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DOCUMENTO DE TRABAJO N° 312

Marzo de 2024

Citar como:
Fioravanti, Federico (2024). Fuzzy Classification Aggregation. Documento de trabajo RedNIE N°312.

Fuzzy Classification Aggregation

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Abstract

We consider the problem where a set of individuals has to classify m objects into p categories and does so by aggregating the individual classifications. We show that if $m \geq 3$, $m \geq p \geq 2$, and classifications are fuzzy, that is, objects belong to a category to a certain degree, then an independent aggregator rule that satisfies a weak unanimity condition belongs to the family of Weighted Arithmetic Means. We also obtain characterization results for $m = p = 2$.
Keywords: Classification Aggregation; Weighted Arithmetic Mean; Fuzzy Setting.
JEL Classification: D71.

1 Introduction

The study of the problem of individuals classifying objects can be traced back to Kasher and Rubinstein (1997). In their paper, they consider a finite

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I am grateful to Fernando Tohmé, Ulle Endriss, Agustín Bonifacio, and Jordi Massó for comments and suggestions that led to an improvement of the paper. The author acknowledges support from the Nederlandse Organisatie voor Wetenschappelijk Onderzoek Vici grant 639.023.811.

society that has to determine which one of its subsets of members consists of exactly those individuals that can be deemed to be part of a group named J. They propose different sets of axioms characterizing three aggregators: the liberal one, where each individual decides by herself whether she belongs or not to the group; the oligarchic one, where a subset of individuals defines who belongs to the group; and the dictatorial, where a unique individual decides who is a J. Kasher and Rubinstein (1997) gave rise to what can be regarded as a subdomain within social choice theory, namely, the study of the Group Identification Problem (see, among others, Samet and Schmeidler, 2003; Miller, 2008; Fioravanti and Tohmé, 2021, for more details).

A more general problem, where a group of individuals has to classify m objects into p different categories, has been considered by Maniquet and Mongin (2016). In their work, they study the case where there are at least as many objects as categories, at least three categories and all the categories must be filled with at least one object.[1] This aspect is combined with other three seemingly natural properties, namely Unanimity, which states that when all individuals make the same classification then the aggregate classification complies; Independence, which is the requirement that an object be classified by the aggregator in the same way at any two classification profiles if every individual classifies it equally in both profiles; and Non-Dictatorship, which requires that there is no individual such that her classification is always selected. The result is that there is no aggregator satisfying these three conditions. Recently, Cailloux et al. (2024) weaken the Unanimity axiom and find a weakening of the impossibility result that holds for $m \geq 3$ and $p \geq 2$, with the existence of an essential dictator. This is an individual such that a permutation of her classification is always selected. They also characterize the unanimous and independent aggregators when $m = p = 2$.

Our work considers the problem where m objects have to be classified into p categories, $m \geq p \geq 2$, and the classifications, both by the individuals and the rule, indicate degrees of membership of the objects to each of the categories. Consider a scenario where a national government allocates equal funds to all regions, with the stipulation that the central government determines how the funds are distributed. For instance, funds might be allocated for security, education, and health, ensuring that each category, across the country, receives at least the amount provided by the government to each

[1] This surjectivity condition becomes relevant, for example, in situations where m workers have to be assigned to p tasks, and no task can be left unassigned.

region. As an illustration, one region could be tasked with allocating half of the funds to security, a quarter to education, and a quarter to health. Another scenario involves assigning individuals to various mandatory tasks with equal time requirements, where proportions dictate how each person's time is allocated. For instance, if tasks include laundry, lawn mowing, and grocery shopping, Bob might spend half of his time on laundry, a quarter on mowing the lawn, and the remaining quarter on grocery shopping.

Fuzzy preferences, those that represent vagueness and uncertainty, are a useful tool that has been used in many aggregation problems. Noteworthy contributions include the work of Dutta (1987), who deals with exact choices under vague preferences, Dutta et al. (1986) who investigate the structure of fuzzy aggregation rules determining fuzzy social orderings, and recent work by Duddy and Piggins (2018), who prove the fuzzy counterpart of Weymark's general oligarchy theorem (Weymark, 1984), and by Raventos-Pujol et al. (2020), who analyze Arrow's (1951) theorem in a fuzzy setting.

Numerous studies have addressed the Group Identification Problem when the preferences or classifications are not crisp. For instance, Cho and Park (2018) present a model of group identification for more than two groups, allowing fractional classifications but no fractional opinions, Ballester and García-Lapresta (2008) deal with fuzzy opinions in a sequential model, and Fioravanti and Tohmé (2022) show that some of the impossibility results of Kasher and Rubinstein (1997) can be avoided. Alcantud et al. (2019) analyze the classification problem in a fuzzy setting, and consider a strong fuzzy counterpart of the surjectivity condition, extending the impossibility result of Maniquet and Mongin (2016).

We avoid both impossibility results, the crisp and the fuzzy ones, by considering a different, but also very natural, fuzzy surjectivity condition. Building on some previous results in functional analysis by Aczél and Wagner (1980); Aczél and Wagner (1981), and Wagner (1982), we show that the only aggregators that are independent and that satisfy a weak version of Unanimity, are the Weighted Arithmetic Means.

Section 2 presents the basic notions and axioms that we use, while we present the results in Section 3. Finally, Section 4 contains some concluding remarks.

2 Basic Notions and Axioms

Let $N = \{1, \ldots, n\}$ be a finite set of agents and let $X = \{x_1, \ldots, x_m\}$ be m objects that need to be classified into the p categories of a set P, with $m \geq p \geq 2$.

The individuals classify each object according to a partial degree of membership to each category. In the crisp setting, classifications are surjective mappings $c : X \to P$. In the fuzzy setting, a *fuzzy classification* is a mapping $c : X \to [0,1]^p$ such that $\sum_{j=1}^m c(x_j)_t \geq 1$ for each $t \in \{1, \ldots, p\}$ and $\sum_{t=1}^p c(x)_t = 1$ for all $x \in X$. The former condition is the fuzzy counterpart of the surjectivity of the classification function, while the latter is the fuzzy counterpart of the assumption that every object x_j must be assigned to exactly one category by each voter. It is easy to see that when $m = p$, then all the inequalities turn into equalities.

We use $\mathcal{C}$ to denote the set of fuzzy classifications, and every $\boldsymbol{c} = (c_1, \ldots, c_n) \in \mathcal{C}^N$ is a *fuzzy classification profile*. Given $\mathbf{c} \in \mathcal{C}^N$ and $x \in X$, we denote with $\mathbf{c}^x \in [0,1]^{N \times P}$ the *fuzzy classification profile restricted to* x, such that the entry $\mathbf{c}^x_{ij}$ indicates the degree of membership of the object x to the category j, according to the agent i. Thus, $\boldsymbol{c}^x_{ij} = c_i(x)_j$.

A *fuzzy classification aggregation function* (FCAF) is a mapping $\alpha : \mathcal{C}^N \to \mathcal{C}$ such that $\alpha(\boldsymbol{c})(x)$ indicates the degrees of membership to the different categories of the object x. We call the outcome of α, the *fuzzy social classification.*

Next, we introduce a particular FCAF, the *Weighted Arithmetic Mean.* Let $\boldsymbol{w} = (w_1, \ldots, w_n)$ be a set of weights such that $w_i \in [0,1]$ for all $i \in N$ and $\sum_{i=1}^n w_i = 1$. Then $\alpha_{\boldsymbol{w}} : \mathcal{C}^n \to \mathcal{C}$ is such that for all $x \in X$,

$$\alpha_{\boldsymbol{w}}(\boldsymbol{c})(x) = w_1 c_1(x) + \ldots + w_n c_n(x).$$

We say that the set of weights is *degenerate* if there is an $i \in N$ such that $w_i = 1$. If we think of the FCAF as a group of experts classifying objects, this FCAF can be appropriate for situations where the individuals differ in expertise, with more experienced voters having higher weights. In particular, if $w_i = \frac{1}{n}$ for all $i \in N$, we call this FCAF the Arithmetic Mean.

Example 1. *Consider a situation where there are $n = 3$ individuals that have to classify $m = 3$ objects into $p = 3$ categories. Let $\boldsymbol{c}$ be a fuzzy classification profile such that:*

$$\boldsymbol{c}^{x_1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix},\ \boldsymbol{c}^{x_2} = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},\ \textit{and}\ \boldsymbol{c}^{x_3} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}.$$

If we consider the Weighted Arithmetic Mean with a set of non-degenerate weights $\boldsymbol{w} = (\frac{1}{2}, 0, \frac{1}{2})$*, we obtain the following social classifications:*

$$\alpha_{\boldsymbol{w}}(\boldsymbol{c})(x_1) = (\tfrac{10}{24}, \tfrac{11}{24}, \tfrac{3}{24}),\ \alpha_{\boldsymbol{w}}(\boldsymbol{c})(x_2) = (\tfrac{14}{24}, 0, \tfrac{10}{24}),\ \textit{and}\ \alpha_{\boldsymbol{w}}(\boldsymbol{c})(x_3) = (0, \tfrac{13}{24}, \tfrac{11}{24}).$$

In the following, we introduce some axioms that an FCAF may satisfy. The first axiom states that the fuzzy social classification of an object in two different fuzzy classification profiles does not change if the classification regarding that object is the same in both profiles for every individual.

Definition 1 (Independence)**.** *Let* $\boldsymbol{c}, \boldsymbol{c'} \in \mathcal{C}^N$ *and* $x \in X$ *be such that* $\boldsymbol{c}_i(x) = \boldsymbol{c}'_i(x)$ *for all* $i \in N$*. If an FCAF is independent, then* $\alpha(\boldsymbol{c})(x) = \alpha(\boldsymbol{c'})(x)$*.*

An independent FCAF α can be seen as a collection of mappings $(\alpha_x)_{x \in X}$, such that $\alpha_x : [0,1]^{N \times P} \to [0,1]^P$ and $\alpha_x(\boldsymbol{c}^x) = \alpha(\boldsymbol{c})(x)$ (Cailloux et al., 2024). We call these mappings *Elementary* FCAFs. A stronger version of Independence requires that if two objects are classified equally in a profile, then the fuzzy social classification must be the same.[2]

Definition 2 (Symmetry)**.** *An FCAF is symmetric if for all* $x, y \in X$ *and all* $\boldsymbol{c} \in \mathcal{C}$ *such that* $\boldsymbol{c}_i(x) = \boldsymbol{c}_i(y)$ *for all* $i \in N$*, it is the case that* $\alpha(\boldsymbol{c})(x) = \alpha(\boldsymbol{c})(y)$*.*

Wagner (1982) shows that a symmetric FCAF is also independent, and thus has the same elementary FCAF for every object. The next property states that if there is an object that is unanimously classified by the individuals, then the FCAF has to classify that object accordingly.

Definition 3 (Unanimity)**.** *An FCAF is unanimous if for all* $\boldsymbol{c} \in \mathcal{C}^N$ *where there is an* $x \in X$ *and a category* $t \in P$ *such that* $c_1(x)_t = \cdots = c_n(x)_t = r$*, it is the case that* $\alpha(\boldsymbol{c})(x)_t = r$*.*

A weaker version of Unanimity states that the only unanimous classification required to be respected is when all the individuals classify an object with a degree of membership 0. The intuition behind this axiom is that if all individuals consider that an object does not belong at all to a certain category, then the social classification should agree.

[2]This property is called Strong Label Neutrality in Wagner (1982).

Definition 4 (Zero Unanimity)**.** *An FCAF is zero unanimous if for all* $\boldsymbol{c} \in \mathcal{C}$ *where there is an* $x \in X$ *and a category* $t \in P$ *such that* $c_1(x)_t = \cdots = c_n(x)_t = 0$, *it is the case that* $\alpha(\boldsymbol{c})(x)_t = 0$.

Now we introduce our first fuzzy axiom, which can be seen as the fuzzy counterpart of Unanimity.[3] It states that the degree to which an object is collectively classified must be between the classification degrees regarding that object for each of the individuals.

Definition 5 (Fuzzy Consensus)**.** *An FCAF satisfies fuzzy consensus if* $\alpha(\mathbf{c})(x)_t \in [\min_{i\in N} c_i(x)_t, \max_{i\in N} c_i(x)_t]$, *for every object* $x \in X$ *and for every category* $t \in \{1, \ldots, p\}$.

It is easy to see that Fuzzy Consensus implies Unanimity, and Unanimity implies Zero Unanimity. These three axioms can be interpreted as the different 'degrees' of consensus that we might require an aggregator to satisfy. The next axiom states that there must not exist an individual that imposes her classification.

Definition 6 (Non-Dictatorship)**.** *An FCAF is Non-Dictatorial if there is no individual* $i \in N$ *such that for all* $x \in X$, *it is the case that* $\alpha(\mathbf{c})(x) = c_i(x)$.

A stronger version of Non-Dictatorship states that the names of the individuals are not important for the aggregation of classifications.

Definition 7 (Anonymity)**.** *For a classification profile* $\boldsymbol{c} \in \mathcal{C}$ *and a permutation* $\sigma : N \to N$ *we define* $\sigma(\boldsymbol{c})$ *as the classification profile such that* $\sigma(\boldsymbol{c}) = \{\boldsymbol{c}_{\sigma(1)}, \ldots, \boldsymbol{c}_{\sigma(n)}\}$. *An FCAF is anonymous if* $\alpha(\sigma(\boldsymbol{c})) = \alpha(\boldsymbol{c})$.

3 Results

In the crisp setting, Maniquet and Mongin (2016) show the following impossibility result.

Theorem 1. (Maniquet and Mongin, 2016) *Let* $m \geq p \geq 3$. *There is no (non-fuzzy) Classification Aggregation Function that satisfies Independence, Unanimity, and Non-Dictatorship.*

[3]This axiom is called *coherence* by Alcantud et al. (2019). Here we use the same name used by Fioravanti and Tohmé (2022).

Alcantud et al. (2019) extend this result to a fuzzy setting, by using Fuzzy Consensus and considering a stronger notion of surjectivity, where for each category there must exist an object with a degree of classification larger than 0.5. They show the existence of a fuzzy dictator, where whenever the dictator classifies an object into a category by more than 0.5, then the object is classified into that category with a degree of more than 0.5. Thus, our main theorem can be seen as an escape from these impossibility results, even in a fuzzy setting.

Theorem 2. *Let $m \geq 3$ and $m \geq p \geq 2$. A Fuzzy Classification Aggregation Function satisfies Independence, Zero Unanimity, and Non-Dictatorship if, and only if, it is a Weighted Arithmetic Mean with a non-degenerate set of weights.*

Now we present a result by Aczél and Wagner (1981) that is useful for our proof.

Theorem. (Aczél and Wagner, 1981) *A family of mappings $\{\boldsymbol{c}^j : [0,1]^N \to [0,1]\}_{j=1}^m$ satisfies the following conditions:*

1. $\boldsymbol{c}^j(0,\ldots,0) = 0$ *for* $j = 1,\ldots,m$, *and*
2. *if* $\sum_{j=1}^m (x_{j1},\ldots,x_{jn}) = (1,\ldots,1)$, *then* $\sum_{j=1}^m \boldsymbol{c}^j(x_{j1},\ldots,x_{jn}) = 1$

if, and only if, there exists a set of weights $(w_1,\ldots,w_n)$ such that for $j = 1,\ldots,m$ we have that

$$\boldsymbol{c}^j(x_{j1},\ldots,x_{jn}) = w_1 x_{j1} + \cdots + w_n x_{jn}.$$

Proof of Theorem 2. That a Weighted Arithmetic Mean with a non-degenerate set of weights satisfies the three axioms is straightforward to see. For the if part, we use the result by Aczél and Wagner (1981). Let α be an FCAF that satisfies Independence and Zero Unanimity. Recall that an independent FCAF can be seen as a collection of elementary FCAFs. So we can consider a family of mappings $\{\alpha_{x_j} : [0,1]^{N\times P} \to [0,1]^P\}_{j=1}^m$. For a fixed category $t \in P$, we have that conditions (1) and (2) are satisfied by $(\alpha_{x_j})_t$.[4] Thus we have that for each category t there is a set of weights $(w_1^t,\ldots,w_n^t)$ such that for $j = 1,\ldots,m$, it is the case that[5]

$$(\alpha_{x_j})_t((x_{j1})_t,\ldots,(x_{jn})_t) = w_1^t (x_{j1})_t + \cdots + w_n^t (x_{jn})_t.$$

[4] $(\alpha_{x_j})_t$ is the projection of α_{x_j} over the coordinate t.
[5] $(x_{ji})_t$ is the projection of x_{ji} over the coordinate t.

What is left to show is that for every $i \in N$, the weights are the same for every category, meaning that $w_i^t = w_i^{t'}$ for all $t, t' \in P$. To avoid excessive notation, we show it for the case of $|N| = 2$, and $m = p = 3$. It is easy to extend it for the general case. Consider the example where $c_1(x_1) = (1, 0, 0)$, $c_1(x_2) = (0, 1, 0)$, $c_1(x_3) = (0, 0, 1)$, $c_2(x_1) = (0, 1, 0)$, $c_2(x_2) = (0, 0, 1)$, and $c_2(x_3) = (1, 0, 0)$. Thus we obtain $\alpha_{x_1}(c_1(x_1), c_2(x_1)) = (w_1^1, w_2^2, 0)$, $\alpha_{x_2}(c_1(x_2), c_2(x_2)) = (0, w_1^2, w_2^3)$, and $\alpha_{x_3}(c_1(x_3), c_2(x_3)) = (w_2^1, 0, w_1^3)$, where every three dimensional vector adds up to one. Thus we obtain the following system of equations:

$$\begin{cases} w_1^1 + w_2^1 = 1 & (1) \\ w_1^2 + w_2^2 = 1 & (2) \\ w_1^3 + w_2^3 = 1 & (3) \\ w_1^1 + w_2^2 = 1 & (4) \\ w_1^2 + w_2^3 = 1 & (5) \\ w_2^1 + w_1^3 = 1 & (6) \end{cases}$$

From (1) and (4) we obtain that $1 - w_2^1 + w_2^2 = 1$, thus $w_2^1 = w_2^2$, and from (2) and (5) we obtain that $1 - w_2^2 + w_2^3 = 1$, thus $w_2^2 = w_2^3$. It is a similar procedure for the weights of agent 1. So the FCAF associates to every individual $i \in N$ a weight w_i such that $w_i \geq 0$ and that $\sum_{i=1}^n w_i = 1$, and as it is non-dictatorial, we have that there is no $i \in N$ such that $w_i = 1$. Thus the FCAF is a Weighted Arithmetic Mean with a non-degenerate set of weights. □

The impossibility result from the crisp setting is obtained when the aggregator is a Weighted Arithmetic Mean and the set of weights is degenerate. As the aggregator must have a crisp outcome, this is the only possible set of weights (except for permutations on the name of the individuals). One notable aspect of this result is that the weight assigned to each individual remains consistent across all categories, with no category receiving a higher weight than the others. An implication of our main result is that if we want all individuals to be considered the same, then the weights must be the same for all individuals.

Corollary 1. *Let $m \geq 3$ and $m \geq p \geq 2$. A Fuzzy Classification Aggregation Function satisfies Independence, Zero Unanimity, and Anonymity if, and only if, it is an Arithmetic Mean.*

As the Weighted Arithmetic Means with non-degenerate weights satisfy Unanimity and Fuzzy Consensus, stronger versions of Zero Unanimity, we have the following two corollaries.

Corollary 2. *Let $m \geq 3$ and $m \geq p \geq 2$. A Fuzzy Classification Aggregation Function satisfies Independence, Unanimity, and Non-Dictatorship if, and only if, it is a Weighted Arithmetic Mean with a non-degenerate set of weights.*

Corollary 3. *Let $m \geq 3$ and $m \geq p \geq 2$. A Fuzzy Classification Aggregation Function satisfies Independence, Fuzzy Consensus, and Non-Dictatorship if, and only if, it is a Weighted Arithmetic Mean with a non-degenerate set of weights.*

Thus, both impossibility results by Maniquet and Mongin (2016) and Alcantud et al. (2019) are avoided in our setting, even if we use their same 'consensual' axioms.

We emphasize that Independence does not impose any condition between objects, in the sense that we can use different elementary FCAFs for each object. But the combination of Independence and Zero Unanimity\Unanimity\Fuzzy Consensus, in this setting with its particular surjectivity conditions, forces the FCAF to also satisfy the symmetry condition, thus obtaining the following collary.

Corollary 4. *Let $m \geq 3$ and $m \geq p \geq 2$. A Fuzzy Classification Aggregation Function that satisfies Fuzzy Consensus, and Non-Dictatorship if, and only if, it is a Weighted Arithmetic Mean with a non-degenerate set of weights.*

It is worth mentioning that the previous results are only valid for $m > 2$. For the case of 2 objects and 2 categories, Independence is trivially satisfied (due to the surjectivity conditions). This forces us to use stronger axioms to obtain a characterization.

For symmetric and Zero Unanimous FCAFs, we can have FCAFs that are not Weighted Arithmetic Means, as the following result shows.

Theorem 3. *Let $m = p = 2$. A symmetric Fuzzy Classification Aggregation Function α satisfies Zero Unanimity if, and only if, there is a function $h : [-\frac{1}{2}, \frac{1}{2}]^n \to [-\frac{1}{2}, \frac{1}{2}]$ where*

$$h(x_1, \ldots, x_n) = -h(-x_1, \ldots, -x_n), \tag{1}$$

and

$$h\left(\frac{1}{2},\dots,\frac{1}{2}\right)=\frac{1}{2}, \tag{2}$$

such that for each $\boldsymbol{c}\in\mathcal{C}^N$*,*

$$\alpha(\boldsymbol{c})(x_t)=\left(h\left(\boldsymbol{c}_{11}^{x_t}-\frac{1}{2},\dots,\boldsymbol{c}_{n1}^{x_t}-\frac{1}{2}\right)+\frac{1}{2},h\left(\boldsymbol{c}_{12}^{x_t}-\frac{1}{2},\dots,\boldsymbol{c}_{n2}^{x_t}-\frac{1}{2}\right)+\frac{1}{2}\right)$$

for $t=1,2$*.*

Before the proof, we present the folowing result by Wagner (1982).

Theorem. (Wagner, 1982) *Let* $m=p=2$*. A symmetric mapping* $\{\boldsymbol{c}:[0,1]^{N\times 2}\to[0,1]^2\}_{j=1}^m$ *satisfies the following conditions:*

1. $\boldsymbol{c}^{x_j}(0,\dots,0)=0$ *for* $j=1,\dots,m$*, and*
2. *if* $\sum_{j=1}^2(x_{j1},\dots,x_{jn})=(1,\dots,1)$*, then* $\sum_{j=1}^2\boldsymbol{c}^{x_j}(x_{j1},\dots,x_{jn})=1$

if, and only if, there exists a function $h:[-\frac{1}{2},\frac{1}{2}]^n\to[-\frac{1}{2},\frac{1}{2}]$ *where*

$$h(x_1,\dots,x_n)=-h(-x_1,\dots,-x_n), \tag{3}$$

and

$$h\left(\frac{1}{2},\dots,\frac{1}{2}\right)=\frac{1}{2}, \tag{4}$$

such that for each $(x_1,x_2)\in[0,1]^2$*,*

$$\boldsymbol{c}(x_1,x_2)=\left(h\left(x_{11}-\frac{1}{2},\dots,x_{n1}-\frac{1}{2}\right)+\frac{1}{2},h\left(x_{12}-\frac{1}{2},\dots,x_{n2}-\frac{1}{2}\right)+\frac{1}{2}\right).$$

Proof of Theorem 3. That such an FCAF satisfies Zero Unanimity is straightforward to see. For the if part, we use the result by Wagner (1982). Thus, we have for each category, a mapping h satisfying Wagner Theorem's conditions, namely h^1 and h^2. What is left to see is that $h^1=h^2$. We have the following conditions that must be satisfied by $\alpha(\boldsymbol{c})(x_1)$ and $\alpha(\boldsymbol{c})(x_2)$:

$$h^1\left(\boldsymbol{c}_{11}^{x_1}-\frac{1}{2},\dots,\boldsymbol{c}_{n1}^{x_1}-\frac{1}{2}\right)+\frac{1}{2}+h^2\left(\boldsymbol{c}_{12}^{x_1}-\frac{1}{2},\dots,\boldsymbol{c}_{n2}^{x_1}-\frac{1}{2}\right)+\frac{1}{2}=1,$$

and

$$h^1\left(\boldsymbol{c}_{11}^{x_1}-\frac{1}{2},\ldots,\boldsymbol{c}_{n1}^{x_1}-\frac{1}{2}\right)+\frac{1}{2}+h^1\left(\boldsymbol{c}_{11}^{x_2}-\frac{1}{2},\ldots,\boldsymbol{c}_{n1}^{x_2}-\frac{1}{2}\right)+\frac{1}{2}=1.$$

It is easy to see that $\boldsymbol{c}_{j2}^{x_1}=\boldsymbol{c}_{j1}^{x_2}$ for all $i=1,\ldots,n$. Thus we obtain that $h^1\left(\boldsymbol{c}_{11}^{x_2}-\frac{1}{2},\ldots,\boldsymbol{c}_{n1}^{x_2}-\frac{1}{2}\right)=h^2(\boldsymbol{c}_{11}^{x_2}-\frac{1}{2},\ldots,\boldsymbol{c}_{n1}^{x_2}-\frac{1}{2})$, concluding our proof. □

Despite its technical flavor, Theorem 3 has as a particular case the class of Weighted Arithmetic Means, and also any other power mean with an odd exponent.[6] Proposition 1 of Cailloux et al. (2024) is a particular case of Theorem 3, where condition 1 is implied by the complementary condition of their result.

A more general result, valid for $m\geq p\geq 2$ can be attained if we slightly change the surjectivity conditions. For this general case, we consider fuzzy classifications $c:X\to\mathbb{R}$, and the surjectivity requirements are such that for all $x\in X$, and for a given $s\in\mathbb{R}$, it is the case that $\sum_{t=1}^{p}c(x)_t=s$, and that $\sum_{j=1}^{m}c(x_j)_t\geq s$ if $s\geq 0$ for all $t\in P$, or $\sum_{j=1}^{m}c(x_j)_t\leq s$ if $s<0$ for all $t\in P$. We can think of this setting as n individuals assigning s hours of use of m machines into p different tasks that need s hours to be finished, or s euros that m persons have to spend in p different projects that need at least s euros to be done. We use $\mathcal{C}^*$ to denote this set of classifications. Thus, the FCAF* is a mapping $\alpha:\mathcal{C}^{*n}\to\mathcal{C}^*$ such that $\alpha(\boldsymbol{c})(x_j)$ indicates, for example, the different hours assigned to the different tasks of the machine x_j.

It is easy to see that when $m=p$, all the inequalities of the surjectivity conditions turn into equalities. This implies that trivially, the FCAF* satisfies the k-*allocation* property for any $k=m\geq 2$ (Aczél and Wagner, 1980), that is, if $\sum_{j=1}^{m}c(x_j)_t=s$, then $(\sum_{j=1}^{m}(\alpha(\boldsymbol{c})(x_j))_1,\ldots,\sum_{j=1}^{m}(\alpha(\boldsymbol{c})(x_j))_p)=\boldsymbol{s}$ where $\boldsymbol{s}$ is a $p-$dimensional vector with an s on every entry. The k-allocation property allows us to characterize the symmetric FCAF*s that satisfy Fuzzy Consensus for any value of m such that $m\geq p\geq 2$.

Theorem 4. *Let $m\geq p\geq 2$. A symmetric FCAF* satisfies Fuzzy Consensus if, and only if, it is a Weighted Arithmetic Mean.*

Proof. Aczél and Wagner (1980) show that a mapping $\boldsymbol{c}:\mathbb{R}^n\to\mathbb{R}$ satisfies the k-allocation property for $k=2,3$ and is bounded if, and only if, it is a

[6] A power mean with an odd exponent is a function $f:\mathbb{R}^n\to\mathbb{R}$ such that $f(x_1,\ldots,x_n)=(\frac{1}{n}\sum_{i=1}^{n}x_i^m)^{\frac{1}{m}}$ where m is an odd number.

Weighted Arithmetic Mean. In our setting, symmetry implies that for a given category $t \in P$, we have the same elementary FCAF* for every object. By Fuzzy Consensus, they are bounded and thus they are Weighted Arithmetic Means. By a similar proof to the one used in Theorem 2, it is easy to see that the elementary FCAF*s are the same for every category, concluding our proof. □

4 Final Remarks

In this work, we present an analysis from a fuzzy point of view, of the challenge of classifying m objects into p different categories. The classifications of the objects by the agents and the rules are no longer crisp statements about to which category the object is assigned. Instead of that, the classifications are expressed in terms of degrees of assignment to each of the categories. In the crisp setting for more than two objects and two categories, requiring the rule to fill each category with at least one object, to be unanimous and independent, implies the existence of an agent such that objects are classified according to the opinions of that agent (Maniquet and Mongin, 2016). Even with a weaker surjectivity condition, this result can not have a major improvement (Cailloux et al., 2024, showing the existence of an essential dictator).

The fuzzy setting proves advantageous, as the surjectivity conditions can assume diverse yet natural interpretations. Strong interpretations, akin to those in Alcantud et al. (2019), can extend the scope of impossibility results. However, by considering weaker surjectivity conditions and different versions of Unanimity, we can circumvent these limitations. Our findings show that rules that belong to the family of Weighted Arithmetic Means are the only ones that satisfy Zero Unanimity and Independence. Under different interpretations of the classification process and the consensual axioms, we can obtain characterization results that hold for $m \geq p \geq 2$.

An intriguing avenue for further inquiry lies in extending our analysis to dynamic scenarios, where object classifications may evolve over time. Exploring the temporal dynamics of fuzzy classifications could provide valuable insights into the adaptability and stability of the proposed framework in real-world applications.

Declarations of interest: none.

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