_k \cup \{j\} \in \mathcal{C}_m^x$. Therefore, $\widehat{h}^k$ is a terminal history and the outcome of the game is $x$. Thus, as $P_j = P_j^y$, agent $j$'s deviation is not profitable. Hence, the truth-telling strategy $\sigma_j^{P_j}$ is obviously dominant with respect to $\mathcal{S}$ in $\Gamma$ for $j$ and $P_j^y$.

Thus, the game $\Gamma \in \mathcal{G}^{\mathcal{S}}$ OSP-implements $f_{\mathcal{C}^x} : \mathcal{P}^N \to \{x, y\}$ with respect to $\mathcal{S}$. This finishes the part ($\Rightarrow$) of the proof of Theorem 2.

($\Leftarrow$) Let $\Gamma \in \mathcal{G}^{\mathcal{S}}$ be a game such that $(\Gamma, (\sigma_i^{P_i})_{P_i \in \mathcal{P}, i \in N}))$ OSP-implements $f_{\mathcal{C}^x}$ with respect to $\mathcal{S}$. By definition of $\mathcal{G}^{\mathcal{S}}$ and the fact that in this context each agent has only two admissible preferences we can assume that each agent plays at most once along any history.

We shall prove that the committee $\mathcal{C}^x$ satisfies the IUP with respect to $\mathcal{S}$.

Let $k = 1$ and $S_1 \in \mathcal{S}$ be the first subset of agents that play in $\Gamma$ at Step 1. Define $\mathcal{ND}^1 = S_1 \cap \mathcal{ND}$.

Let $k \geq 2$. Given a non-terminal history $h^{k-1}$, outcome of Step $k-1$ (in what follows we give more details of $h^{k-1}$), there exists an element of the partition $S_k \in \mathcal{S}$, whose agents play in $\Gamma$ at Step $k \geq 2$ after history $h^{k-1}$.

The proof is by induction on $k$, the number of steps of the game $\Gamma$, of the following statement.

**Claim B.k** Let $P \in \mathcal{P}^N$ be an arbitrary profile and let $1 \leq k \leq K-1$ be a fixed step of $\Gamma \in \mathcal{G}^{\mathcal{S}}$ such that, for each $1 \leq t \leq k$, each $X_t \in \mathcal{X}^t$ and each history $h^t = (h^{t-1}, (a_i)_{i \in S_t})$ of $\Gamma$ have the property that $a_i = \{P_i^x\}$ if and only if $i \in X_t$. Then,
(i) the history $h^k$ of $\Gamma$ is non-terminal and
(ii) for each $j \in \mathcal{ND}^k \setminus X^k$, $X^k \cup \{j\} \in \mathcal{C}_m^{x,k}$.

**Proof of Claim B.k.** Let $P \in \mathcal{P}^N$ be a profile.

Suppose $k = 1$. Let $S_1$ be the set of agents that play in $\Gamma$ at Step 1. If $\mathcal{X}^1 = \emptyset$ the proof is trivial. Suppose otherwise, and fix $X \in \mathcal{X}^1$.

Then, there exists $M \in \mathcal{C}_m^x$ such that $X = M \cap S_1$. Hence, $X \subseteq \mathcal{ND}^1$. Let $h^1 = (a_i)_{i \in S_1}$ be the history of Step 1, where $a_i = \{P_i^x\}$ if and only if $i \in X$. As $X \in \mathcal{X}^1$, $X \notin \mathcal{C}_m^x$. Furthermore, $(S_1 \setminus X) \cap M = \emptyset$. By (2), $S_1 \setminus X \notin \mathcal{C}_m^y$. Therefore, as $\Gamma$ OSP-implements $f_{\mathcal{C}^x}$ with respect to $\mathcal{S}$, $h^1$ is non-terminal and both outcomes $x$ and $y$ can follow after $h^1$.

We show that $S_1$ is the set required by the IUP for $k=1$. For each $i \in S_1$, we denote by $I_i$ the information set that agent $i$ has in Step 1.[8] To obtain a contradiction, suppose there exist $X \in \mathcal{X}^1$ and $j \in \mathcal{ND}^1 \setminus X$ such that $X \cup \{j\} \notin \mathcal{C}^x$. Since $j \in \mathcal{ND}^1$, the set of available choices of agent $j$ at $j$'s information set $I_j$ at Step 1 is equal to $\{\{P_i^x\}, \{P_i^y\}\}$. Since $j \notin X$, the choice consistent with the truth-telling strategy of agent $j$ is $\{P_j^y\}$; namely, $\sigma_j^{P_j}(I_j) = \{P_j^y\}$. Let $\sigma_i$ be a fixed strategy for $i \in S_1 \setminus \{j\}$, where $\sigma_i(I_i) = \{P_i^x\}$ for all $i \in X$ and $\sigma_i(I_i) \neq \{P_i^x\}$ for all $i \in S_1 \setminus (X \cup \{j\})$. Observe that $\{i \in S_1 \mid \sigma_i(I_i) = \{P_i^x\}\} = X$. By (1.3) in Step 1 of the definition of $\Gamma$, Step 2 follows; accordingly, $\overline{h}^1 = (\sigma_i(I_i))_{i \in S_1}$ is a non-terminal history (*i.e.*, condition (i) in Claim B.1 holds) and the result $x$ can follow after $\overline{h}^1$; *i.e.*, after agent $j$ truth-tells and $(\sigma_i)_{i \in S_1 \setminus \{j\}}$ is played. Suppose that agent $j$ deviates and plays $\sigma_j'(I_j) = \{P_j^x\}$. Let $\widehat{h} = (\sigma_j'(I_j), (\sigma_i(I_i))_{i \in S_1 \setminus \{j\}})$. As $X \cup \{j\} \notin \mathcal{C}^x$ and $\Gamma$ implements $f_{\mathcal{C}^x}$, the outcome $y$ is feasible under the deviation. Therefore, truth-telling is not an obviously dominant strategy with respect to $\mathcal{S}$. Thus, $S_1$ is the subset whose existence is required by the IUP for $k=1$.

Now suppose that $k > 1$ and that the statement of Claim B.t holds for $t = 1, \ldots, k-1$. We prove that it holds for $k$ as well. Let $X_1, \ldots, X_{k-1}$ be such that $X_t \in \mathcal{X}^t$ for each $t = 1, \ldots, k-1$, let $h^{k-1}$ be the corresponding non-terminal history at Step $k-1$, and let $S_k$ be the set of agents that play in $\Gamma$ at Step $k$, after the history $h^{k-1}$. Consider the families of subsets $\mathcal{C}_m^{x,k}$ and $\mathcal{X}^k$, identified in the recursive definition of the IUP. If $\mathcal{X}^k = \emptyset$ the proof is trivial. Suppose otherwise, and fix $X \in \mathcal{X}^k$.

Then, there exists $M^k \in \mathcal{C}^{x,k}$ such that $X = M^k \cap S_k$. Hence, $X \subseteq \mathcal{ND}^k$. Let $h^k = (h^{k-1}, (a_i)_{i \in S_k})$ be the history after Step k, where agents in $(X_1 \cup \cdots \cup X_{k-1} \cup X)$ and only them choose $P^x$ along $h^k$. As $X \in \mathcal{X}^k$, $X \notin \mathcal{C}_m^{x,k}$. Therefore, $X_1 \cup \cdots \cup X_{k-1} \cup X \notin \mathcal{C}_m^x$. Denote $M = X_1 \cup \cdots \cup X_{k-1} \cup M^k$. By definition of $M$ and $\mathcal{C}_m^{x,k}$, $((S_1 \setminus X_1) \cup \cdots \cup (S_{k-1} \setminus X_{k-1}) \cup (S_k \setminus X)) \cap M = \emptyset$ and $M \in \mathcal{C}^x$. By (2), $(S_1 \setminus X_1) \cup \cdots \cup (S_{k-1} \setminus X_{k-1}) \cup (S_k \setminus X) \notin \mathcal{C}^y$. Since $\Gamma$ OSP-implements $f_{\mathcal{C}^x}$ with respect to $\mathcal{S}$, $h^k$ is non-terminal and both results $x$ and $y$ can follow from $h^k$.

We show that $S_k$ is the set required by the IUP for $k$. For each $i \in S_k$, we denote by $I_i$ the information set that agent $i$ has in Step k after $h^{k-1}$.[9] To obtain a contradiction, suppose there exist $X \in \mathcal{X}^k$ and $j \in \mathcal{ND}^k \setminus X$ such that $X \cup \{j\} \notin \mathcal{C}_m^{x,k}$. Since $j \in \mathcal{ND}^k$, the set of available choices of agent $j$ at $j$'s information set $I_j$ in Step k is equal to $\{\{P_i^x\}, \{P_i^y\}\}$.

[8] By definition of $\Gamma$, agents in $S_1$ only have an information set at Step 1.

[9] By definition of $\Gamma$, agents in $S_k$ only have an information set in Step k.

Since $j \notin X$, the choice consistent with the truth-telling strategy of agent $j$ is $P_i^y$; namely, $\sigma_j(I_j) = \{P_i^y\}$. Let $\sigma_i$ be a fixed strategy for $i \in S_k \setminus \{j\}$, where $\sigma_i(I_i) = \{P_i^x\}$ for all $i \in X$ and $\sigma_i(I_i) \neq \{P_i^x\}$ for all $i \in S_k \setminus (X \cup \{j\})$. Observe that $\{i \in S_k \mid \sigma_i(I_i) = \{P_i^x\}\} = X$. By (k.3) in Step k of the definition of $\Gamma$, Step k+1 follows; accordingly, $\overline{h}^k = (h^{k-1}, (\sigma_i(I_i))_{i \in S_k}$ is a non-terminal history (*i.e.*, condition (i) in Claim B.k holds) and $x$ can follow after $\overline{h}^k$; *i.e.*, after agent $j$ truth-tells and $(\sigma_i)_{i \in S_k \setminus \{j\}}$ is played. Suppose that agent $j$ deviates and plays $\sigma'_j(I_j) = \{P_i^x\}$. Let $\widehat{h} = (h^{k-1}, (\sigma'_j(I_j), (\sigma_i(I_i))_{i \in S_k \setminus \{j\}})$. As $X \cup \{j\} \notin \mathcal{C}^{x,k}$, $(X_1 \cup \cdots \cup X_{k-1}, \cup X \cup \{j\}) \notin \mathcal{C}^x$. Since $\Gamma$ OSP-implements $f_{\mathcal{C}^x}$ with respect to $\mathcal{S}$, the outcome $y$ is feasible under the deviation. Therefore, truth-telling is not an obviously dominant strategy with respect to $\mathcal{S}$. Thus, $S_k$ is the subset whose existence is required by the IUP for $k$.

This finishes the proof of Theorem 2. ■

## 4.2 Anonymity

We characterize the committees that satisfy the IUP with respect to a partition and two alternative notions of anonymity: Theorem 3 for strong anonymity (the chosen alternative does not change after agents' names are permuted in any way), and Theorems 4 and 5 for anonymity relative to a partition (the chosen alternative does not change after agents' names are permuted only among the members belonging to the same subset of the partition). Of course, by Theorem 2, all these results identify anonymous subclasses of social choice functions in this setting (*i.e.*, EMVRs) that are obviously strategy-proof relative to a partition.

### 4.2.1 Strong anonymity

A committee $\mathcal{C}^x$ is *strongly anonymous* if for all bijections $\pi : N \to N$ and all $M \in \mathcal{C}^x$, $\pi(M) \in \mathcal{C}^x$. This is the straightforward definition of anonymous committee that does not take into account the partition.

**Remark 5** *Let $\mathcal{C}^x$ be an strongly anonymous committee. Then, there exists an integer $q \in \{1, \ldots, n\}$, called the* quota*, such that, $M \in \mathcal{C}_m^x$ if and only if $|M| = q$.*

**Theorem 3** *Let $\mathcal{S}$ be partition and let $\mathcal{C}^x$ be a strongly anonymous committee with quota $q$. Then, $\mathcal{C}^x$ satisfies the IUP with respect to $\mathcal{S}$ if and only if one of the following statements hold:*

*(i)* $q = 1$.

*(ii)* $q = n$.

*(iii)* $\mathcal{S} = \{\widehat{S}, \overline{S}\}$, *where* $|\widehat{S}| = n - 1$.

**Proof.** Let $\mathcal{S}$ be a partition and let $\mathcal{C}^x$ be a strongly anonymous committee with quota $q$.

($\Leftarrow$) Assume $q = 1$. Then, it is easy to check that, for any $1 \leq k < K$ and independently of $\mathcal{S}$, $\mathcal{X}^k = \{\emptyset\}$ and $\mathcal{ND}^k = S_k$. Accordingly, for all $i \in \mathcal{ND}^k$, $\{\emptyset\} \cup \{i\} \in \mathcal{C}_m^{x,k}$ holds because $q = 1$. Hence, the IUP with respect to $\mathcal{S}$ is satisfied.

Assume $q = n$. Suppose first that $K = 1$. Then, the IUP does not impose any restriction, and so it holds trivially. Suppose now that $K > 1$. Then, for any $1 \leq k \leq K-1$, $\mathcal{X}^k = \{S_k\}$ and $\mathcal{ND}^k = S_k$. Accordingly, since $\mathcal{ND}^k \setminus S_k = \emptyset$, the IUP with respect to $\mathcal{S}$ is immediately satisfied.

Assume $q \notin \{1, n\}$ and $\mathcal{S} = \{\widehat{S}, \overline{S}\}$, where $|\widehat{S}| = n - 1$. To show that the IUP with respect to $\mathcal{S}$ holds, consider $S_1 = \widehat{S}$. If $\mathcal{X}^1 = \emptyset$ the proof is trivial. Let $X \in \mathcal{X}^1$. Then, there exists $M \in \mathcal{C}_m^x$ such that $X = M \cap \widehat{S}$ and $X \notin \mathcal{C}_m^x$. Since $1 < |M| = q < n$ and $|\widehat{S}| = n - 1$, $q - 1 \leq |X| \leq q$. But $X \notin \mathcal{C}_m^x$ implies $|X| = q - 1$. Let $i \in \mathcal{ND}^1 \setminus X$. Hence $|X \cup \{i\}| = q$, which implies that $X \cup \{i\} \in \mathcal{C}_m^x$, and the IUP with respect to $\mathcal{S}$ is satisfied.

($\Rightarrow$) To obtain a contradiction, suppose that $\mathcal{C}^x$ satisfies the IUP with respect to $\mathcal{S}$ and neither (i), (ii) nor (iii) hold. Let $S_1 \in \mathcal{S}$ be the subset of the partition identified at the first step of the IUP with respect to $\mathcal{S}$. We proceed by distinguishing between two cases.

Case 1: $|S_1| < q$. Then, there exists $M \in \mathcal{C}_m^x$ such that $S_1 \subsetneq M$. Since (ii) does not hold, there exists $j \notin M$. Consider $i \in S_1$ and define $\overline{M} = (M \setminus \{i\}) \cup \{j\}$. Hence, $\overline{M} \in \mathcal{C}_m^x$ because $|\overline{M}| = q$. Then, $\overline{M} \cap S_1 = S_1 \setminus \{i\}$, and since $|S_1 \setminus \{i\}| < q$, we have that $S_1 \setminus \{i\} \in \mathcal{X}^1$. This implies, by (3), that $S_1 \in \mathcal{C}_m^x$, a contradiction.

Case 2: $q \leq |S_1|$. Since neither (i), (ii) nor (iii) hold, $1 < |S_1| < n - 1$. Then, there exist $i, j \notin S_1$. Consider $M \in \mathcal{C}_m^x$ such that $M \subseteq S_1$. Since (i) does not hold, there exist $i', j' \in M$. Define $\overline{M} = (M \setminus \{i', j'\}) \cup \{i, j\}$ and let $X = \overline{M} \cap S_1 = M \setminus \{i', j'\}$. Then, $|X| = q-2$ which means that $X \in \mathcal{C}_m^x$ and so $X \in \mathcal{X}^1$. Since $i' \in \mathcal{ND}^1 \setminus X$ and $X \cup \{i'\} \notin \mathcal{C}_m^x$ hold, (3) is not satisfied, contradicting the hypothesis that $\mathcal{C}^x$ satisfies the IUP with respect to $\mathcal{S}$. ■

Theorems 2 and 3 together identify a family of social choice functions in this setting (*i.e.*, EMVRs) that are OSP with respect to a partition and strongly anonymous.[10]

[10] A social choice function $f : \mathcal{P}^N \to \{x, y\}$ is *strongly anonymous* if, for all bijections $\pi : N \to N$ and all $P = (P_1, \ldots, P_n) \in \mathcal{P}^N$, $f(P_1, \ldots, P_n) = f(P_{\pi(1)}, \ldots, P_{\pi(n)})$.

#### 4.2.2 Anonymity relative to a partition

Let $\mathcal{S}$ be a partition of $N$ and let $\Pi^{\mathcal{S}}$ be the set of all bijections $\pi^{\mathcal{S}} : N \to N$ that only swap agents within each element of $\mathcal{S}$; namely, $\pi^{\mathcal{S}} \in \Pi^{\mathcal{S}}$ if and only if, for each $S \in \mathcal{S}$, $\pi^{\mathcal{S}}(S) = S$.

A committee $\mathcal{C}^x$ is *anonymous relative to a partition* $\mathcal{S}$ if (i) it does not have dummy agents and (ii) for all $\pi^{\mathcal{S}} \in \Pi^{\mathcal{S}}$ and $M \in \mathcal{C}^x$, $\pi^{\mathcal{S}}(M) \in \mathcal{C}^x$.[11]

To characterize all committees that are anonymous relative to a partition and satisfy the IUP respect to the same partition, we need some additional notation.

Given an ordered partition, denoted by $\mathcal{S}^o = \{S_1, \dots, S_K\}$, and a vector of quotas $Q = (q_1, \dots, q_K) \in \mathbb{Z}^K_{++}$ such that, for all $1 \leq k \leq K$, $q_k \leq |S_k|$, define, for each $1 \leq k \leq K$, the committee (of minimal winning coalitions) $\mathcal{C}^x_{Q,k}$ as follows:

$$\mathcal{C}^x_{Q,k} = \begin{cases} \{\bigcup_{t=1}^k T_t \cup \{i_k\} \mid T_t \subset S_t,\ |T_t| = q_t \text{ and } i_k \in S_k \setminus T_k\} & \text{if } q_k < |S_k| \\ \emptyset & \text{if } q_k = |S_k|. \end{cases} \tag{10}$$

Moreover, set

$$\mathcal{C}^x_Q = \bigcup_{k=1}^{K} \mathcal{C}^x_{Q,k}.$$

Example 5 illustrates how to obtain the committee $\mathcal{C}^x_Q$ from a given ordered partition $\mathcal{S}^o$ and vector of quotas $Q$.

**Example 5** Let $N = \{1, \dots, 10\}$ be the set of agents, $\mathcal{S}^o = \{\{1,2,3\},\{4,5,6,7,8\},\{9,10\}\}$ be the ordered partition of $N$ and $Q = (q_1, q_2, q_3) = (2, 3, 2)$ be the vector of quotas. Then,

$\mathcal{C}^x_{Q,1} = \{1, 2, 3\}$,

$\mathcal{C}^x_{Q,2} = \{\{1,2,4,5,6,7\},\{1,3,4,5,6,7\},\{2,3,4,5,6,7\},$
$\{1,2,4,5,6,8\},\{1,3,4,5,6,8\},\{2,3,4,5,6,8\},$
$\{1,2,4,5,7,8\},\{1,3,4,5,7,8\},\{2,3,4,5,7,8\},$
$\{1,2,4,6,7,8\},\{1,3,4,6,7,8\},\{2,3,4,6,7,8\},$
$\{1,2,5,6,7,8\},\{1,3,5,6,7,8\},\{2,3,5,6,7,8\}\}$, and

$\mathcal{C}^x_{Q,3} = \emptyset$.

Hence,

$\mathcal{C}^x_Q = \mathcal{C}^x_{Q,1} \cup \mathcal{C}^x_{Q,2}$. □

[11] Observe that a committee satisfying only (ii) in this definition could have dummy agents (for example, all agents in $S$ for some $S \in \mathcal{S}$) and (i) excludes explicitly this possibility. Of course, to attribute to a committee with dummy and non-dummy agents any property of anonymity would sound weird.

**Theorem 4** *Let $\mathcal{C}^x$ be an anonymous committee relative to a partition $\mathcal{S}$. Then, $\mathcal{C}^x$ satisfies the IUP with respect to $\mathcal{S}$ if and only if there exist an order in $\mathcal{S}$, written as $\mathcal{S}^o = \{S_1, \ldots, S_K\}$, and a vector of quotas $Q = (q_1, \ldots, q_K)$ such that*

$$\mathcal{C}_m^x = \mathcal{C}_Q^x.$$

In the proof of Theorem 4 we will use Lemma 1.

**Lemma 1** *Let $\mathcal{C}^x$ be an anonymous committee relative to a partition $\mathcal{S}$ elements that satisfies the IUP with respect to $\mathcal{S}$. Then, for every $1 \leq k < K$, there exists an order of up to $k$ elements of $\mathcal{S}$, denoted as $\mathcal{S}^{k,o} = \{S_1, \ldots, S_k\}$, such that, for all $t \in \{1, \ldots, k\}$, $\mathcal{X}^t \neq \emptyset$ and $|X_t'| = |X_t|$ for all $X_t', X_t \in \mathcal{X}^t$.*[12]

**Proof of Lemma 1.** If $K = 1$ the statement follows trivially. Assume $K > 1$. The proof is by induction on $k$.

First, set $k = 1$. Let $S_1$ be the subset of $\mathcal{S}$ identified at the first step of the IUP with respect to $\mathcal{S}$. Furthermore, as there are no dummy agents and $k = 1 < K$, there is $M_1^* \in \mathcal{C}_m^x$ such that $M_1^* \cap S_1 \neq \emptyset$ and $M_1^* \cap S_1 \notin \mathcal{C}_m^x$. Hence, $X_1^* := M_1^* \cap S_1 \in \mathcal{X}^1$. Assume, to obtain a contradiction, that there exits $X_1' \in \mathcal{X}^1$ such that $|X_1'| \neq |X_1^*|$. Suppose first that $|X_1'| < |X_1^*|$. Let $M' \in \mathcal{C}_m^{x,1}$ be such that $X_1' = M' \cap S_1$ and consider a bijection $\pi^S \in \Pi^{\mathcal{S}}$ with the property that $\pi^{\mathcal{S}}(X_1') \subsetneq X_1^*$ and $\pi^{\mathcal{S}}(j) = j$ for every $j \in N \backslash S_1$. By anonymity relative to $\mathcal{S}$, $\pi^{\mathcal{S}}(M') \in \mathcal{C}_m^{x,1}$. Moreover, $X_1' \in \mathcal{X}^1$ implies $\mathcal{X}_1' \notin \mathcal{C}^x$. By anonymity relative to $\mathcal{S}$, $\pi^{\mathcal{S}}(M') \cap S_1 \notin \mathcal{C}^x$. Then, by definition of $\mathcal{X}^1$, $\pi^{\mathcal{S}}(M') \cap S_1 = \pi^{\mathcal{S}}(X_1') \in \mathcal{X}^1$. Let $i \in X_1^* \setminus \pi^{\mathcal{S}}(X_1')$. Then, since $i \in \mathcal{ND}^1$, (3) in the definition of the IUP implies that, $\pi^{\mathcal{S}}(X_1') \cup \{i\} \in \mathcal{C}_m^x$. Then, by the coalition monotonicity of the committee, $X_1^* \in \mathcal{C}^{x,1}$ which is a contradiction to the fact that $X_1^* \in \mathcal{X}^1$. Proceed similarly to obtain a contradiction for the case where $|X_1'| > |X_1^*|$ holds. Set $\mathcal{S}^{1,o} = \{S_1\}$. Hence, the necessary condition of Lemma 1 holds for $k = 1$.

Now, assume that the necessary condition of Lemma 1 holds for $1 \leq k < K$. Then, by hypothesis, there exists an order $\mathcal{S}^{k,o} = \{S_1, \ldots, S_k\}$ such that, for all $t \in \{1, \ldots, k\}$, $\mathcal{X}^t \neq \emptyset$ and $|X_t'| = |X_t|$ for all $X_t', X_t \in \mathcal{X}^t$. Hence, there exists $M^* \in \mathcal{C}_m^x$ such that $X_t^* := M^* \cap S_t \in \mathcal{X}^t$ for all $t = 1, \ldots, k$. We shall show that the necessary condition of Lemma 1 holds for $k+1$ as well, where $k + 1 < K$.

[12] Given $\mathcal{S}^{K-1,o} = \{S_1, \ldots, S_{K-1}\}$, define $\mathcal{S}^o = \{S_1, \ldots, S_{K-1}, S_K\}$.

Now, set $k+1 \leq K-1$. Let $S_{k+1}$ be the element of $\mathcal{S}$ identified at the step $k+1$ of the IUP. Then, exists $i \in N \setminus (S_1 \cup \cdots \cup S_{k+1})$. As there is no dummy agents, there is $M' \in \mathcal{C}^x_m$ such that $i \in M'$ and, for all $t \leq k+1$, $X'_t := M' \cap S_t \in \mathcal{X}^t$. By the Induction Hypothesis, $|X'_t| = |X^*_t|$ for all $t \leq k$. To obtain a contradiction, assume that there exists $X_{k+1} \in \mathcal{X}^{k+1}$ such that $|X_{k+1}| \neq |X'_{k+1}|$. Suppose first that $|X'_{k+1}| < |X_{k+1}|$. Then, by (3) in the definition of the IUP, for all $i \in S_{k+1} \setminus X'_{k+1}$,

$$M'_{k+1} := X^*_1 \cup \cdots \cup X^*_k \cup X'_{k+1} \cup \{i\} \in \mathcal{C}^x_m. \tag{11}$$

Since $X_{k+1} \in \mathcal{X}^{k+1}$, there exists $M^{k+1} \in \mathcal{C}^{x,k+1}_m$ such that $X_{k+1} = S_{k+1} \cap M^{k+1} \notin \mathcal{C}^{x,k+1}_m$. Hence, by the definition of $\mathcal{C}^{x,k+1}_m$, there exists $M \in \mathcal{C}^x_m$ such that

$$M \cap (\bigcup_{t=1}^{k+1} S_t) = X^*_1 \cup \cdots \cup X^*_k \cup X_{k+1} \notin \mathcal{C}^x_m. \tag{12}$$

Consider a bijection $\pi^{\mathcal{S}} \in \Pi^{\mathcal{S}}$ with the property that $\pi^{\mathcal{S}}(M'_{k+1}) \subsetneq M \cap (\bigcup_{t=1}^{k+1} S_t)$, and the identity otherwise. By anonymity, (11) implies $\pi^{\mathcal{S}}(M'_{k+1}) \in \mathcal{C}^x_m$ which, together with the monotonicity of the committee, it contradicts (12). Proceed similarly to obtain the contradiction for the case where $|X'_{k+1}| > |X_{k+1}|$ holds. □

**Proof of Theorem 4.** Let $\mathcal{C}^x$ be an anonymous committee relative to a partition $\mathcal{S}$.

Suppose $K = 1$.

($\Rightarrow$) Since $\mathcal{S} = \{N\}$, the committee is strongly anonymous. By Remark 5, let $q \in \{1, \ldots, n\}$, be the quota such that, $M \in \mathcal{C}^x_m$ if and only if $|M| = q$. Let $S_1 = N$ and $Q = (q_1)$ where $q_1 = q - 1 < n$. Then,

$$\mathcal{C}^x_Q = \{T \cup \{i\} \mid |T| = q_1 \text{ and } i \in N \setminus T\} = \mathcal{C}^x_m,$$

and so condition (10) holds.

($\Leftarrow$) Since the IUP is vacuous, it holds trivially.

Suppose $K > 1$.

($\Rightarrow$) Assume that $\mathcal{C}^x$ satisfies the IUP with respect to $\mathcal{S}$.

**Claim** There exist $M^* \in \mathcal{C}^x_m$ and an order $S_1, \ldots, S_K$ of $\mathcal{S}$ such that, for all $1 \leq t < K$,

(i) $X^*_t := M^* \cap S_t \in \mathcal{X}^t$ and

(ii) if $X_t \in \mathcal{X}^t$, then $|X_t| = |X^*_t|$.

**Proof of the Claim.** Let $S_1, \ldots, S_K$ be the ordered partition, where $S_1, \ldots, S_{K-1}$ is identified in Lemma 1 for the particular case where $k = K-1$. Since $S_K$ does not contain dummy agents, by Lemma 1 again, there exists $M^* \in \mathcal{C}_m^x$ with properties (i) and (ii) stated in the Claim. $\square$

Let $M^* \in \mathcal{C}_m^x$ be given by the Claim and consider, for each $1 \leq t < K$, $X_t^* = M^* \cap S_t \in \mathcal{X}^t$. Observe that $X_K^* := M^* \cap S_K \neq \emptyset$. Otherwise, $X_{K-1}^* = M^* \setminus \cup_{t=1}^{K-2} S_t \in \mathcal{C}_m^{x,K-1}$, which would imply that $X_{K-1}^* \notin \mathcal{X}^{K-1}$, a contradiction with part (i) in the Claim. Then, $M^*$ can be written as $M^* = \bigcup_{t=1}^{K} X_t^* \in \mathcal{C}_m^x$. Define $q_t = |M^* \cap S_t|$ for each $t = 1, \ldots, K-1$, $q_K = |M^* \cap S_K| - 1$ and $Q = (q_1, \ldots, q_K)$.

We finish this part of the proof of Theorem 4 by showing that $\mathcal{C}_m^x = \mathcal{C}_Q^x$ holds.

First, we show that $\mathcal{C}_m^x \subseteq \mathcal{C}_Q^x$. Let $M \in \mathcal{C}_m^x$ be arbitrary. Let $1 \leq k \leq K$ be such that $M \cap S_k \neq \emptyset$ and, for all $k < t \leq K$, $M \cap S_t = \emptyset$. Define, for every $1 \leq t \leq k$,

$$\overline{X}_t := M \cap S_t.$$

Assume that $t < k$. Then, $\overline{X}_1 \in \mathcal{X}^1$ , and by the Claim, $|\overline{X}_1| = |X_1^*|$. By anonymity, $M_1 := X_1^* \cup (M \setminus S_1) \in \mathcal{C}_m^x$.

Similarly, we get that $M_t = X_1^* \cup \cdots \cup X_t^* \cup [M \setminus (S_1 \cup \cdots \cup S_t)] \in \mathcal{C}_m^x$. Therefore, by the Claim, $\left|\overline{X}_t\right| = q_t$ for all $t < k$.

Now we prove that $|M \cap S_k| = q_k + 1$ holds for all $k \geq 1$. First, assume that $k = K$. Then, $M \cap S_K \neq \emptyset$. Therefore, by anonymity and Claim, $|M \cap S_K| = |M^* \cap S_K| = q_K + 1$. Second, assume that $k < K$. Let $i^k \in \mathcal{ND}^k \setminus X_k^*$. By the IUP,

$$M^{*k} = (\cup_{t=1}^{k} X_t^*) \cup \{i^k\} \in \mathcal{C}_m^x. \tag{13}$$

Furthermore, as $M_{k-1} = X_1^* \cup \cdots \cup X_{k-1}^* \cup (M \cap S_k) \in \mathcal{C}_m^x$ if $k > 1$ and $M_0 = M \cap S_1 = M$ if $k = 1$, anonymity implies $|M \cap S_k| = \left|X_k^* \cup \{i^k\})\right| = q_k + 1$. To see that the first equality holds, suppose first $|M \cap S_k| < \left|X_k^* \cup \{i^k\})\right|$. Consider the permutation $\pi^{\mathcal{S}}$ such that $\pi^{\mathcal{S}}(M \cap S_k) \subsetneq X_k^* \cup \{i^k\}$ and $\pi^{\mathcal{S}}(j) = j$ for all $j \notin S_k$. Then, $\pi^{\mathcal{S}}(M_{k-1}) \subsetneq M^{*k}$. By anonymity, $\pi^{\mathcal{S}}(M_{k-1}) \in \mathcal{C}_m^x$, which contradicts that $M^{*k} \in \mathcal{C}_m^x$. Proceed similarly to obtain the contradiction for the case where the other strict inequality holds.

Therefore, $M \in \mathcal{C}_Q^x$.

Now, we will prove that $\mathcal{C}_Q^x \subseteq \mathcal{C}_m^x$.

Let $M \in \mathcal{C}_Q^x$. Then, there is $k$ such that $M = \cup_{t=1}^{k} T_t \cup \{\bar{i}_k\}$ where $T_t \subset S_t$, $|T_t| = q_t$ and for all $t = 1, \cdots, k$, $\bar{i}_k \in S_k \setminus T_k$. Then, by definition of $q$, $q_t = |T_t| = |X_t^*|$ for all $t = 1, \ldots, K-1$ if $k = K$ or all $t = 1, \ldots, k$ if $k < K$. Then, there exits $\pi^{\mathcal{S}}$ such that $\pi^{\mathcal{S}}(X_t^*) = T_t$ and $\pi^{\mathcal{S}}(i^k) = \bar{i}_k$ for all $t = 1, \ldots, K-1$ if $k = K$ or all $t = 1, \ldots, k$ if $k < K$.

First, assume that $k = K$. Then, $M \cap S_K \neq \emptyset$. Therefore, by anonymity and Claim, $|M \cap S_K| = |M^* \cap S_K|$. Then, there exist $\pi^{\mathcal{S}}$ such that $\pi^{\mathcal{S}}(M) = M^*$. Therefore, by anonymity, $M \in \mathcal{C}_m^x$. Second, assume that $k < K$. By (13), $M^{*k} \in \mathcal{C}_m^x$. Then, anonymity implies that $M \in \mathcal{C}_m^x$.

($\Leftarrow$) The statement follows by the definitions of $\mathcal{C}_Q^x$ and the IUP with respect to $\mathcal{S}$.

This finishes the proof of the Theorem 4. ■

Theorems 2 and 4 together characterize the family of all social choice functions in this setting (*i.e.*, EMVRs) that are OSP with respect to a partition and anonymous relative to the same partition.[13]

Theorem 4 may also be used to describe in an alternative way a given EMVR $f_{\mathcal{C}^x}$, whose committee $\mathcal{C}^x$ is anonymous relative to $\mathcal{S}$ and satisfies the IUP with respect to $\mathcal{S}$. The description is as follows. By Theorem 4, let $\mathcal{S} = \{S_1, \ldots, S_K\}$ be the ordered partition of $N$ and let $Q = (q_1, \ldots, q_K)$ be its associated vector of quotas. Fix an arbitrary profile $P \in \mathcal{P}^N$, and let $A(P) = \{i \in N \mid P_i = P_i^x\}$ be the set of agents that approve (or vote for) $x$ at $P$. Then, $f_{\mathcal{C}^x}(P)$ is the alternative identified by the following step-wise process.

Step 1:

(1.1) if $|A(P) \cap S_1| < q_1$, then $f_{\mathcal{C}^x}(P) = y$,

(1.2) if $|A(P) \cap S_1| > q_1$, then $f_{\mathcal{C}^x}(P) = x$,

(1.3) if $|A(P) \cap S_1| = q_1$, then go to Step 2.

Step $k$ ($1 < k < K$):

(k.1) if $|A(P) \cap S_k| < q_k$, then $f_{\mathcal{C}^x}(P) = y$,

(k.2) if $|A(P) \cap S_k| > q_k$, then $f_{\mathcal{C}^x}(P) = x$,

(k.3) if $|A(P) \cap S_k| = q_k$, then go to Step k+1.

Step $K$:

(K.1) if $|A(P) \cap S_K| \leq q_K$, then $f_{\mathcal{C}^x}(P) = y$,

(k.2) if $|A(P) \cap S_K| > q_K$, then $f_{\mathcal{C}^x}(P) = x$,

Let $\mathcal{C}^x$ be an anonymous committee relative to a partition $\mathcal{S}$ that satisfies the IUP with respect to $\mathcal{S}$. Theorem 2 guarantees that there exists a game $\Gamma \in \mathcal{G}^{\mathcal{S}}$ such that $(\Gamma, (\sigma_i^{P_i})_{P_i \in \mathcal{P}, i \in N}))$ OSP-implements $f_{\mathcal{C}^x}$ with respect to $\mathcal{S}$. The description of $f_{\mathcal{C}^x}$ by means of the above step-wise process, applied to each $P \in \mathcal{P}$, allows to identify a much simple game $\Gamma_Q$ to be used to the OSP-implementation of $f_{\mathcal{C}^x}$ with respect to $\mathcal{S}$.

[13] A social choice function $f : \mathcal{P}^N \to \{x, y\}$ is *anonymous relative to the partition* $\mathcal{S}$ if, for all bijections $\pi^{\mathcal{S}} \in \Pi^{\mathcal{S}}$ and $P = (P_1, \ldots, P_n) \in \mathcal{P}^N$, $f(P_1, \ldots, P_n) = f(P_{\pi(1)}, \ldots, P_{\pi(n)})$.

Given an ordered partition $\mathcal{S}^o = \{S_1, \ldots, S_K\}$ and a vector of quotas $Q = (q_1, \ldots, q_K)$, define the extensive game form $\Gamma_Q \in \mathcal{G}^{\mathcal{S}}$ through the following finite sequence of steps, to which we refer to as the $[\mathcal{S}^o, Q]$-process

- Step 1: Agents in $S_1$ play only once and simultaneously, and the set of available choices of each $i \in S_1$ is the partition $\{\{P^x\}, \{P^y\}\}$. Let $h^1$ be a given history at the end of Step 1. Then, (i) $h^1$ is terminal and the outcome of $\Gamma_Q$ is $x$ if strictly more than $q_1$ agents in $S_1$ have chosen $\{P^x\}$ along $h^1$, (ii) $h^1$ is terminal and the outcome of $\Gamma_Q$ is $y$ if strictly less than $q_1$ agents in $S_1$ have chosen $\{P^x\}$ along $h^1$, and (iii) $h^1$ is non-terminal if $q_1$ agents in $S_1$ have chosen $\{P^x\}$ along $h^1$, in which case go to Step 2.

- $\ldots$

  Given $S_1, \ldots, S_{k-1}$, with $1 < k < K$.

- Step $k$: For each non-terminal and commonly known history $h^{k-1}$ at the end of Step $k-1$, agents in $S_k$ play only once and simultaneously, and the set of available choices of each $i \in S_k$ is the partition $\{\{P^x\}, \{P^x\}\}$. Let $h^k$ be a given history at the end of Step k. Then, (i) $h^k$ is terminal and the outcome of $\Gamma_Q$ is $x$ if strictly more than $q_k$ agents in $S_k$ have chosen $\{P^x\}$ along $h^k$, (ii) $h^k$ is terminal and the outcome of $\Gamma_Q$ is $y$ if strictly less than $q_k$ agents in $S_k$ have chosen $\{P^x\}$ along $h^k$, and (iii) $h^k$ is non-terminal if $q_k$ agents in $S_k$ have chosen $\{P^x\}$ along $h^k$, in which case go to Step $k+1$.

- Step $K$: For each non-terminal and commonly known history $h^{K-1}$ at the end of Step $K-1$, agents in $S_K$ play only once and simultaneously, and the set of available choices of each $i \in S_K$ is the partition $\{\{P^x\}, \{P^y\}\}$. Let $h^K$ be a given terminal history at the end of Step $K$. Then, (i) $h^K$ is terminal and the outcome of $\Gamma_Q$ is $x$ if strictly more than $q_K$ agents in $S_K$ have chosen $\{P^x\}$ along $h^K$ and (ii) $h^K$ is terminal and the outcome of $\Gamma_Q$ is $y$ if less than or equal to $q_K$ agents in $S_K$ have chosen $\{P^x\}$ along $h^K$.

Given and ordered partition $\mathcal{S}^o = \{S_1, \ldots, S_K\}$ and a vector of quotas $Q = (q_1, \ldots, q_K)$ for which, for all $k = 1, \ldots, K$, $q_k \leq |S_k|$, denote by $\mathcal{F}^{\mathcal{S}^o, Q}$ the subclass of $\mathcal{G}^{\mathcal{S}}$ containing all extensive game forms that can be obtained as a $[\mathcal{S}^o, Q]$-process.

**Theorem 5** *Let $\mathcal{C}^x$ be an anonymous committee relative to the partition $\mathcal{S}$. Then, $\mathcal{C}^x$ satisfies the IUP with respect to $\mathcal{S}$ if and only if there exists $\Gamma \in \mathcal{F}^{\mathcal{S}^o,Q}$ such that $(\Gamma, (\sigma_i^{P_i})_{i \in N, P_i \in \mathcal{P}})$ OSP-implements $f_{\mathcal{C}^x} : \mathcal{P}^N \to \{x, y\}$ with respect to $\mathcal{S}$.*

**Proof.** Let $\mathcal{C}^x$ be an anonymous committee relative to the partition $\mathcal{S}$.

($\Leftarrow$) Assume $\Gamma \in \mathcal{F}^{\mathcal{S}^o,Q}$ is such that $(\Gamma, (\sigma_i^{P_i})_{i \in N, P_i \in \mathcal{P}})$ OSP-implements $f_{\mathcal{C}^x}$ with respect to $\mathcal{S}$. Since $\mathcal{F}^{\mathcal{S}^o,Q} \subsetneq \mathcal{G}^{\mathcal{S}}$, $\Gamma \in \mathcal{G}^{\mathcal{S}}$. Then, by Theorem 2, the committee $\mathcal{C}^x$ satisfies the IUP with respect to $\mathcal{S}$.

($\Rightarrow$) Assume $\mathcal{C}^x$ satisfies the IUP with respect the $\mathcal{S}$. By Theorem 4, there exist an order in $\mathcal{S}$, written as $\mathcal{S}^o = \{S_1, \ldots, S_K\}$, and a vector of quotas $Q = (q_1, \ldots, q_K)$ such that, for each $1 \leq k \leq K$, $q_k \leq |S_k|$ and

$$\mathcal{C}^x_m = \mathcal{C}^x_Q.$$

Let $\Gamma_Q \in \mathcal{F}^{\mathcal{S}^o,Q}$ be the extensive game form obtained by a [$\mathcal{S}^o$, $Q$]-process. For each $i \in N$, consider the truth-telling type-strategy $(\sigma_i^{P_i})_{P_i \in \mathcal{P}}$ where, for every $z_i \in Z_i$ such that $|Ch(z_i)| = 2$, $\sigma_i^{P_i}(z_i) = \{P_i^x\}$ if $P_i = P_i^x$ and $\sigma_i^{P_i}(z_i) = \{P_i^y\}$ if $P_i = P_i^y$.

We shall show that $(\Gamma_Q, (\sigma_i^{P_i})_{i \in N, P_i \in \mathcal{P}})$ OSP-implements $f_{\mathcal{C}^x}$ with respect to $\mathcal{S}$.

First, we show that $\Gamma_Q$ and $(\sigma_i^{P_i})_{i \in N, P_i \in \mathcal{P}}$ induce $f_{\mathcal{C}^x}$ by going through the sequence of steps defining $\Gamma_Q$. Fix an arbitrary profile $P \in \mathcal{P}^N$.

- Step 1: Agents in $S_1$ play only once and simultaneously, and the set of their available choices is the partition $\{\{P^x\}, \{P^y\}\}$. Let $h^1$ be the history at the end of Step 1. We distinguish among three different cases. depending on the feature of $h^1$.

  (i) $h^1$ is terminal and $g(z^{\Gamma_Q}(z_0, \sigma^P)) = x$. By the definition of $\Gamma_Q$, strictly more than $q_1$ agents in $S_1$, a wining coalition of $x$, have chosen $\{P^x\}$. Accordingly, $f_{\mathcal{C}^x}(P) = x$.

  (ii) $h^1$ is terminal and $g(z^{\Gamma_Q}(z_0, \sigma^P)) = y$. By the definition of $\Gamma_Q$, strictly less than $q_1$ agents in $S_1$, a coalition that is not winning for $x$, have chosen $x$, which means that agents of a winning coalition for $y$ have chosen $\{P^y\}$. Accordingly, $f_{\mathcal{C}^x}(P) = y$.

  (iii) $h^1$ is non-terminal. This means that exactly $q_1$ agents in $S_1$ have chosen $\{P^x\}$. By the definition of $\Gamma_Q$, the [$\mathcal{S}^o$ $Q$]-process moves to Step 2.

- ... Let $1 < k < K$.

- Step $k$: Agents in $S_k$ play only once and simultaneously, and the set of their available choices is the partition $\{\{P^x\}, \{P^y\}\}$. Let $h^k$ be the history at the end of Step $k$. We distinguish among three different cases, depending on the feature of $h^k$.

(i) $h^k$ is terminal and $g(z^{\Gamma_Q}(z_0, \sigma^P)) = x$. By the definition of $\Gamma_Q$, for each $1 \leq t < k$, exactly $q_t$ agents in $S_t$ have chosen $\{P^x\}$ and strictly more than $q_k$ agents in $S_k$ has also chosen $\{P^x\}$. According to its definition in (10), this set belongs to $\mathcal{C}^x_{Q,k}$ and so a winning coalition of $x$ has chosen $\{P^x\}$. Accordingly, $f_{\mathcal{C}^x}(P) = x$.

(ii) $h^k$ is terminal and $g(z^{\Gamma_Q}(z_0, \sigma^P)) = y$. By the definition of $\Gamma_Q$, for each $1 \leq t < k$, exactly $q_t$ agents in $S_t$ have chosen $\{P^x\}$ and strictly less than $q_k$ agents in $S_k$ have also chosen $\{P^x\}$. According to its definition in (10), this set does not belong to $\mathcal{C}^x_{Q,k}$ and so a winning coalition of $y$ has chosen $\{P^y\}$. Accordingly, $f_{\mathcal{C}^x}(P) = y$.

(iii) $h^k$ is non-terminal. By the definition of $\Gamma_Q$, for each $1 \leq t \leq k$, exactly $q_t$ agents in $S_t$ have chosen $\{P^x\}$. According to the definition of $\Gamma_Q$, the $[\mathcal{S}^o\, Q]$-process goes to Stage $k+1$.

- ......

- Step $K$: Agents in $S_K$ play only once and simultaneously, and the set of their available choices is the partition $\{\{P^x\}, \{P^y\}\}$. Let $h^K$ be the history at the end of Step $K$. Since $K$ is the last step of the $[\mathcal{S}^o\, Q]$-process, $h^K$ is terminal. We distinguish between two different cases, depending on the outcome associated to $h^K$.

  (i) $g(z^{\Gamma_Q}(z_0, \sigma^P)) = x$. By the definition of $\Gamma_Q$, for each $1 \leq t < k$, exactly $q_t$ agents in $S_t$ have chosen $\{P^x\}$ and strictly more than $q_K$ agents in $S_K$ have also chosen $\{P^x\}$. According to its definition in (10), this set belongs to $\mathcal{C}^x_{Q,K}$ and so a winning coalition of $x$ has chosen $\{P^x\}$. Accordingly, $f_{\mathcal{C}^x}(P) = x$.

  (ii) $g(z^{\Gamma_Q}(z_0, \sigma^P)) = y$. By the definition of $\Gamma_Q$, for each $1 \leq t < k$, exactly $q_t$ agents in $S_t$ have chosen $\{P^x\}$ and less than or equal to $q_K$ agents in $S_K$ have also chosen $\{P^x\}$. According to its definition in (10), this set does not belong to $\mathcal{C}^x_{Q,K}$ and so a winning coalition of $y$ has chosen $\{P^y\}$. Accordingly, $f_{\mathcal{C}^x}(P) = y$.

Therefore, $f_{\mathcal{C}^x}(P) = g(z^{\Gamma_Q}(z_0, \sigma^P))$.

We now prove that the truth-telling strategy $\sigma_i^{P_i}$ is obviously dominant with respect to $\mathcal{S}$ in $\Gamma_Q$ for $i$ and $P_i$.

Assume agent $j$ has to choose, at information set $I_j^k$ of Step $k$ that starts after history $h^{k-1}$, one from the set $Ch(I_j^k) = \{\{P_j^x\}, \{P_j^y\}\}$. By definition of $\Gamma_Q$, $j \in \mathcal{ND}^k$ and the history $h^{k-1}$ can be identified with a sequence $X_1, \ldots, X_{k-1}$ where, for each $t = 1, \ldots, k-1$, $X_t$ is the subset of agents in $S_t$ that have chosen $\{P^x\}$ along the history $h^{k-1}$. Notice that, since the $[\mathcal{S}^o\, Q]$-process has reached Step $k$, $|X_t| = q_t$ for all $1 \leq t \leq k$. We distinguish between two general cases which, in turn, each is divided into three subcases.

*Case A.* Assume $P_j = P_j^x$. The choice consistent with $j$'s truth-telling strategy is $\overline{a}_j = P_j^x$. Let $\sigma_i$ be a fixed strategy for each $i \in S_k \setminus \{j\}$. Denote, for each $i \in S_k \setminus \{j\}$, $\sigma_i(I_i^k) = \overline{a}_i$, where $I_i^k$ is agent $i$'s information set that goes across the history that starts at $h^{k-1}$ and it is played by agents in $S_k$ along Step $k$. Let $\overline{h}^k = (h^{k-1}, (\overline{a}_i)_{i\in S_k})$ and $\overline{X}_k = \{i \in \mathcal{ND}^k \mid \overline{a}_i = P_i^x\}$. We distinguish among three subcases.

*Case A.1.* $|\overline{X}_k| < q_k$. Then, $\overline{h}^k$ is a terminal history and the outcome of the game is $y$ because $(S_1 \setminus X_1) \cup \cdots \cup (S_{k-1} \setminus X_{k-1}) \cup (S_k \setminus \overline{X}_k) \in \mathcal{C}_Q^y$. Suppose agent $j$ deviates and plays $\widehat{a}_j = P_j^y$. Let $\widehat{a} = (\widehat{a}_j, (\overline{a}_i)_{i\in S_k\setminus\{j\}})$, $\widehat{h}^k = (h^{k-1}, (\widehat{a}_i)_{i\in S_k})$, $\widehat{X}_k = \{i \in \mathcal{ND}^k \mid \widehat{a}_i = P_i^x\}$, and $|\widehat{X}_k| < |\overline{X}_k| < q_k$. Then, $(S_1 \setminus X_1) \cup \cdots \cup (S_{k-1} \setminus X_{k-1}) \cup (S_k \setminus \widehat{X}_k) \in \mathcal{C}_Q^y$. Hence, the outcome of the game after $j$'s deviation continues to be $y$. Therefore, as $P_j = P_j^x$, the truth-telling strategy $\sigma_j^{P_j}$ is an obvious dominant strategy with respect to $\mathcal{S}$.

*Case A.2.* $|\overline{X}_k| > q_k$. Then $X_1 \cup \cdots \cup X_{k-1} \cup \overline{X}_k \in \mathcal{C}_Q^x$. Therefore, $\overline{h}^k$ is a terminal history and the outcome of the game is $x$ and, as $P_j = P_j^x$, the truth-telling strategy $\sigma_j^{P_j}$ is an obvious dominant strategy with respect to $\mathcal{S}$.

*Case A.3.* $|\overline{X}_k| = q_k$. Suppose agent $j$ deviates and plays $\widehat{a}_j = P_j^y$. Let $\widehat{a} = (\widehat{a}_j, (\overline{a}_i)_{i\in S_k\setminus\{j\}})$, $\widehat{h}^k = (h^{k-1}, (\widehat{a}_i)_{i\in S_k})$, $\widehat{X}_k = \{i \in \mathcal{ND}^k \mid \widehat{a}_i = P_i^x\}$, $\overline{X}_k = \widehat{X}_k \cup \{j\}$, and $|\widehat{X}_k| < q_k$. Then, $(S_1 \setminus X_1) \cup \cdots \cup (S_{k-1} \setminus X_{k-1}) \cup (S_k \setminus \widehat{X}_k) \in \mathcal{C}_Q^y$. Therefore, $\widehat{h}^k$ is a terminal history and the outcome of the game is $y$. Thus, as $P_j = P_j^x$, the truth-telling strategy $\sigma_j^{P_j}$ is an obvious dominant strategy with respect to $\mathcal{S}$.

*Case B.* Assume $P_j = P_j^y$. The choice consistent with $j$'s truth-telling strategy is $\overline{a}_j = P_j^y$. Let $\sigma_i$ be a fixed strategy for each $i \in S_k \setminus \{j\}$. Denote, for each $i \in S_k \setminus \{j\}$, $\sigma_i(I_i^k) = \overline{a}_i$, where $I_i$ is agent $i$'s information set that goes across the history that starts at $h^{k-1}$ and it is played by agents in $S_k$ along Step $k$. Let $\overline{h}^k = (h^{k-1}, (\overline{a}_i)_{i\in S_k})$ and $\overline{X}_k = \{i \in \mathcal{ND}^k \mid \overline{a}_i = P_i^x\}$. We distinguish among three subcases.

*Case B.1.* $|\overline{X}_k| > q_k$. Then, $\overline{h}^k$ is a terminal history and the outcome of the game is $x$ because $X_1 \cup \cdots \cup X_{k-1} \cup \overline{X}_k \in \mathcal{C}^x$. Suppose agent $j$ deviates and plays $\widehat{a}_j = P_j^x$. Let $\widehat{a} = (\widehat{a}_j, (\overline{a}_i)_{i\in S_k\setminus\{j\}})$, $\widehat{h}^k = (h^{k-1}, (\widehat{a}_i)_{i\in S_k})$, $\widehat{X}_k = \{i \in \mathcal{ND}^k \mid \widehat{a}_i = P_i^x\}$, $\overline{X}_k = \widehat{X}_k \setminus \{j\}$, and $|\widehat{X}_k| > q_k$. Then, $X_1 \cup \cdots \cup X_{k-1} \cup \widehat{X}_k \in \mathcal{C}_Q^x$, $\widehat{h}^k$ is a terminal history and the outcome of the game is $x$. Therefore, as $P_j = P_i^y$, the truth-telling strategy $\sigma_j^{P_j}$ is an obvious dominant strategy with respect to $\mathcal{S}$.

*Case B.2.* $|\overline{X}_k| < q_k$. Then $(S_1 \setminus X_1) \cup \cdots \cup (S_{k-1} \setminus X_{k-1}) \cup (S_k \setminus \overline{X}_k) \in \mathcal{C}^y$. Then, $\overline{h}^k$ is a terminal history and the outcome of the game is $y$. Therefore, as $P_j = P_j^y$, the truth-telling strategy $\sigma_j^{P_j}$ is an obvious dominant strategy with respect to $\mathcal{S}$.

*Case B.3* $|\overline{X}_k| = q_k$. Suppose agent $j$ deviates and plays $\widehat{a}_j = P_j^x$. Let $\widehat{a} = (\widehat{a}_j, (\overline{a}_i)_{i\in S_k\setminus\{j\}})$, $\widehat{h}^k = (h^{k-1}, (\widehat{a}_i)_{i\in S_k})$, $\widehat{X}_k = \{i \in \mathcal{ND}^k \mid \widehat{a}_i = P_i^x\}$, $\widehat{X}_k = \overline{X}_k \cup \{j\}$ and $|\widehat{X}_k| > q_k$. Then, $X_1 \cup \cdots \cup X_{k-1} \cup \widehat{X}_k \in \mathcal{C}_q^x$. Therefore, $\widehat{h}^k$ is a terminal history and the outcome of the game is $x$. Thus, as $P_j = P_j^y$, the truth-telling strategy $\sigma_j^{P_j}$ is an obvious dominant strategy with respect to $\mathcal{S}$.

Thus, the game $\Gamma_Q \in \mathcal{F}^{\mathcal{S}^o,Q}$ OSP-implements $f_{\mathcal{C}^x}$ with respect to $\mathcal{S}$. This finishes the proof of Theorem 5. ■

# 5 Two final remarks

## 5.1 Round table mechanisms

Before finishing, we want to comment that in general, as it is the case for the OSP-implementation, to OSP-implement a social choice function with respect to a partition one can restrict attention to the class of round table mechanisms with or without perfect information (see Mackenzie (2020) for the case of perfect information). The reason follows from two ideas, which adapt the arguments for OSP-implementation to OSP-implementation with respect to a partition.

The first idea is related to the pruning principle (see, for instance, Li (2016) and Ashlagi and Gonczarowski (2018)). Namely, assume the pair $(\Gamma, (\sigma^R)_{R\in\mathcal{D}})$, composed by the game and the type-strategy profile, OSP-implements the social choice function $f; \mathcal{D}^N \to A$. Delete the last parts of the paths of $\Gamma$ that are never played when agents use $(\sigma^R)_{R\in\mathcal{D}}$ and denote this pruned game by $\widehat{\Gamma}$ and the restriction of $(\sigma^R)_{R\in\mathcal{D}}$ to $\widehat{\Gamma}$ by $(\widehat{\sigma}^R)_{R\in\mathcal{D}}$. It is evident that the pair $(\widehat{\Gamma}, (\widehat{\sigma}^R)_{R\in\mathcal{D}})$ also OSP-implements $f$, since after pruning $\Gamma$ the worst-case from continuing can only get better and the best-case from deviating can only get worse.

The second idea is related to the relabeling of the choices of $\widehat{\Gamma}$ proposed by Mackenzie (2020); namely, for each agent $i$, each history at which $i$ has to play, and each choice available to $i$ there, relabel that choice with the collection of preferences $\overline{R}_i \in \mathcal{D}_i$ whose corresponding part of the type-strategy $(\sigma_i^{R_i})_{R_i\in\mathcal{D}_i}$ are compatible with the history and the strategies $\sigma_i^{\overline{R}_i}$ select that choice.

The extensive game form obtained after the pruning and the relabeling of choices is called a round table mechanism, which is therefore an extensive game form, now potentially with imperfect information, where the sets of choices are non-empty subsets of preferences satisfying the following properties: (a) the set of choices at any information set are disjoint subsets of preferences, (b) when player $i$ has to play for the first time the set of choices is a partition of $\mathcal{D}_i$, and (c) later, at an information set $I_i$, the union of available choices is the intersection of the choices taken by agent $i$ at all predecessor nodes that lead to $I_i$.

Observe that the extensive game forms used in Theorem 1 and in the application to extended majority voting rules with two alternatives (Theorems 2 and 5) are all round table mechanisms with imperfect information.

In general, the extensive game form $\Gamma$ that OSP-implements the social choice function $f$ with respect to a partition $\mathcal{S}$ requires that $\Gamma$ has imperfect information. To understand why, consider the following argument. By definition, if the $\Gamma$ that OSP-implements $f$ with respect to $\mathcal{S}$ would have perfect information, then $\Gamma$ would SP-implement $f$ as well. However, Mackenzie (2020) establishes that SP-implementation with perfect information is equivalent to OSP-implementation. Since OSP-implementation with respect to a partition is strictly stronger than just OSP-implementation, $\Gamma$ can not have perfect information; in particular, the application of Subsection 4.2 with two alternatives contains instances of anonymous social choice functions that are OSP-implementable with respect to $\mathcal{S}$ but, according to Arribillaga, MassÃ[3] and Neme (2020), they are not OSP-implementable. This points out that certain imperfect information is required to OSP-implement with respect to $\mathcal{S}$.

## 5.2 Group obvious strategy-proofness

Subsets of agents (coalitions), organized in a partition, play a crucial role in the definition of obvious strategy-proofness relative to a partition. The literature contains other notions of implementation in which strategic incentives are imposed not only on individual agents but also on coalitions of agents; for example, implementation in strong Nash equilibria or group strategy-proofness. Therefore, it is natural to extend the original Li (2017)'s notion of obvious strategy-proofness based on individual incentives to a notion that addresses coalitional incentives as well.

This subsection contains a natural definition of *group obvious strategy-proofness*, that merges group strategy-proofness and obvious strategy-proofness. Theorem 6 establishes that group obvious strategy-proofness coincides with obvious strategy-proofness.

Let $\Gamma$ be an extensive game form with set of agents $N$ and outcomes in $A$. Fix a subset of agents $S \subset N$. Given $\sigma_S$ and $\sigma'_S$ such that $\sigma'_j \in \Sigma_j \setminus \{\sigma_j\}$ for all $j \in S$, an earliest point of departure of $i \in S$ for $\sigma_S$ and $\sigma'_S$ is a subset of nodes of an information set $I_i$ with the properties that all nodes in $I_i$ are compatible with $\sigma_S$, and $\sigma_i$ and $\sigma'_i$ prescribe different actions at each of them but $\sigma_S$ and $\sigma'_S$ prescribe identical actions at all its previous information sets that come across to each of their paths.

**Definition 4** Let $\sigma_S$ and $\sigma'_S$ be such that $\sigma'_j \in \Sigma_j \setminus \{\sigma_j\}$ for all $j \in S$ and let $i \in S$. Given $i$'s information set $I_i \in \mathcal{I}_i$, we say that the set of all nodes $z \in I_i$ compatible with $\sigma_S$, denoted by $I_i(\sigma_S, \sigma'_S)$, is an *earliest point of departure of agent $i$ for $\sigma_S$ and $\sigma'_S$* if
(i) $\sigma_i(I_i) \neq \sigma'_i(I_i)$,
(ii) for every $j \in S$, $\sigma_j(I'_j) = \sigma'_j(I'_j)$ for all $I'_j \in \mathcal{I}_j$ such that $I'_j \prec I_i$.

Observe that an earliest point of departure is a subset of an information set of a single agent $i$.

Given $i \in S$, $\sigma_S$ and $\sigma'_S$, denote the set of earliest points of departures of $i$ for $\sigma_S$ and $\sigma'_S$ by $\alpha_i(\sigma_S, \sigma'_S)$.

Given $S$, $\sigma_S$, $\sigma'_S$ and $i \in S$, let $O_i(\sigma_S, \sigma'_S)$ and $O'_i(\sigma_S, \sigma'_S)$ be the two sets of options left respectively by $\sigma_S$ and $\sigma'_S$ at the earliest point of departure $I_i(\sigma_S, \sigma'_S)$ of $i$; namely,

$$O_i(\sigma^i_S, \sigma'_S) = \{x \in A \mid \exists \overline{\sigma}_{-S} \in \Sigma_{-S} \text{ and } z \in I_i(\sigma_S, \sigma'_S) \text{ s.t. } x = g(z^\Gamma(z, (\sigma_S, \overline{\sigma}_{-S})))\}$$

and

$$O'_i(\sigma_S, \sigma'_S) = \{y \in A \mid \exists \overline{\sigma}_{-S} \in \Sigma_{-S} \text{ and } z \in I_i(\sigma_S, \sigma'_S) \text{ s.t. } y = g(z^\Gamma(z, (\sigma'_S, \overline{\sigma}_{-S})))\}.$$

We are now ready to define the notion of group obviously dominant strategy.

**Definition 5** We say that $\sigma_S$ is *group obviously dominant* in $\Gamma$ for $R_S$ if for all $\sigma'_S \in \Sigma_S$ such that $\sigma'_j \in \Sigma_j \setminus \{\sigma_j\}$ for all $j \in S$, all $i \in S$ and all $I_i(\sigma_S, \sigma'_S) \in \alpha_i(\sigma_S, \sigma'_S)$,

$$x \, R_i \, y$$

holds, for all $x \in O_i(\sigma_S, \sigma'_S)$ and all $y \in O'_i(\sigma_S, \sigma'_S)$.[14]

**Definition 6** A social choice function $f : \mathcal{D} \to A$ is *group obviously strategy-proof* (GOSP) if there exist an extensive game form $\Gamma \in \mathcal{G}$ and a type-strategy profile $(\sigma_i^{R_i})_{R_i \in \mathcal{D}_i, i \in N}$ for $\Gamma$ such that, for each $R \in \mathcal{D}$, (i) $f(R) = g(z^\Gamma(z_0, \sigma^R))$ and (ii) for all $S \subseteq N$, $\sigma_S^{R_S}$ is group obviously dominant in $\Gamma$ for $R_S$.

[14]Namely, given $R_S$, $\sigma_S$ and $\sigma'_S$, from the point of view of $i \in S$ the worst alternative that can be reached when agents in $S$ are playing $\sigma_S$ is at least as preferred according to $R_i$ as the best alternative that can be reached when agents in $S$ are playing $\sigma'_S$; in this sense, for every $i \in S$, $\sigma_S$ is undoubtedly better than $\sigma'_S$.

As in the case of obvious strategy-proofness, when (i) holds we say that $\Gamma$ and $(\sigma_i^{R_i})_{R_i \in \mathcal{D}_i, i \in N}$ induce $f$. When (i) and (ii) hold we say that $\Gamma$ GOSP-*implements* $f$.

Mackenzie (2020) contains a general revelation principle stating that the extensive game form used to implement a social choice function in obviously dominant strategies may have, without loss of generality, perfect information, That is, $I_i$ is a singleton set for every $i$.

**Theorem 6** *A social choice function $f : \mathcal{D} \to A$ is group obviously strategy-proof if and only if $f$ is obviously strategy-proof.*

**Proof.** Let $f : \mathcal{D} \to A$ be a social choice function.

($\Rightarrow$) From the two definitions, if $f$ is GOSP, then $f$ is OSP.

($\Leftarrow$) Let $f$ be OSP. Then, there exist $\Gamma \in \mathcal{G}$ and $(\sigma_i^{R_i})_{R_i \in \mathcal{D}_i, i \in N}$ that OSP-implement $f$. Therefore, conditions (i) in Definitions 6 and 3 coincide; that is, $\Gamma$ and $(\sigma_i^{R_i})_{R_i \in \mathcal{D}_i, i \in N}$ induce $f$. By Mackenzie (2022), we may assume that $\Gamma$ has perfect information. To obtain a contradiction, suppose condition (ii) in Definition 6 does not hold for $\Gamma$ and $(\sigma_i^{R_i})_{R_i \in \mathcal{D}_i, i \in N}$. Then, there exist $S \subseteq N$ and $R_S$ such that $\sigma_S^{R_S}$ is not group obviously dominant in $\Gamma$ for $R_S$. That is, there exist $\sigma'_S \in \Sigma_S$ such that $\sigma'_j \in \Sigma_j \setminus \{\sigma_j\}$ for all $j \in S$, $i \in S$ and $\{z_i\} = I_i(\sigma_S, \sigma'_S) \in \alpha_i(\sigma_S, \sigma'_S)$, such that

$$y \, P_i \, x \tag{14}$$

holds, for some $x \in O_i(\sigma_S, \sigma'_S)$ and some $y \in O'_i(\sigma_S, \sigma'_S)$. Fix such pair of alternatives $x$ and $y$. Then,

$$\begin{aligned} &\max_{R_i}\{w' \in X \mid \text{there exists } \sigma_{-S} \in \Sigma_{-S} \text{ such that } w' = g(z^\Gamma(z_i, (\sigma'_S, \sigma_{-S})))\} \\ P_i &\min_{R_i}\{w \in X \mid \text{there exists } \sigma_{-S} \in \Sigma_{-S} \text{ such that } w = g(z^\Gamma(z_i, (\sigma_S, \sigma_{-S})))\} \end{aligned}$$

hold because $y$ and $x$ belong respectively to the first and second sets where the maximum and the minimum are obtained according to $R_i$. Therefore,

$$\begin{aligned} &\max_{R_i}\{w' \in X \mid \text{there exists } \sigma_{-i} \in \Sigma_{-i} \text{ such that } w' = g(z^\Gamma(z_i, (\sigma'_i, \sigma_{-i})))\} \\ R_i &\max_{R_i}\{w' \in X \mid \text{there exists } \sigma_{-S} \in \Sigma_{-S} \text{ such that } w' = g(z^\Gamma(z_i, (\sigma'_S, \sigma_{-S})))\} \end{aligned}$$

and

$$\begin{aligned} &\min_{R_i}\{w \in X \mid \text{there exists } \sigma_S \in \Sigma_{-S} \text{ such that } w = g(z^\Gamma(z_i, (\sigma_S, \sigma_{-S})))\} \\ R_i &\min_{P_i}\{w \in X \mid \text{there exists } \sigma_{-i} \in \Sigma_{-i} \text{ such that } w = g(z^\Gamma(z_i, (\sigma_i, \sigma_{-i})))\}. \end{aligned}$$

Thus, there exist $i \in N$, $\sigma'_i \in \Sigma_i$, and a node $z_i$, which by Mackenzie (2020) it coincides with an earliest point of departure $\{z_i\} = I_i(\sigma_i, \sigma'_i) \in \alpha_i(\sigma_i, \sigma'_i)$ for $\sigma_i$ and $\sigma'_i$, such that

$$\max_{R_i}\{w' \in X \mid \text{there exists } \sigma_{-i} \in \Sigma_{-i} \text{ such that } w' = g(z^{\Gamma}(z_i, (\sigma'_i, \sigma_{-i})))\}$$
$$P_i \min_{R_i}\{w \in X \mid \text{there exists } \sigma_{-i} \in \Sigma_{-i} \text{ such that } w = g(z^{\Gamma}(z_i, (\sigma_i, \sigma_{-i})))\},$$

which means that $\Gamma$ does not OSP-implement $f$ with respect to the partition $\{\{1\}, \ldots, \{n\}\}$. According to Remark 1, this contradicts that $\Gamma$ OSP-implements $f$. ■

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