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Equilibrium existence in a discrete-time endogenous growth model with physical and human capital

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Abstract

This paper studies a discrete-time version of the Lucas-Uzawa endogenous growth model with physical and human capital. Equilibrium existence is proved applying tools of dynamic programming with unbounded returns. The proofs rely on properties of homogeneous functions and also apply well-known inequalities in real analysis, seldom used in the literature, which significantly simplifies the task of verifying certain assumptions that are rather technical in nature.

Keywords: Endogenous Growth, Equilibrium, Human capital, Dynamic Programming

JEL Classification: C61, C63, O41

1 Introduction

This paper studies a discrete-time version of a two-sector endogenous growth model with physical and human capital in the tradition of Lucas (1988) and Uzawa (1965). Equilibrium existence for the social planner's problem is proved based on earlier work in dynamic programming with unbounded returns by Alvarez and Stokey (1998) and further results from Le Van and Morhaim (2002) and Le Van (2006).

The main results are developed for the model without human capital externalities. It is later shown that the functional form used to represent output technology allows for a straightforward extension of all results to the case where the model includes externalities. The proofs rely on certain useful properties of homogeneous functions and also apply well-known inequalities in real analysis, seldom used in the literature, which significantly simplifies the task of verifying certain assumptions that are rather technical in nature.

2 A two-sector model of endogenous growth

Time is discrete and denoted by $t \in \mathbb{Z}_+$ (the set of nonnegative integers). The economy is populated by a large number of identical, infinitely-lived agents with unit mass. In each period t, there is a single good that is produced using two inputs: physical capital, $k_t \in \mathbb{R}_+$, and human capital $h_t \in \mathbb{R}_+$. These inputs depreciate every period at constant rates, given by $0 < \delta_k < 1$ and $0 < \delta_h < 1$, respectively. Each agent has an endowment of one time unit per period, from which $u_t \geq 0$ is allocated to market activities and $v_t \geq 0$ to human capital accumulation.

For simplicity, the main results of the paper are presented for the model without human capital externalities. Later we show that these results can be extended to a model with externalities in a straightforward manner. Output per capita is produced with a Cobb-Douglas technology, given by $Ak_t^{\alpha}(u_t h_t)^{1-\alpha}$ with $A > 0$ and $0 < \alpha < 1$. Let $n \geq 0$ denote the exogenous population growth rate. The per capita aggregate resource restriction for each $t = 0, 1, \ldots$ is then

$$c_t + (1+n)k_{t+1} \leq Ak_t^{\alpha}(u_t h_t)^{1-\alpha} + (1-\delta_k)k_t.$$

Human capital accumulation is linear in h_t and given by

$$h_{t+1} = [B\phi(v_t) + (1-\delta_h)]\, h_t,$$

where $B > 0$ and $\phi : [0,1] \to \mathbb{R}_+$ is continuous, strictly increasing and concave with $\phi(0) = 0$ and $\phi(1) > \delta_h/B$. Note that Lucas (1988) assumes that $\phi(v) = v$, for all $0 \leq v \leq 1$, while Uzawa (1965) uses a similar specification, but both authors ignore human capital depreciation.

The instantaneous utility of a representative agent $U : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ satisfies

$$U(c_t) = \begin{cases} \dfrac{c_t^{1-\sigma} - 1}{1-\sigma}, & \text{if } 0 < \sigma < +\infty,\ \sigma \neq 1, \\ \log(c_t), & \text{if } \sigma = 1. \end{cases} \tag{1}$$

Suppose that $0 < \beta < 1$ is the discount factor. Along an *optimal path*, a social planner chooses a path $\{(c_t, u_t, v_t, k_{t+1}, h_{t+1})\}_{t=0}^{\infty}$ to solve the following problem

$$\max \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{2}$$

$$\text{s.t. } c_t + (1+n)k_{t+1} \leq Ak_t^{\alpha}(u_t h_t)^{1-\alpha} + (1-\delta_k)k_t, \qquad t = 0, 1, \ldots, \tag{3}$$

$$h_{t+1} = [B\phi(v_t) + (1-\delta_h)]\, h_t, \qquad t = 0, 1, \ldots, \tag{4}$$

$$u_t + v_t \leq 1, \qquad t = 0, 1, \ldots, \tag{5}$$

$$c_t, u_t, v_t, k_{t+1}, h_{t+1} \geq 0, \qquad t = 0, 1, \ldots, \tag{6}$$

$$(k_0, h_0) \in \mathbb{R}_+^2 \text{ given.}$$

Since the objective of this problem is strictly monotone in each c_t, u_t, v_t, and both k_t and h_t are essential for production, the resource constraint (3) and time constraint (5) in the planner's problem will hold with equality. Moreover, standard Inada conditions on preferences and technology are satisfied, which imply interior optima, provided that $(k_0, h_0) \gg 0$. Using these properties of the solution, problem (2) can be reformulated in terms of the choice of (k_{t+1}, h_{t+1}) only, for all t. The following lemma shows that there is a *minimum* fraction of time $\underline{v}$ in the human capital sector needed for this economy to exhibit sustained growth. This result will be useful to reformulate the problem and for later proofs .

Lemma 1. *There exists a unique value* $\underline{v} \in (0, 1)$*, implicitly defined by* $\phi(\underline{v}) = \delta_h/B$*, such that* $[B\phi(v) + (1 - \delta_h)] \leq 1$*, for all* $v \in [0, \underline{v}]$*, and* $[B\phi(v) + (1 - \delta_h)] > 1$*, for all* $v \in (\underline{v}, 1]$*.*

Proof. By the assumptions made on ϕ, we have that $B[\phi(0) + (1 - \delta_h)] = B(1 - \delta_h) < 1$ and $[B\phi(1) + (1 - \delta_h)] > 1$. Hence, the intermediate value theorem implies that there is a $\underline{v} \in (0, 1)$ such that $[B\phi(\underline{v}) + (1 - \delta_h)] = 1$. Since ϕ is strictly increasing, the value of $\underline{v}$ must be unique. The remaining properties follow from the continuity of ϕ. ■

Given that (5) holds with equality, substituting $v_t = 1 - u_t$ into (4) allows to define a function $\psi : \mathbb{R}^2_+ \to [0, 1]$ by

$$\psi(h_t, h_{t+1}) = \begin{cases} 1 & \text{if } 0 \leq h_{t+1} \leq (1 - \delta_h)h_t, \\ 1 - \phi^{-1}\left[\frac{1}{B}\left(\frac{h_{t+1}}{h_t} - (1 - \delta_h)\right)\right] & \text{if } (1 - \delta_h)h_t \leq h_{t+1} \leq [B\phi(1) + (1 - \delta_h)]h_t, \end{cases} \tag{7}$$

for all $h_t > 0$, and $\psi(0, h_{t+1}) = 1$, for all $h_{t+1} \geq 0$. It follows directly from the definition of ψ in (7) and Lemma 1 that there is a *maximum* fraction of time $\overline{u}$ devoted to market activities compatible with sustained growth, which is defined by $\overline{u} := \psi(h, h) = 1 - \phi^{-1}(\delta_h/B)$, for all $h > 0$. Then, in each period t where $u_t = \overline{u}$ and $h_t > 0$, human capital is kept constant at its current level, i.e., $h_{t+1} = h_t$, and, for any $0 \leq u_t < \overline{u}$, human capital increases next period, i.e., $h_{t+1} > h_t$.

Given $(k_t, h_t) \in \mathbb{R}^2_+$, combining the restrictions for c_t, u_t, and v_t yields the feasible choices for the state next period (k_{t+1}, h_{t+1}) as

$$\frac{(1 - \delta_k)k_t}{(1 + n)} \leq k_{t+1} \leq \frac{Ak_t^\alpha h_t^{1-\alpha} + (1 - \delta_k)k_t}{(1 + n)}, \tag{8}$$

$$(1 - \delta_h)h_t \leq h_{t+1} \leq [B\phi(1) + (1 - \delta_h)]\, h_t, \tag{9}$$

for each $t = 0, 1, \ldots$ This defines a feasibility correspondence $\Gamma : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ given by

$$\Gamma(k_t, h_t) := \left\{(k_{t+1}, h_{t+1}) \in \mathbb{R}^2_+ : (8) \text{ and } (9) \text{ hold for some } (k_t, h_t) \in \mathbb{R}^2_+\right\}. \tag{10}$$

The planner's problem (2) can be written as

$$\max_{\{k_{t+1},h_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F\left(k_t, h_t, k_{t+1}, h_{t+1}\right) \tag{11}$$

$$\text{s.t.} \quad (k_{t+1}, h_{t+1}) \in \Gamma(k_t, h_t), \qquad t = 0, 1, \ldots,$$

$$(k_0, h_0) \in \mathbb{R}_+^2 \text{ given,}$$

where the return function $F : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R} \cup \{-\infty\}$ is defined as

$$F(k_t, h_t, k_{t+1}, h_{t+1}) := U\left[Ak_t^{\alpha}\left(\psi(h_t, h_{t+1})\, h_t\right)^{1-\alpha} + (1-\delta_k)k_t - (1+n)k_{t+1}\right], \tag{12}$$

for all $(k_t, h_t, k_{t+1}, h_{t+1}) \in \mathbb{R}_{++}^2 \times \mathbb{R}_+^2$ for which $U(\cdot) > -\infty$. In the cases where any of k_t or h_t is zero, $F(k_t, h_t, k_{t+1}, h_{t+1}) := U\left[\max\{0, (1-\delta_k)k_t - (1+n)k_{t+1}\}\right]$, for all $(k_{t+1}, h_{t+1}) \in \mathbb{R}_+^2$.

For any $(k_0, h_0) \in \mathbb{R}_+^2$, let

$$\Pi(k_0, h_0) := \{\{(k_t, h_t)\}_{t=0}^{\infty} : (k_{t+1}, h_{t+1}) \in \Gamma(k_t, h_t),\ t = 0, 1, \ldots\} \tag{13}$$

be the set of all sequences of the state variables that are feasible from (k_0, h_0). A typical element of Π will be denoted by $(\underset{\sim}{k}, \underset{\sim}{h})$.

3 Equilibrium existence and characterization

Equilibrium existence for the model developed in the previous section is mainly based on Le Van and Morhaim (2002) and Le Van (2006). For ease of notation, the Euclidean norm of any vector $(k, h) \in \mathbb{R}_+^2$ will be written as $\|k, h\|$. If the current state of the system is given by (k, h), the next period value is represented by (k', h'). Also, the graph of Γ is denoted as $\mathrm{gr}(\Gamma)$.

A number of assumptions on Γ, F, and β will be used to state and prove the results.

(A1) Γ is a nonempty, continuous, compact-valued correspondence and $(0, 0) \in \Gamma(0, 0)$.

(A2) There exist constants $\zeta \geq 0$ and $\zeta' \geq 0$, with $\zeta \neq 1$, such that $(k', h') \in \Gamma(k, h)$ implies $\|k', h'\| \leq \zeta\|k, h\| + \zeta'$.

(A3) For all $(k, h, k', h') \in \mathrm{gr}(\Gamma)$, there exist constants $\eta \geq 0$ and $\eta' \geq 0$ such that

$$F(k, h, k', h') \leq \eta\left(\|k, h\| + \|k', h'\|\right) + \eta'.$$

(A4) If $\zeta > 0$ in (A2), then $\beta\zeta < 1$.

(A5) F is continuous at any point in $\mathrm{gr}(\Gamma)$ such that $F(k, h, k', h') > -\infty$. If $F(k, h, k', h') = -\infty$, then for any sequence $\{(k_n, h_n, k_n', h_n')\}_{n=0}^{\infty}$ in $\mathrm{gr}(\Gamma)$ that converges to (k, h, k', h'), it follows that $\lim\limits_{n\to\infty} F(k_n, h_n, k_n', h_n') = -\infty$;

(A6) For all $(k, h, k', h') \in \mathrm{gr}(\Gamma)$ and for all $\lambda \in (0, 1]$,

(a) $(\lambda k, \lambda h, \lambda k', \lambda h') \in \mathrm{gr}(\Gamma)$;

(b) there exist continuous functions $\Phi_1 : (0, 1] \to \mathbb{R}$ and $\Phi_2 : (0, 1] \to \mathbb{R}$, with $\Phi_1(1) = 1$ and $\Phi_2(1) = 0$, such that $F(\lambda k, \lambda h, \lambda k', \lambda h') \geq \Phi_1(\lambda)F(k, h, k', h') + \Phi_2(\lambda)$;

(c) if $(\tilde{k}, \tilde{h}) \in \mathbb{R}^2_+$, then for all $(k', h') \in \Gamma(\tilde{k}, \tilde{h})$ and $\varepsilon > 0$ sufficiently small, there exists a neighborhood $\mathcal{N}$ of $(\tilde{k}, \tilde{h})$ in $\mathbb{R}^2_+$, such that $((1 - \varepsilon)k', (1 - \varepsilon)h') \in \Gamma(k, h)$ for each $(k, h) \in \mathcal{N}(\tilde{k}, \tilde{h})$.

The next two lemmas, although somewhat technical, are needed to develop the main results.

Lemma 2. *Assume that (A1) and (A2) are satisfied. Then*

(a) $\Pi(k_0, h_0)$ *is compact in the product topology, for all* $(k_0, h_0) \in \mathbb{R}^2_+$*;*

(b) *The mapping* $\Pi : \mathbb{R}^2_+ \to \left(\mathbb{R}^2_+\right)^\infty$ *is continuous in the product topology.*

Proof. First, it is shown that (A1) and (A2) are verified. From the definition given in (10), it is clear that $(0, 0) \in \Gamma(0, 0)$. In fact, Γ satisfies the stronger condition $\Gamma(0, 0) = \{(0, 0)\}$ in this case. For each $(k, h) \in \mathbb{R}^2_+$, by (8) and (9), the feasibility correspondence can be written as the Cartesian product of two closed intervals

$$\left[(1 + n)^{-1}(1 - \delta_k)k, (1 + n)^{-1}\left(Ak^\alpha h^{1-\alpha} + (1 - \delta_k)k\right)\right] \quad \text{and} \quad \left[(1 - \delta_h)h, (B\phi(1) + (1 - \delta_h))\, h\right],$$

which is a compact set. Thus, Γ is compact-valued. The continuity of Γ can be proved with standard arguments, hence it is omitted. Therefore, (A1) is satisfied.

Now suppose that $(k', h') \in \Gamma(k, h)$ for some $k, h \geq 0$. Applying the (weighted) generalized mean inequality[1] to the righ-hand side of (8) yields

$$k' \leq \frac{Ak^\alpha h^{1-\alpha} + (1 - \delta_k)k}{(1 + n)} \leq \frac{A\left(\alpha k + (1 - \alpha)h\right) + (1 - \delta_k)k}{(1 + n)} = \frac{\left(\alpha A + (1 - \delta_k)\right)k + (1 - \alpha)Ah}{(1 + n)}.$$

It immediately follows that $k' \leq \theta\|k, h\|$, where

$$\theta := \max\left\{\frac{\alpha A + (1 - \delta_k)}{(1 + n)}, \frac{(1 - \alpha)A}{(1 + n)}\right\}. \tag{14}$$

On the other hand, by (4), we have that $h' \leq [B\phi(1) + (1 - \delta_h)]\,\|k, h\|$. Set ζ to be the greater between θ and $B\phi(1) + (1 - \delta_h)$, and let $\zeta' = 0$. Clearly, $\zeta > 0$. The condition $\zeta \neq 1$ is also satisfied, for if $\zeta = 1$ would contradict that $B\phi(1) + (1 - \delta_h) > 1$. Hence, (A2) holds.

The result given in part (a) of this Lemma follows from Tychonoff's theorem, which states that the product of any collection of compact sets is compact with respect to the product topology. The proof for (b) can be found in Lemma 2 on Le Van and Morhaim (2002). ■

[1] Let $x_1, \ldots, x_n$ be positive real numbers and $w_i \in [0, 1]$, for each $i = 1, \ldots, n$, such that $\sum_i w_i = 1$. Then,

$$\prod_{i=1}^n x_i^{w_i} \leq \left(\sum_{i=1}^n w_i x_i^p\right)^{1/p} \quad \text{for any } p > 0.$$

Lemma 3. *Suppose that (A1), (A2), (A3), and (A4) are satisfied. Then, for all* $(k_0, h_0) \in \mathbb{R}^2_+$ *and for all* $(\underset{\sim}{k}, \underset{\sim}{h}) \in \Pi(k_0, h_0)$*, the sum* $\sum_{t=0}^{\infty} \beta^t F(k_t, h_t, k_{t+1}, h_{t+1})$ *is well defined and equals*

$$\lim_{T \to \infty} \sum_{t=0}^{T} \beta^t F(k_t, h_t, k_{t+1}, h_{t+1}).$$

Proof. Given that the validity of (A1) and (A2) has already been established in the proof of the previous lemma, it will suffice to show that (A3) and (A4) also hold. A detailed proof of the result for a general case is given in Le Van (2006) (Lemma 2.1.1 and Remark 2.2.1).

It is easily verified from (1) that U is pointwise bounded on $\mathbb{R}_+$. In particular, as $\sigma \to 0^+$, it follows that $U(c) \leq (c - 1)$, for all $c \geq 0$. From this fact, and given that $\psi(h, h') \leq 1$ for all $(h, h') \in \mathbb{R}^2_+$, we have from (12) that

$$\begin{aligned} F(k, h, k', h') &\leq Ak^{\alpha} \left(\psi(h, h')h\right)^{1-\alpha} + (1 - \delta_k)k - (1 + n)k' - 1, \\ &\leq Ak^{\alpha}h^{1-\alpha} + (1 - \delta_k)k - (1 + n)k' - 1, \end{aligned}$$

for every $(k, h, k', h') \in \text{gr}(\Gamma)$. Therefore,

$$\begin{aligned} F(k, h, k', h') &\leq A\left[\alpha k + (1 - \alpha)h\right] + (1 - \delta_k)k - (1 + n)k' - 1, \\ &\leq \left[\alpha A + (1 - \delta_k)\right]k + (1 - \alpha)Ah - (1 + n)k' - 1, \\ &\leq \zeta(1 + n)\|k, h\|, \\ &\leq \zeta(1 + n)\left(\|k, h\| + \|k', h'\|\right), \end{aligned}$$

where the generalized mean inequality is applied on the first line. Thus, (A3) is satisfied with $\eta := \zeta(1 + n)$ and $\eta' := 0$.

In order for (A4) to hold, we simply impose the condition $\beta\zeta < 1$, which is equivalent to

$$\beta \cdot \max\left\{\frac{\alpha A + (1 - \delta_k)}{(1 + n)}, \frac{(1 - \alpha)A}{(1 + n)}, B\phi(1) + (1 - \delta_h)\right\} < 1.$$

This completes the proof. ■

From Lemma 3, it is possible to define total discounted returns $J : \Pi \to \mathbb{R} \cup \{-\infty\}$ as

$$J(\underset{\sim}{k}, \underset{\sim}{h}) := \sum_{t=0}^{\infty} \beta^t F(k_t, h_t, k_{t+1}, h_{t+1}). \tag{15}$$

Then, problem (11) can be written more compactly as

$$\max\ \left\{J(\underset{\sim}{k}, \underset{\sim}{h}) : (\underset{\sim}{k}, \underset{\sim}{h}) \in \Pi(k_0, h_0)\right\}. \tag{16}$$

Proposition 4. *If (A1), (A2), (A3), and (A4) are satisfied, then a solution to problem* (16) *exists.*

Proof. Applying the result from Lemma 2.2.1 in Le Van (2006), the function $J(\underset{\sim}{k}, \underset{\sim}{h})$ given in (15) is upper semi-continuous in the product topology for each $(\underset{\sim}{k}, \underset{\sim}{h}) \in \Pi(k_0, h_0)$. Since $\Pi(k_0, h_0)$ is compact in the same topology, by Lemma 2, then J attains its maximum on $\Pi(k_0, h_0)$. ■

Let $V(k_0, h_0)$ be the value of problem (16). To further characterize the solution, it is useful to define the set of feasible paths for which total discounted returns are finite, that is,

$$\Pi'(k_0, h_0) := \left\{(\underset{\sim}{k}, \underset{\sim}{h}) \in \Pi(k_0, h_0) : J(\underset{\sim}{k}, \underset{\sim}{h}) > -\infty\right\}.$$

Lemma 5. $\Pi'(k_0, h_0)$ *is nonempty for all* $(k_0, h_0) \neq (0, 0)$.

Proof. If $0 < \sigma < 1$, then F is bounded below by $F(0, 0, 0, 0) = -(1-\sigma)^{-1} > -\infty$, so the result is immediate. Some analysis is required for the case $1 \leq \sigma < +\infty$. Let $(k_0, h_0) \neq (0, 0)$. Moreover, assume $u_t = \overline{u}$ for all $t = 0, 1, \ldots$, and

$$k_0 < \left(\frac{A}{n + \delta_k}\right)^{\frac{1}{1-\alpha}} \overline{u} h_0. \tag{17}$$

By Lemma 1 and the definition of $\overline{u}$, $h_t = h_0$, for all t. Then, it is possible to choose a constant consumption sequence $(c_0, c_0, \ldots)$ with $c_0 := A k_0^\alpha (\overline{u} h_0)^{1-\alpha} - (n + \delta_k) k_0 > 0$, such that

$$k_{t+1} = \frac{A k_0^\alpha (\overline{u} h_0)^{1-\alpha} + (1-\delta_k) k_0 - c_0}{1+n} = k_0, \qquad \text{for all } t = 0, 1, \ldots$$

Clearly, the constant sequences $\underset{\sim}{k}_0 := (k_0, k_0, \ldots)$ and $\underset{\sim}{h}_0 := (h_0, h_0, \ldots)$ are in $\Pi(k_0, h_0)$, and

$$J(\underset{\sim}{k}_0, \underset{\sim}{h}_0) = \frac{U[A k_0^\alpha (\overline{u} h_0)^{1-\alpha} - (n + \delta_k) k_0]}{1 - \beta} > -\infty.$$

Now, if (17) is not satisfied, i.e., $k_0 \geq (A/(n + \delta_k))^{\frac{1}{1-\alpha}} (\overline{u} h_0)$, set $k_1 > 0$ so that

$$\begin{aligned} c_0 &= A k_0^\alpha (\overline{u} h_0)^{1-\alpha} + (1-\delta_k) k_0 - (1+n) k_1 > 0, \quad \text{and} \\ c_1 &= A k_1^\alpha (\overline{u} h_0)^{1-\alpha} - (n + \delta_k) k_1 > 0, \end{aligned} \tag{18}$$

which in turn requires that

$$k_1 < \min\left\{\frac{A k_0^\alpha (\overline{u} h_0)^{1-\alpha} + (1-\delta_k) k_0}{(1+n)}, \left(\frac{A}{n + \delta_k}\right)^{\frac{1}{1-\alpha}} \overline{u} h_0\right\}.$$

Note that, by construction, $c_t = c_1$ implies $k_t = k_1$, for all $t \geq 1$. Then, there exist sequences $\underset{\sim}{h}_0 = (h_0, h_0, h_0, \ldots)$ and $\underset{\sim}{k}_1 = (k_0, k_1, k_1, \ldots)$ that are feasible from (k_0, h_0), whose total discounted value $J(\underset{\sim}{k}_1, \underset{\sim}{h}_0)$ is given by

$$U\left[Ak_0^{\alpha}\left(\overline{u}h_0\right)^{1-\alpha} + (1-\delta_k)k_0 - (1+n)k_1\right] + \frac{\beta}{1-\beta}U\left[Ak_1^{\alpha}\left(\overline{u}h_0\right)^{1-\alpha} - (n+\delta_k)k_1\right] > -\infty.$$

This completes the proof. ■

Let S be the space of functions $f : \mathbb{R}_+^2 \to \mathbb{R} \cup \{-\infty\}$ that are upper semicontinuous and satisfy:

(i) for all $(k_0, h_0) \in \mathbb{R}_+^2$ and for all $(\underset{\sim}{k}, \underset{\sim}{h}) \in \Pi(k_0, h_0)$, $\limsup_{T\to+\infty} \beta^T f(k_T, h_T) \leq 0$;

(ii) for all $(k_0, h_0) \in \mathbb{R}_+^2$ such that $\Pi'(k_0, h_0)$ is not empty, and for all $(\underset{\sim}{k}, \underset{\sim}{h}) \in \Pi'(k_0, h_0)$,

$$\lim_{T\to+\infty} \beta^T f(k_T, h_T) = 0.$$

Proposition 6. *Suppose that (A1), (A2), (A3), (A4), and (A5) hold.*

(a) *The value function $V : \mathbb{R}_+^2 \to \mathbb{R} \cup \{-\infty\}$ satisfies the following Bellman equation*

$$V(k, h) = \max\left\{F(k, h, k', h') + \beta V(k', h') : (k', h') \in \Gamma(k, h)\right\}, \tag{19}$$

for all $(k, h) \in \mathbb{R}_+^2$.

(b) *V is the unique solution to the Bellman equation (19) in the space S.*

(c) *If, in addition, (A6) is satisfied, then V is continuous for any $(\underset{\sim}{k}, \underset{\sim}{h}) \in \Pi'(k_0, h_0)$, and if $V(k_0, h_0) = -\infty$, then for any sequence $\{(k_n, h_n)\}_{n=0}^{\infty}$ that converges to (k_0, h_0), if follows that $\lim\limits_{n\to\infty} V(k_n, h_n) = -\infty$.*

Proof. In order to prove parts (a) and (b) of this Proposition, it remains to verify that (A5) holds, and the results follow from Theorem 2 in Le Van and Morhaim (2002). Standard continuity arguments apply, hence the proof is omitted. Next we show that (A6) is satisfied.

Note from (7) that ψ is homogeneous of degree zero in (h, h'). Moreover, from (8) and (9), if $(k', h') \in \Gamma(k, h)$ then $(\lambda k', \lambda h') \in \Gamma(\lambda k, \lambda h)$, for every $\lambda \geq 0$; that is, Γ is a cone. Hence, (A6)(a) holds. By the definition of U in (1), it follows that for every $c > 0$ and $\lambda > 0$,

$$U(\lambda c) = \frac{(\lambda c)^{1-\sigma} - 1}{1-\sigma} = \frac{\lambda^{1-\sigma}(c^{1-\sigma} - 1)}{1-\sigma} + \frac{\lambda^{1-\sigma} - 1}{1-\sigma} = \lambda^{1-\sigma}U(c) + U(\lambda). \tag{20}$$

Let $\Phi_1(\lambda) := \lambda^{1-\sigma}$ and $\Phi_2(\lambda) := U(\lambda)$ for all $\lambda \in (0, 1]$. Then (20) implies that

$$F(\lambda k, \lambda h, \lambda k', \lambda h') = \Phi_1(\lambda)F(k, h, k', h') + \Phi_2(\lambda),$$

with $\Phi_1(1) = 1$ and $\Phi_2(1) = 0$, for all $(k, h, k', h') \in \operatorname{gr}\Gamma$, which also implies that (A6)(b) holds. Now, assume that $(k, h) \gg (0, 0)$,

$$\frac{(1-\delta_k)k}{(1+n)} < k' \leq \frac{Ak^{\alpha}h^{1-\alpha} + (1-\delta_k)k}{(1+n)}, \quad \text{and} \quad (1-\delta_h)h < h' \leq [B\phi(1) + (1-\delta_h)]h.$$

Then, there exists $\varepsilon > 0$ such that

$$\frac{(1-\delta_k)k}{(1+n)} < (1-\varepsilon)k' < \frac{Ak^{\alpha}h^{1-\alpha} + (1-\delta_k)k}{(1+n)}, \quad \text{and}$$
$$(1-\delta_h)h < (1-\varepsilon)h' < [B\phi(1) + (1-\delta_h)]h.$$

Since both lower and upper bounds are continuous functions of k and h, the above inequalities also hold for every $(\tilde{k}, \tilde{h}) \in \mathcal{N}(k, h)$. If $k \geq 0$, $h \geq 0$,

$$\frac{(1-\delta_k)k}{(1+n)} = k' < \frac{Ak^{\alpha}h^{1-\alpha} + (1-\delta_k)k}{(1+n)}, \quad \text{and} \quad (1-\delta_h)h = h' < [B\phi(1) + (1-\delta_h)]h,$$

simply multiply both sides of the left-hand side equalities by $(1-\varepsilon)$ and both conditions are also verified for all $(\tilde{k}, \tilde{h}) \in \mathcal{N}(k, h)$. Part (c) follows. ■

Remark 7. Note that every result in Proposition 6 remains valid if $\sigma = 1$, i.e., utility is logarithmic. In particular, for $U(c) = \log c$, it follows that for all $\lambda \in (0, 1]$,

$$\log(\lambda c) = \log c + \log \lambda = \lambda^0 \log c + \log \lambda,$$

thus (20) is satisfied. This in turn implies that $\Phi_1(\lambda) = 1$ and $\Phi_2(\lambda) = \log \lambda$. Hence, a separate treatment for logarithmic utility, as in Alvarez and Stokey (1998) or Le Van and Morhaim (2002) is not needed with our set of assumptions.

4 Concluding remarks

To conclude, it is shown that all the results presented in the previous section extend to the model with externalities à la Lucas (1988). Let h_{at} denote the average level of human capital in the population. Output technology is now given by $Ak_t^{\alpha}(u_t h_t)^{1-\alpha}(h_{at})^{\gamma}$, with $\gamma \geq 0$. In this case, the social planner's problem is similar to (11), with an additional equilibrium condition given by $h_{at} = h_t$, for all t. Then, the resource constraint takes the form

$$c_t + (1+n)k_{t+1} \leq Ak_t^{\alpha}u_t^{1-\alpha}h_t^{1-\alpha+\gamma} + (1-\delta_k)k_t, \qquad t = 0, 1, \dots, \tag{21}$$

which is no longer homogeneous of degree one in (k_t, h_t), provided that $\gamma > 0$.[2] However, it is possible to keep this convenient feature of the model and apply the methods developed above, by making a change of variables. Let

$$\hat{h}_t := h_t^{\frac{1-\alpha+\gamma}{1-\alpha}}. \tag{22}$$

[2] It is convenient to allow $\gamma = 0$, which reduces the model to the case without externalities, for ease of comparison.

Clearly, output technology is homogeneous of degree one in k_t and the transformed variable $\hat{h}_t$, i.e., $Ak_t^{\alpha}u_t^{1-\alpha}\hat{h}_t^{1-\alpha}$. Substituting (22) in (4) yields

$$\hat{h}_{t+1} = [B\phi(v_t) + (1-\delta_h)]^{\frac{1-\alpha+\gamma}{1-\alpha}}\,\hat{h}_t,$$

hence the human capital accumulation technology is still linear in $\hat{h}_t$ and concave in v_t.[3] Finally, define a function $\widehat{\psi} : \mathbb{R}_+^2 \to [0,1]$ by

$$\widehat{\psi}(\hat{h}_t, \hat{h}_{t+1}) = \begin{cases} 1 & \text{if } \ 0 \le \hat{h}_{t+1} \le d\hat{h}_t, \\ 1 - \phi^{-1}\left\{ \frac{1}{B}\left[\left(\frac{\hat{h}_{t+1}}{\hat{h}_t} \right)^{\frac{1-\alpha}{1-\alpha+\gamma}} - d^{\,\frac{1-\alpha}{1-\alpha+\gamma}} \right] \right\} & \text{if } \ d\hat{h}_t \le \hat{h}_{t+1} \le D\hat{h}_t, \end{cases} \tag{23}$$

where

$$d := (1-\delta_h)^{\frac{1-\alpha+\gamma}{1-\alpha}} \quad \text{and} \quad D := [B\phi(1) + (1-\delta_h)]^{\frac{1-\alpha+\gamma}{1-\alpha}},$$

to play the same role as $\psi(h_t, h_{t+1})$ in the problem without externalities. Changing the definitions of $\Gamma(k_t, \hat{h}_t)$ and $F(k_t, \hat{h}_t, k_{t+1}, \hat{h}_{t+1})$ accordingly, it is easy to verify that all arguments developed in the previous section apply to the social planner's problem (11) reformulated appropriately with the transformed variable.

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[3] Let $g(v) := B[\phi(v) + (1-\delta_h)]$. Then $h_{t+1} = g(v_t)h_t$ is clearly concave in v_t for all $h_t \ge 0$. Define $G : \mathbb{R}_+ \to \mathbb{R}_+$ by $G(y) = y^{\frac{1-\alpha+\gamma}{1-\alpha}}$. The change of variable implies that $\hat{h}_{t+1} = G(g(v_t))\,\hat{h}_t$ with G being strictly increasing, so the concavity of $(G \circ g)(v_t)$ is preserved.