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A Theory of Labor Markets with Inefficient Turnover*

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Abstract

We develop a theory of labor markets with four features: search frictions, worker productivity shocks, wage rigidity, and two-sided lack of commitment. Inefficient job separations occur in the form of endogenous quits and layoffs that are unilaterally initiated whenever a worker's wage-to-productivity ratio moves outside an inaction region. We derive sufficient statistics for the labor market response to aggregate shocks based on the distribution of workers' wage-to-productivity ratios. These statistics crucially depend on the incidence of inefficient job separations, which we show how to identify using readily available microdata on wage changes and worker flows between jobs.

Keywords: Wage Rigidity, Directed Search, Limited Commitment, Job Separations, Quits, Layoffs, Inflation

JEL Classification: E31, E52, J64

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1 Introduction

The classical idea that inflation "greases the wheels of the labor market" (Keynes, 1936; Tobin, 1972) forms the bedrock of many theories of macroeconomic fluctuations: After the onset of a recession, nominal wage rigidities lead to inefficiently high wages and depressed employment, which creates a role for inflation to restore the economy to its optimal employment level by reducing real wages. While models in the Keynesian tradition (e.g., Erceg *et al.*, 2000; Christiano *et al.*, 2005) take seriously the proposition that frictions prevent the efficient adjustment of wages, they are usually silent on the micromechanics of the labor market, including the effects of inflation on *turnover*—i.e., which jobs are saved, destroyed, and created. Conversely, models in the search-theoretic tradition (e.g., Diamond, 1982; Pissarides, 1985; Mortensen and Pissarides, 1994) yield rich predictions for the distribution of wages and employment flows in the labor market, but abstract from inefficient turnover due to the simplifying assumption that contracts can be continuously and costlessly renegotiated.

In this paper, we develop a theoretical model bridging these two traditions. Motivated by empirical evidence linking wage rigidity to the employment sensitivity to aggregate shocks, bilaterally inefficient job separations, and real effects of monetary policy in the labor market, we explore the implications of inefficient turnover in a labor market model with four features. First, job search is frictional. Second, workers are subject to idiosyncratic productivity shocks. Third, wage rigidity due to staggered renegotiations. Fourth, neither workers nor firms can commit to remaining in a match. In our environment, all four features are necessary to generate inefficient turnover. At the center stage of our model are endogenous quits and layoffs that are unilaterally initiated whenever a worker's wage-to-productivity ratio moves outside an inaction region, giving rise to inefficient job separations.[1] In turn, inefficiencies on the separation margin feed back into job creation through workers' and firms' search decisions. In summary, our contribution is to analytically characterize inefficient turnover and to derive *sufficient statistics* for the economy's response to aggregate shocks in an equilibrium labor market model.

We first consider a stationary economy set in continuous time. The labor market is populated by a unit mass of heterogeneous workers and an endogenous mass of homogeneous firms. Workers' incomes depend on their employment state and idiosyncratic productivity, which follows a Brownian motion in logarithms. Unemployed workers and idle firms direct their search across submarkets indexed by their wage rate and worker productivity, as in Shimer (1996) and Moen (1997). Once matched, a worker-firm

[1]Ronald Coase explicitly pointed out that realistic transaction costs may prevent bilaterally efficient outcomes since *"it is necessary to discover who it is that one wishes to deal with, to inform people that one wishes to deal and on what terms, to conduct negotiations leading up to a bargain, to draw up the contract, to undertake the inspection needed to make sure that the terms of the contract are being observed, and so on. These operations are often extremely costly, sufficiently costly at any rate to prevent many transactions that would be carried out in a world in which the pricing system worked without cost"* (Coase, 1960). See also Sections 4 and 5 of Barro (1977).

pair is subject to two contractual frictions. First, wages are rigid in between staggered renegotiations à la Calvo (1983).[2] Second, neither workers nor firms can commit to remaining in a match, which can be endogenously dissolved in the form of unilateral quits and layoffs.

Once matched, a worker and a firm play a *nonzero-sum stochastic differential game with stopping times* (Bensoussan and Friedman, 1977). Their interaction forms a *game* due to their strategic choices of their own *stopping times*, defining when to unilaterally separate from the match. The game is *stochastic* and *differential* because worker productivity evolves according to a Brownian motion. It is *nonzero-sum* because the equilibrium match surplus is positive. To characterize the solution to this problem, we leverage powerful tools based on *variational inequalities* (Lions and Stampacchia, 1967; Øksendal, 2007).

While workers and firms engage in complex forward-looking behavior, we show that their decisions depend only on a single state variable: the wage-to-productivity ratio. A match is dissolved when the wage-to-productivity ratio falls outside an inaction region bounded by two thresholds. On one side, workers resign when their wage-to-productivity ratio falls below a quit threshold. On the other side, firms dismiss workers whose wage-to-productivity ratio exceeds a layoff threshold. Endogenous job separations due to quits and layoffs are unilateral in the sense that they occur voluntarily in the eyes of one party, even if they are involuntary in the eyes of the other party (cf. McLaughlin, 1991).

Our analysis yields three main results. First, we prove the existence and uniqueness of a *block-recursive equilibrium* (BRE). This result requires substantially different methods than those in the seminal work of Menzio and Shi (2010a,b, 2011), which we extend to a continuous-time setting with two-sided lack of commitment. Second, we provide a novel characterization of match surplus, entry wages, and job-separation under inefficient turnover by linking them to the expected discounted duration of a match, which here—unlike in models with flexible wages or full commitment—distinctly depends on rent sharing between a worker and a firm. Third, we demonstrate that two-sided lack of commitment has implications for labor markets that are profoundly different from prominent models of product pricing and investment. Unlike in models of inaction (e.g., Barro, 1972; Bernanke, 1983; Alvarez *et al.*, 2011), workers' and firms' ability to unilaterally separate bounds the option value of a match, even as the volatility of productivity shocks grows unboundedly. Compared to Sheshinski and Weiss (1977), the quit threshold and entry wage in our environment are less responsive to expected productivity growth and trend inflation.

Having characterized the stationary economy, we then introduce aggregate shocks. To this end, we

[2] We follow a long macroeconomic tradition in modeling wage rigidity through staggered renegotiations, as in Erceg *et al.* (2000), Christiano *et al.* (2005), Gertler and Trigari (2009), and Broer *et al.* (2023). While our model abstracts from their specific microfoundations, we think of such rigidities as capturing a list of reasons surveyed by Bewley (1999), including transaction costs (Barro, 1977), wage norms (Hall, 2005), fairness concerns (Akerlof, 1982), or information asymmetries (Hall and Lazear, 1984). On the equivalence between time- and state-dependent pricing models, see Alvarez *et al.* (2016a,b) and Auclert *et al.* (2023).

assume that incumbents' wages are nominally rigid while there are fluctuations in *aggregate revenue productivity* (TFPR)—i.e., either *aggregate physical productivity* (TFPQ) or the price level. Such aggregate shocks shift incumbents' TFPR-adjusted wages, leading to movements in the rate of endogenous job separations in the form of quits and layoffs. Under flexible entry wages, the wage that unemployed workers search for responds to the aggregate shock. Motivated by the allocative role of new-hire wages (Pissarides, 2009) and the limited cyclicality of reservation wages (Koenig *et al.*, 2023), we also study rigid entry wages, under which unemployed workers search for the same nominal wage schedule as before the aggregate shock, thereby changing firms' vacancy posting incentives. In this environment, inflation can "grease the wheels of the labor market" by affecting both job-separation and job-finding rates.

To study the effects of an economy-wide TFPR shock on aggregate employment, we analyze the economy's *cumulative impulse response* (CIR), defined as the area under an *impulse response function* (IRF). To this end, we extend the seminal work of Alvarez *et al.* (2016a) on sufficient statistics in the product market to a labor market context. Under flexible entry wages, the CIR of aggregate employment is fully described by three data moments: the job-finding rate, the variance of workers' wage changes across jobs, and a measure of the skewness of wage changes across jobs. That skewness appears in the sufficient statistic is a novel result. Intuitively, wage changes between jobs reflect workers' wage-to-productivity ratios, the skewness of which reflects the relative mass of workers near the quit versus layoff thresholds.

Under rigid entry wages, the CIR of aggregate employment additionally depends on the job-finding rate's elasticity with respect to the aggregate shock, which itself is a function of the share of inefficient job separations. Intuitively, an increase in TFPR incentivizes firms to post more vacancies but the magnitude of this effect is decreasing in the share of inefficient job separations: Firms choose when to lay off workers but do not control workers' quit decisions, which limit a firm's expected returns from vacancies.

While our theory highlights the relevant mechanisms at play in labor markets with inefficient turnover, we also lay the foundation for quantifying these mechanisms. To this end, we provide a formal identification proof, which allows us to recover the distribution of unobserved wage-to-productivity ratios as well as the parameters of the productivity process—and thus the share of inefficient job separations—from conventional labor market microdata on wage changes between job spells.

In a first pass at confronting our theory with the data, we draw on monthly employment and wage records from the *Survey of Income and Program Participation* (SIPP) for the U.S. The calibrated model predicts a CIR of aggregate employment of 3.1, indicating that a positive TFPR shock has expansionary effects in the labor market. Intuitively, this points toward layoffs being the most important source of unemployment risk, consistent with empirical evidence by Elsby *et al.* (2010).

Related Literature. Relative to the existing literature, we make two contributions. Our first contribution is to develop an equilibrium framework with inefficient turnover, in which nominal fluctuations affect both job-finding and job-separation rates through the split of match surplus. This approach sets us apart from the two traditions. On one hand, models in the Keynesian tradition have highlighted wage rigidity is the key friction for quantitative models to generate a realistic transmission of aggregate shocks (Christiano *et al.*, 2005; Blanchard and Galí, 2010; Schmitt-Grohé and Uribe, 2016). We add to this literature an equilibrium model of endogenous job creation and destruction with inefficient turnover due to wage rigidity, which naturally connects to labor market microdata on quits and layoffs (Graves *et al.*, 2023).

On the other hand, models in the search-theoretic tradition have studied the role of wage rigidity in amplifying unemployment fluctuations, following Shimer (2005a). These models restrict attention to bilaterally efficient contracts, as in Hall (2003) and Elsby *et al.* (2023), where costless wage renegotiations prevent the dissolution of matches with positive surplus. Similarly, the wage-setting protocols assumed by Hall (2005), Hall and Milgrom (2008), Michaillat (2012), Christiano *et al.* (2016), Gornemann *et al.* (2021), Birinci *et al.* (2023), and Moscarini and Postel-Vinay (2023) result only in efficient job separations. In related work by Gertler and Trigari (2009) and Gertler *et al.* (2020, 2022), inefficient job separations arise from wage rigidity and productivity shocks in theory, but are ignored in practice.

All aforementioned models steer clear of the Barro (1977) critique of inefficient outcomes under long-term contracts. Models in this tradition have produced many important insights. At the same time, there is mounting empirical evidence linking wage rigidity to the employment sensitivity to aggregate shocks,[3] bilaterally inefficient job separations,[4] and real effects of monetary policy in the labor market.[5] Our work represents a stark departure from this tradition in that we explicitly model inefficient turnover as a result of search frictions, productivity shocks, wage rigidity, and two-sided lack of commitment. In doing so, our theory yields a novel characterization of job creation, job separation, and wage determination under inefficient turnover in steady state and over the business cycle. In this sense, our work connects with a theoretical literature's conclusion that *"regrettable layoffs when demand is weak and regrettable quits when*

[3]Schmieder and von Wachter (2010) and Murray (2021) document that workers with higher wages and more rigid wages face increased layoff risk in the U.S. Kaur (2019) finds that wage rigidities distort employment levels in the presence of labor demand shocks in India. Ehrlich and Montes (2024) show that wage rigidity increases layoffs but decreases quits and hiring in Germany.

[4]Davis and Krolikowski (2023) find that many unemployment insurance (UI) recipients would accept significant wage cuts instead of being laid off. Yet employers do not consider pay cuts a viable substitute for layoffs, as documented by Bertheau *et al.* (2023) using a matched worker-firm survey. Jäger *et al.* (2022) provide quasi-experimental evidence of inefficient job separations exploiting changes in UI policies. See Bewley (1999) for extensive qualitative evidence on wage rigidity and job separations.

[5]Olivei and Tenreyro (2007, 2010) show that staggered wage contracts explain the effects of monetary policy shocks on output in the U.S. and other countries. Coglianese *et al.* (2023) document that wage rigidity mediates the unemployment response to a monetary policy quasi-experiment in Sweden. Faia and Pezone (2023). Broer *et al.* (2022) and Graves *et al.* (2023) find that worker flows are key to understanding the labor market response to identified monetary policy shocks.

demand is strong are the outcome of practical limitations on contracts" (p. 255 of Hall and Lazear, 1984).[6] Recent work along these lines includes Mueller (2017) who calibrates a model with inefficient separations due to wage rigidity, Carlsson and Westermark (2022) who develop a model of inefficient layoffs, and Heathcote and Cai (2023) who study the implications of inefficient quits for optimal UI design. More broadly, our theory opens up the door to a new research agenda studying the propagation of aggregate shocks in frictional labor markets subject to wage rigidity.

Our second contribution is methodological in nature and adds to two prominent literatures. Relative to the search-theoretic literature, we introduce the powerful tools of nonzero-sum stochastic differential games with stopping times (Bensoussan and Friedman, 1977). Such continuous-time methods are well suited to our environment because they offer three distinct benefits. First, they allow us to prove the existence and uniqueness of a BRE under inefficient turnover. Second, they yield convenient properties of value functions (e.g., continuity) and policy functions (e.g., connectedness), allowing us to study equilibrium conditions using variational inequalities. Third, they allow us to derive sharp comparative statics (e.g., anticipatory and option value effects). The foundational work of Menzio and Shi (2010a) studies BRE in a discrete-time model under efficient turnover. We complement their work by leveraging new tools to characterize BRE in a continuous-time model under inefficient turnover.[7]

There are important differences between our approach and existing results in the product pricing literature.[8] Specifically, we introduce new methods to extend the sufficient statistic approach to an environment with no commitment on behalf of two strategically interacting parties (i.e., workers and firms) and endogenous transitions between discrete states (i.e., employment and unemployment) in our labor market setting. A notable contribution of Alvarez *et al.* (2016a) has been to show that the CIR of output is linked to the ratio of the kurtosis and frequency of price changes in a large class of product pricing models. We complement their important insights by deriving the novel result that the CIR of employment in our labor market context is proportional to a measure of skewness of wage changes.

Outline. The rest of the paper is organized as follows. Section 2 develops a model of labor markets with inefficient turnover and characterizes its equilibrium. Section 3 derives sufficient statistics for the economy's response to aggregate shocks. Section 4 extends the baseline model in important dimensions. Section 5 proves the identification of the model based on conventional labor market microdata. Section 6 takes a first pass at confronting our theory with the data. Finally, Section 7 concludes.

[6] See also Perry and Solon (1985), MacLeod and Malcomson (1993), Malcomson (1997), and Delacroix and Wasmer (2009).

[7] Existing search models restrict attention to efficient turnover under full commitment (e.g., Shimer, 1996; Moen, 1997; Acemoglu and Shimer, 1999a,b), one-sided commitment (e.g., Shi, 2009; Menzio and Shi, 2010b, 2011; Schaal, 2017; Fukui, 2020; Balke and Lamadon, 2022), and two-sided lack of commitment (e.g., Sigouin, 2004; Rudanko, 2009, 2021; Bilal *et al.*, 2022, 2023).

[8] See, for example, Carvalho and Schwartzman (2015), Alvarez *et al.* (2021), and Baley and Blanco (2021, 2022).

2 A Model of Labor Markets with Inefficient Turnover

In this section, we develop a model of labor markets with inefficient turnover arising from the combination of search frictions, idiosyncratic productivity shocks, wage rigidity, and two-sided lack of commitment.

2.1 Environment

Time is continuous and indexed by t. A unit mass of heterogeneous workers and an endogenously determined mass of homogeneous firms meet in a frictional labor market.

Preferences. Both workers and firms discount the future at a common rate $\rho > 0$. Firms maximize profits. Workers have risk-neutral preferences over consumption streams $\{C_t\}_{t=0}^{\infty}$ given by $\mathbb{E}\left[\int_0^\infty e^{-\rho t} C_t \, \mathrm{d}t\right]$.

Technology. A worker's flow income depends on their employment state E_t, which can be either employed (h) or unemployed (u), and their productivity Z_t. While employed, a worker produces $Y_t = Z_t$ and consumes their wage W_t. While unemployed, a worker consumes $\tilde{B} Z_t$ from home production, with $\tilde{B} \in (0, 1)$. Henceforth, lower-case letters denote the natural logarithm of variables in upper-case letters.

Stochastic Process. A worker's idiosyncratic productivity follows a Brownian motion in logarithms: $\mathrm{d}z_t = \gamma \, \mathrm{d}t + \sigma \, \mathrm{d}\mathcal{W}_t^z$, where γ is the drift, σ is the volatility, and $\mathcal{W}_t^z$ is a Wiener process.

Search Frictions. Unemployed workers and idle firms direct their search across segmented submarkets indexed by worker productivity z and the wage w, as in Shimer (1996) and Moen (1997). In each submarket $(z; w)$, firms post vacancies at flow cost $\tilde{K} e^z$. Given $\mathcal{U}(z; w)$ unemployed workers and $\mathcal{V}(z; w)$ vacancies, a Cobb-Douglas matching function with constant returns to scale produces $m(z; w) = \mathcal{U}(z; w)^\alpha \mathcal{V}(z; w)^{1-\alpha}$ matches, where α is the elasticity of matches with respect to the unemployed. Given market tightness $\theta(z; w) := \mathcal{V}(z; w)/\mathcal{U}(z; w)$, workers' job-finding rate is $f(\theta(z; w)) = m(z; w)/\mathcal{U}(z; w) = \theta(z; w)^{1-\alpha}$ and firms' job-filling rate is $q(\theta(z; w)) = m(z; w)/\mathcal{V}(z; w) = \theta(z; w)^{-\alpha}$. Existing matches can end for any of three reasons: they can be exogenously dissolved at Poisson rate δ, or they can be endogenously and unilaterally dissolved by either the worker or the firm.

Wage Determination. While wages are competitively set at match formation, they are intermittently rigid thereafter, with staggered wage renegotiations à la Calvo (1983) occurring at rate $\delta^r \geq 0$ and following a Nash bargaining protocol with worker weight α. We present the limiting case with $\delta^r = 0$ in the main text. All results extend to the case of $\delta^r > 0$, shown in Supplementary Material IV.1.[9]

[9] We take no stance on the sources of wage rigidity but think of it as technological in nature, similar to adjustment costs in models of product pricing (Barro, 1972) and investment (Cooper and Haltiwanger, 2006).

Agents' Choices. An unemployed worker's choice of submarket $(z; w)$ is associated with a job-finding rate $f(\theta(z; w))$. Exogenous separations occur at rate δ, inducing a stopping time τ^{δ}. Given the wage w, a matched worker chooses a continuation productivity set $\mathcal{Z}^h(w)$, inducing the worker's stopping time $\tau^h(z; w) = \inf\{t \geq 0 : z_t \in \mathcal{Z}^h(w)^c, z_0 = z\}$, where $X^c := \mathbb{R} \backslash X$. Similarly, given w, a matched firm chooses a continuation productivity set $\mathcal{Z}^j(w)$, inducing the firm's stopping time $\tau^j(z; w) = \inf\{t \geq 0 : z_t \in \mathcal{Z}^j(w)^c, z_0 = z\}$. Naturally, agents' stopping times must be measurable with respect to their productivity history. Given the worker's and the firm's continuation sets and the exogenous separation hazard, the match duration is the first stopping time in $\vec{\tau}^m = (\tau^h, \tau^j, \tau^{\delta})$, denoted $\tau^m = \min\{\tau^h, \tau^j, \tau^{\delta}\}$.

2.2 Block-Recursive Equilibrium

A BRE can be described in two steps.[10] In the first step, we describe the optimal search behavior of unmatched workers and firms. Let $u(z)$ be the value of an unemployed worker under their optimal search policy with current productivity z. Let $\theta(z; w)$ denote market tightness in submarket $(z; w)$. Let $h(z; w)$ and $j(z; w)$ be the equilibrium values of an employed worker and a filled job. The problem of an unemployed worker is characterized by the Hamilton-Jacobi-Bellman (HJB) equation,

$$\rho u(z) = \tilde{B}e^z + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 u(z)}{\partial z^2} + \max_{w} f(\theta(z; w))[h(z; w) - u(z)], \tag{1}$$

with corresponding optimal search strategy $w^*(z)$. Equation (1) states that the unemployed worker's flow value is that of an asset with return equal to the sum of flow dividends (i.e., the value of home production) and expected capital gains (i.e., productivity fluctuations and finding a job). Free entry requires that

$$\min\left\{\tilde{K}e^z - q(\theta(z; w))j(z; w), \theta(z; w)\right\} = 0. \tag{2}$$

Equation (2) says that firms make zero profits in open submarkets and nonpositive profits in closed ones.

In the second step, which is the novel focus of this paper, we describe the strategic interaction that forms part of the *game* between a matched worker-firm pair, which has three features. First, payoffs are *nonzero-sum*, since the match flow value, e^z, exceeds the flow value of separating, $\tilde{B}e^z$. Second, agents' payoffs are *stochastic* and *differential*, since worker productivity z follows a Brownian motion. Third, agents' strategies consist of when to unilaterally separate from the match; i.e., the *stopping times* implied by their continuation sets ($\mathcal{Z}^h(w)$ and $\mathcal{Z}^j(w)$). Thus, the interaction between a worker-firm pair can be formulated as a nonzero-sum stochastic differential game with stopping times (Bensoussan and Friedman,

[10] Equilibrium objects generally depend on the distribution of productivities, wages, and employment states. BRE objects, however, are independent of this distribution, allowing us to omit it from all notation, as in Menzio and Shi (2010a).

1977). The application of these mathematical methods in a labor market context is a key contribution of this paper.

Value Functions. As long as one agent stays in the match at the current state z, the other agent chooses whether to stay in the match or to separate, reflecting the two-sided lack of commitment. Thus, we use *variational inequalities* (Øksendal, 2007) to characterize the values of both agents. The HJB equation of a worker employed at wage w with productivity z inside the firm's optimal continuation set $\mathcal{Z}^{j*}(w)$ is

$$\rho h(z;w) = \max\left\{ e^w + \gamma \frac{\partial h(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 h(z;w)}{\partial z^2} + \delta\left[u(z) - h(z;w)\right],\ \rho u(z) \right\}. \tag{3}$$

Equation (3) reflects the employed worker's choice between staying in the match and quitting the firm. The flow value of staying in the match is that of an asset for which the return is the sum of flow dividends (i.e., the wage) and expected capital gains (i.e., productivity fluctuations and separation). The flow value of quitting the firm is simply that of unemployment. The variational inequality in equation (3) satisfies $h(\cdot;w) \in \mathbb{C}^1(\mathcal{Z}^{j*}(w)) \cap \mathbb{C}(\mathbb{R})$. That is, the value of the employed worker is continuously once-differentiable inside the firm's optimal continuation set and continuous everywhere. These continuity and differentiability conditions correspond to the *value matching* condition and the *smooth pasting* condition of the worker's value function under their own best response. Importantly, a smooth pasting condition characterizes the optimal boundary of the worker's continuation region.

Analogously, the HJB equation of a firm employing a worker at wage w with productivity z inside the worker's optimal continuation set $\mathcal{Z}^{h*}(w)$ is

$$\rho j(z;w) = \max\left\{ e^z - e^w + \gamma \frac{\partial j(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 j(z;w)}{\partial z^2} - \delta j(z;w),\ 0 \right\}. \tag{4}$$

Equation (4) reflects the firm's choice between staying in the match and laying off the worker. The flow value of staying in the match is that of an asset for which the return is the sum of flow dividends (i.e., profits) and expected capital gains (i.e., productivity fluctuations and separation). The flow value of laying off the worker is simply that of being idle. The variational inequality in equation (4) satisfies $j(\cdot;w) \in \mathbb{C}^1(\mathcal{Z}^{h*}(w)) \cap \mathbb{C}(\mathbb{R})$. That is, the value of the matched firm is continuously once-differentiable inside the worker's optimal continuation set and continuous everywhere. Again, a smooth pasting condition characterizes the optimal boundary of the firm's continuation region.

If either of the two agents dissolves the match, then the other agent receives the value of their outside option. Therefore, the worker's and the firm's values of a match with productivity z and wage w satisfy:

$$h(z;w) = u(z) \quad \forall z \in (\mathcal{Z}^{j*}(w))^c, \tag{5}$$

$$j(z;w) = 0 \qquad \forall z \in (\mathcal{Z}^{h*}(w))^c. \tag{6}$$

Equations (5)–(6) define each agent's payoff outside the other agent's continuation set. Value-matching conditions imply the continuity of each agent's value function at the boundaries of the other agent's continuation set. However, smooth pasting conditions *do not* apply to either agent's value at the boundary of the other agent's continuation set. This is because the HJB equations (3)–(4) do not hold when an agent has no optimization problem to solve, which happens outside the other agent's continuation set.[11] For the same reason, we do not require value functions to be differentiable in the entire domain, but only in the part of the domain where an agent has a choice between staying in the match or not.

Continuation Sets. Two conditions characterize agents' optimal continuation sets. First, agents optimally choose to continue whenever the value of doing so exceeds that of separating:

$$h(z;w) > u(z), \tag{7}$$

$$j(z;w) > 0. \tag{8}$$

Second, to resolve any ambiguity in the strategic choice of a party that is indifferent between continuing and separating, we focus on the nontrivial equilibrium by invoking an equilibrium refinement based on weakly dominant strategies. Specifically, we assume that agents choose to continue whenever staying in the match is a weakly dominant strategy. For *any* policy of the worker, the firm strictly prefers to continue the match if flow profits are strictly positive—i.e., $e^z - e^w > 0$—because the firm always has the option of firing the worker in the future. Therefore, the firm's optimal continuation set is

$$\mathcal{Z}^{j*}(w) := int \left\{ z \in \mathbb{R} : j(z;w) > 0 \text{ or } e^z - e^w > 0 \right\}. \tag{9}$$

Analogously, the worker's optimal continuation set includes all productivity levels for which the sum of the current wage and the discounted capital gains from unemployment is positive:

$$\mathcal{Z}^{h*}(w) := int \left\{ z \in \mathbb{R} : h(z;w) > u(z) \text{ or } 0 < e^w - \rho u(z) + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 u(z)}{\partial z^2} \right\}. \tag{10}$$

[11]For example, for any $z \in (\mathcal{Z}^{h*}(w))^c$, we have $0 = \rho j(z;w) < \max\{e^z - e^w + \gamma \frac{\partial j(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 j(z;w)}{\partial z^2} - \delta j(z;w)\,, 0\}$.

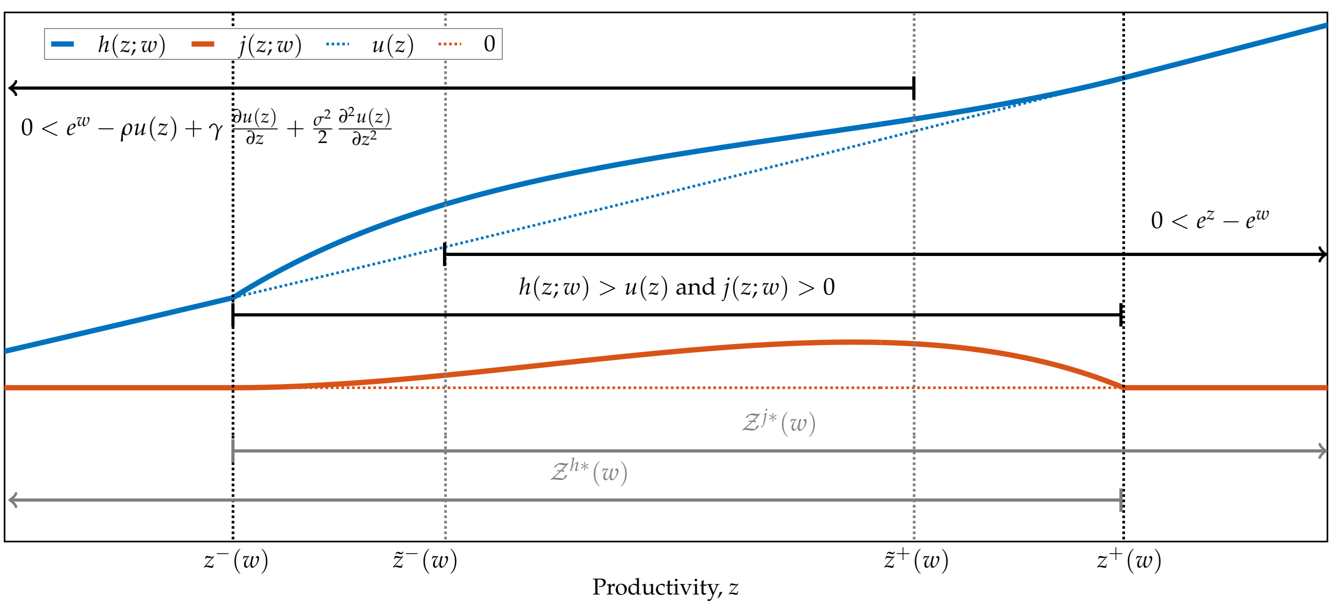
Intuitively, the HJB equation of the unemployed worker in (1) implies that

$$0 < e^w - \rho u(z) + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 u(z)}{\partial z^2} \iff \underbrace{\tilde{B}e^z + \max_{w'} f(\theta(z;w'))[h(z;w') - u(z)]}_{\text{flow opportunity cost}} < e^w.$$

Thus, if the wage strictly exceeds the flow opportunity cost, then continuing strictly dominates quitting.

Figure 1 illustrates the equilibrium values and optimal policies of a worker-firm match. The firm's continuation set is $\mathcal{Z}^{j*}(w) = (z^-(w), \infty)$, which contains productivities for which the firm makes strictly positive flow profits—i.e., $z > \tilde{z}^-(w) := w$—as well as productivities for which the worker and the firm continue despite negative flow profits due to a positive and large enough continuation value—i.e., $z \in (z^-(w), \tilde{z}^-(w))$. Analogously, the worker's continuation set is $\mathcal{Z}^{h*}(w) = (-\infty, z^+(w))$, which contains productivities for which the worker's wage strictly exceeds the flow opportunity cost—i.e., $z < \tilde{z}^+(w)$, where $\tilde{z}^+$ satisfies $0 = e^w - \rho u(\tilde{z}^+) + \gamma \partial u(\tilde{z}^+)/\partial z + (\sigma^2/2)\partial^2 u(\tilde{z}^+)/\partial z^2$—as well as productivities for which the worker and the firm continue despite the worker's negative net flow value due to a positive and large enough continuation value—i.e., $z \in (\tilde{z}^+(w), z^+(w))$. The existence and uniqueness of a threshold characterizing each agent's separation policy is not an assumption but a result formally derived below.

FIGURE 1. EQUILIBRIUM VALUES AND OPTIMAL POLICIES



Notes: The figure plots the value functions of workers (blue lines) and firms (red lines) for a given log wage w as a function of log productivity z. Solid lines show the values in the match, which is $h(z;w)$ for the worker and $j(z;w)$ for the firm. Dashed lines show the values outside of a match, which is $u(z)$ for the worker and 0 for the firm. The equilibrium continuation sets of the worker and the firm are $\mathcal{Z}^{h*}(w) = (-\infty, z^+(w))$ and $\mathcal{Z}^{j*}(w) = (z^-(w), \infty)$, respectively. The worker has positive net flow payoff for any productivity level $z < \tilde{z}^+(w)$, where $\tilde{z}^+$ satisfies $0 = e^w - \rho u(\tilde{z}^+) + \gamma \partial u(\tilde{z}^+)/\partial z + (\sigma^2/2)\partial^2 u(\tilde{z}^+)/\partial z^2$. The firm makes strictly positive flow profits for any productivity level $z > \tilde{z}^-(w) := w$. *Source:* Model simulations.

A Markov perfect equilibrium of this game is a fixed point between the two agents' best-response mappings involving continuation productivity levels z, given wage w.[12] To address the multiplicity of equilibria, our equilibrium definition implicitly incorporates a restriction to weakly dominant strategies.

Definition 1. *A BRE consists of a set of value functions $\{u(z), h(z;w), j(z;w)\}$, a market tightness function $\theta(z;w)$, the matched worker's and the matched firm's continuation sets $\{\mathcal{Z}^{h*}(w), \mathcal{Z}^{j*}(w)\}$, and the unemployed worker's search strategy function $w^*(z)$ such that:*

1. *Given $h(z;w)$ and $\theta(z;w)$, $u(z)$ solves (1) with optimal search strategy $w^*(z)$.*
2. *Given $j(z;w)$, the market tightness function $\theta(z;w)$ solves the free-entry condition (2).*
3. *Given $u(z)$ and $\mathcal{Z}^{j*}(z)$, $h(z;w) \in \mathbb{C}^1(\mathcal{Z}^{j*}(w)) \cap \mathbb{C}(\mathbb{R})$ solves (3) and (5). Given $\mathcal{Z}^{h*}(z)$, $j(z;w) \in \mathbb{C}^1(\mathcal{Z}^{h*}(w)) \cap \mathbb{C}(\mathbb{R})$ solves (4) and (6).*
4. *Given $u(z)$, the continuation set of the firm $\mathcal{Z}^{j*}(z)$ is given by (9) and that of the worker $\mathcal{Z}^{h*}(z)$ by (10).*

Part 1 of Definition 1 requires the optimality of unemployed workers' search strategies. Part 2 imposes the free-entry condition. The remaining parts of the definition describe agents' best responses in two steps. First, given the other agent's optimal continuation set, Part 3 describes the associated value function under the optimal continuation policy. Given these value functions, Part 4 describes the continuation sets.

Equilibrium Refinement. Our equilibrium definition incorporates an equilibrium refinement based on weakly dominant strategies. For illustration, suppose time is discrete, a period lasts $\mathrm{d}t$, and the match will end in the following period with certainty. If continuation is optimal today in expectation of match separation next period, which is the worst possible outcome from the next period onward, then continuation must be optimal under any possible outcome from next period onward. Table 1 lists the payoffs in the period game. Suppose that productivity z is such that flow payoffs in the match exceed flow payoffs from the outside options for both the worker and the firm—i.e., $(e^z - e^w)\,\mathrm{d}t > 0$ and $e^w\,\mathrm{d}t + \mathbb{E}_{z'}[e^{-\rho\,\mathrm{d}t}u(z')|z] > u(z)$. Then, there are two equilibria: one in which both agents choose to separate and one in which both players decide to continue. However, the first equilibrium does not survive the *iterated elimination of weakly dominated strategies* since, independent of what the other agent does, it is weakly better to continue. As $\mathrm{d}t \to 0$, we recover the continuation sets in equations (9)–(10), which incorporate a restriction to weakly dominant strategies in continuous time. That is, $(e^z - e^w)\,\mathrm{d}t > 0$ and $e^w\,\mathrm{d}t + \mathbb{E}_{z'}[e^{-\rho\,\mathrm{d}t}u(z')|z] > u(z)$ imply $e^z - e^w > 0$ and $0 < e^w - \rho u(z) + \gamma \partial u(z)/\partial z + (\sigma^2/2)\partial^2 u(z)/\partial z^2$ as $\mathrm{d}t \to 0$.

[12] The Markovian nature of the equilibrium reflects the two-sided lack of commitment. Supplementary Material II.3 derives the recursive equilibrium in continuous time from its discrete-time counterpart.

TABLE 1. ILLUSTRATING THE EQUILIBRIUM REFINEMENT USING PAYOFFS IN THE PERIOD GAME

	Worker stops	**Worker continues**
Firm stops	$(0, u(z))$	$(0, u(z))$
Firm continues	$(0, u(z))$	$((e^z - e^w)\,\mathrm{d}t, e^w\,\mathrm{d}t + \mathbb{E}_{z'}[e^{-\rho\,\mathrm{d}t}u(z')\vert z])$

Notes: This table shows the payoffs in a discrete-time approximation of the game played between a worker and a firm under the assumption that in the next period, match separat.

Inefficient Turnover. The flow benefit of a match, net of its opportunity cost, is given by $e^z - (\tilde{B}e^z + \max_w f(\theta(z;w))[h(z;w) - u(z)]) > 0$, reflecting the positive social value of a match. At the same time, wages are *allocative* in the sense that their level, given wage rigidity, affects the expected match duration. For this reason, *inefficient job separations* occur whenever a match is endogenously dissolved by either the worker or the firm. The lack of commitment is reflected in the equilibrium definition: Endogenous separations are optimal at each point of the state space for at least one of the agents. Importantly, inefficiencies on the job separation margin also imply *inefficient job match creation* due to their effects on unemployed workers' search decisions through $h(z;w)$ and on firms' vacancy posting decisions through $j(z;w)$. Thus, *inefficient turnover* results from the interaction between search frictions, worker productivity shocks, wage rigidity, and two-sided lack of commitment in our framework.

2.3 Equilibrium Characterization

To understand the dependence of equilibrium objects on state variables, we recast the model in terms of a reduced state space. It turns out that the relevant state variable for both workers and firms is the log-wage-to-productivity ratio, $\hat{w} := w - z$. We can express agents' values and policies as functions of the scalar $\hat{w}$ instead of the duplet $(z;w)$. To simplify notation, we define the transformed drift $\hat{\gamma} := \gamma + \sigma^2$ and the transformed discount factor $\hat{\rho} := \rho - \gamma - \sigma^2/2$. The following Lemma characterizes the equilibrium.

Lemma 1. *Suppose that the set $(u(z), h(z;w), j(z;w), \theta(z;w))$ satisfies the equilibrium conditions (1)–(6), given the continuation sets $\mathcal{Z}^{h*}(w)$ and $\mathcal{Z}^{j*}(w)$ defined in (9)–(10) and search policy $w^*(z)$. Then, the transformed set*

$$\left(\hat{U}\,,\,\hat{J}(w-z)\,,\,\hat{W}(w-z)\,,\,\hat{\theta}(w-z)\right) = \left(\frac{u(z)}{e^z}\,,\,\frac{j(z;w)}{e^z}\,,\,\frac{h(z;w)-u(z)}{e^z}\,,\,\theta(z;w)\right)$$

equivalently characterizes the equilibrium if the following conditions are satisfied:

1. *Given $\hat{W}(\hat{w})$ and $\theta(\hat{w})$, $\hat{U}$ satisfies*

$$\hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}} f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w}), \tag{11}$$

where the optimal choice of submarket for an unemployed worker to search in is $\hat{w}^* = w^*(z) - z$.

2. *Given* $\hat{J}(\hat{w})$*, free entry is satisfied:* $\min\left\{\tilde{K} - q(\hat{\theta}(\hat{w}))\hat{J}(\hat{w})\,,\,\hat{\theta}(\hat{w})\right\} = 0$.

3. *Given* $\hat{\mathcal{Z}}^{h*} := int\left\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) > 0 \text{ or } e^{\hat{w}} > \hat{\rho}\hat{U}\right\}$ *and* $\hat{\mathcal{Z}}^{j*} := int\left\{\hat{w} \in \mathbb{R} : \hat{J}(\hat{w}) > 0 \text{ or } e^{\hat{w}} < 1\right\}$*, the transformed value functions* $\hat{W}(\hat{w})$ *and* $\hat{J}(\hat{w})$ *satisfy the variational inequalities*

$$\hat{\rho}\hat{W}(\hat{w}) = \begin{cases} \max\left\{e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}) - \delta\hat{W}(\hat{w})\,,\,0\right\} & \forall \hat{w} \in \hat{\mathcal{Z}}^{j*}, \\ 0 & \forall \hat{w} \in (\hat{\mathcal{Z}}^{j*})^c, \end{cases} \tag{12}$$

$$\hat{\rho}\hat{J}(\hat{w}) = \begin{cases} \max\left\{1 - e^{\hat{w}} - \hat{\gamma}\hat{J}'(\hat{w}) + \frac{\sigma^2}{2}\hat{J}''(\hat{w}) - \delta\hat{J}(\hat{w})\,,\,0\right\} & \forall \hat{w} \in \hat{\mathcal{Z}}^{h*}, \\ 0 & \forall \hat{w} \in (\hat{\mathcal{Z}}^{h*})^c, \end{cases} \tag{13}$$

with $\hat{W} \in \mathbb{C}^1(\hat{\mathcal{Z}}^{j*}) \cap \mathbb{C}(\mathbb{R})$ *and* $\hat{J} \in \mathbb{C}^1(\hat{\mathcal{Z}}^{h*}) \cap \mathbb{C}(\mathbb{R})$*. Finally, the optimal stopping times are given by* $\tau^{h*} = \inf\{t \geq 0 : \hat{w}_t \in (\hat{\mathcal{Z}}^{h*})^c, w_0 = \hat{w}^*\}$ *and* $\tau^{j*} = \inf\{t \geq 0 : \hat{w}_t \in (\hat{\mathcal{Z}}^{j*})^c, w_0 = \hat{w}^*\}$.

Proof. See Online Appendix A.1. □

The equilibrium conditions in Lemma 1 are transformed versions of those of the original problem stated above. Part 1 gives the value of unemployment under the optimal search strategy in equation (11). Part 2 states the transformed free-entry condition. Part 3 describes a nontrivial equilibrium, with equations (12)–(13) referencing agents' optimal continuation regions such that workers' wages are above the flow value of unemployment whenever $e^{\hat{w}} > \hat{\rho}\hat{U}$ and firms' flow profits are positive whenever $e^{\hat{w}} < 1$.

We now state a key result on equilibrium existence and uniqueness.

Proposition 1. *There exists a unique BRE.*

Proof. See Supplemental Material I. □

Although equilibrium existence and uniqueness are important properties of models of directed search, in our context they do not follow from previous work. Standard arguments in discrete time with only exogenous job separations involve Schauder's fixed-point theorem (e.g., Menzio and Shi, 2010a,b; Schaal, 2017), which critically relies on two conditions: continuity in the value functions and continuity in the mapping between value functions that characterize the BRE. These standard arguments no longer apply to variants of the above-referenced models in discrete time after the inclusion of endogenous separations, nor do they carry over to our continuous-time setup. Instead, we leverage techniques from the literature on variational inequalities to prove the existence and uniqueness of a nontrivial equilibrium in our model. The proof proceeds in three steps. In the first step, we represent the equilibrium conditions (12)–(13) in terms of quasi-variational inequalities (cf. Antman, 1983). In the second step, we use the existence and uniqueness results in Lions and Stampacchia (1967) to show the existence of the agents' best response functions and their associated value functions. In the third step, we define a functional equation $Q(\cdot)$ that maps the worker's value function to itself using both agents' best response functional equations. Thus, proving the existence of a unique nontrivial Nash equilibrium becomes equivalent to finding a fixed point $\hat{W}^*$ such that $Q(\hat{W}^*) = \hat{W}^*$. To this end, we show that the operator $Q(\cdot)$ is monotonic, thus allowing us to establish the existence of the fixed point by invoking the Birkhoff-Tartar theorem (Aubin, 2007), which applies under relatively weak regularity conditions. Finally, we show that the operator $Q(\cdot)$ satisfies a type of concavity property, which allows us to establish the uniqueness of the fixed point. This uniqueness result is nontrivial given the complementarity in agents' continuation decisions based on strategic worker-firm interactions within a match. Importantly, our continuous-time setup also allows us to leverage properties of the employed worker's and the firm's value functions—e.g., continuity with respect to $\hat{U}$—which are necessary to find a unique equilibrium of this economy.

Next, we characterize properties of the BRE. Recalling the definition of the transformed state variable $\hat{w} := w - z$, we postulate that there exist optimal policies $\hat{w}^- < \hat{w}^* < \hat{w}^+$, where $\hat{w}^-$ is the worker's optimal job-separation threshold, $\hat{w}^*$ is the optimal search strategy at match formation, and $\hat{w}^+$ is the firm's optimal job separation threshold. We define the transformed surplus of the match as $\hat{S}(\hat{w}) := \hat{J}(\hat{w}) + \hat{W}(\hat{w})$ and the worker's share of the transformed surplus as $\eta(\hat{w}) := \hat{W}(\hat{w})/\hat{S}(\hat{w})$.

Proposition 2. *The BRE has the following properties:*

1. *The joint match surplus satisfies*

$$\hat{S}(\hat{w}) = (1 - \hat{\rho}\hat{U})\mathcal{T}(\hat{w}, \hat{\rho}), \quad \textit{where} \quad \tilde{B} < \hat{\rho}\hat{U} < 1 \tag{14}$$

and the expected discounted match duration is given by

$$\mathcal{T}(\hat{w}, \hat{\rho}) := \mathbb{E}\left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t}\, \mathrm{d}t | \hat{w}_0 = \hat{w}\right]. \tag{15}$$

2. *The competitive entry wage, i.e.,* $\hat{w}^* = \arg\max_{\hat{w}} f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w})$ *exists and it is unique. Moreover, it solves*

$$\hat{w}^* = \arg\max_{\hat{w}} \left\{\hat{W}(\hat{w})^{\alpha}\hat{J}(\hat{w})^{1-\alpha}\right\} = \arg\max_{\hat{w}} \left\{\eta(\hat{w})^{\alpha}(1 - \eta(\hat{w}))^{1-\alpha}\mathcal{T}(\hat{w}, \hat{\rho})\right\}, \tag{16}$$

with the unique solution characterized by the optimality condition given by

$$\underbrace{\eta'(\hat{w}^*)\left(\frac{\alpha}{\eta(\hat{w}^*)} - \frac{1-\alpha}{1-\eta(\hat{w}^*)}\right)}_{\textit{share channel}} = -\underbrace{\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}}_{\textit{surplus channel}}. \tag{17}$$

3. *The equilibrium job-finding rate $f(\hat{\theta}(\hat{w}^*))$ and the flow opportunity cost of employment $\hat{\rho}\hat{U}$ are given by*

$$f(\hat{\theta}(\hat{w}^*)) = \left[(1-\eta(\hat{w}^*))(1-\hat{\rho}\hat{U})\mathcal{T}(\hat{w}^*,\hat{\rho})/\tilde{K}\right]^{\frac{1-\alpha}{\alpha}}, \tag{18}$$

$$\hat{\rho}\hat{U} = \tilde{B} + \left(\tilde{K}^{\alpha-1}\left(1-\eta(\hat{w}^*)\right)^{1-\alpha}\eta(\hat{w}^*)^{\alpha}\left(1-\hat{\rho}\hat{U}\right)\mathcal{T}(\hat{w}^*,\hat{\rho})\right)^{\frac{1}{\alpha}}. \tag{19}$$

4. *If $\gamma \neq 0$ or $\sigma \neq 0$, then agents' continuation sets are connected and the game's continuation set is bounded:*

$$\hat{\mathcal{Z}}^{h*} = (\hat{w}^-,\infty) \quad \textit{and} \quad \hat{\mathcal{Z}}^{j*} = (-\infty,\hat{w}^+), \quad \textit{where} \quad -\infty < \hat{w}^- \leq \log(\hat{\rho}\hat{U}) < 0 \leq \hat{w}^+ < \infty. \tag{20}$$

Workers' and firms' value functions satisfy the following smooth pasting conditions: $\hat{W}'(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0$.

Proof. See Online Appendix A.2. □

Starting with Part 1 of Proposition 2, equation (14) states that the match surplus equals the product of the transformed flow surplus $1-\hat{\rho}\hat{U}$ and the expected discounted match duration $\mathcal{T}(\hat{w},\hat{\rho})$ defined in equation (15), which depends on the wage $\hat{w}$ and the width of the match's continuation set $(\hat{w}^-,\hat{w}^+)$. Also, the flow opportunity cost of employment $\hat{\rho}\hat{U}$ is bounded between 1 (i.e., the transformed value of flow output in the match) and $\tilde{B}$ (i.e., the transformed value of home production). Since $1 > \hat{\rho}\hat{U}$, the joint match surplus is always strictly positive, so that all endogenous job separations are inefficient.

Equations (16)–(17) of Part 2 show that the competitive entry wage $\hat{w}^*$ balances a *share channel* and a *surplus channel*. Unemployed workers search for wages that are competitively set as if they were the outcome of a Nash bargaining problem with worker weight α, thereby satisfying the well-known Hosios (1990) condition. This result derives from free entry, which implies that workers' job-finding rate is proportional to the firm's value. A larger entry wage increases the worker's surplus share by $\eta'(\hat{w}^*)\alpha/\eta(\hat{w}^*)$ but reduces the job-finding probability by $\eta'(\hat{w}^*)(1-\alpha)/(1-\eta(\hat{w}^*))$. This trade-off is reflected in the share channel and standard in models of directed search (e.g., Shimer, 1996; Moen, 1997).

In our setting, a novel *surplus channel* arises, which captures the dependence of expected match duration and surplus on the wage set at match formation. The higher (lower) the entry wage, the sooner the firm (worker) will dissolve the match in expectation. Only if $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho}) = 0$ will the worker's surplus share $\eta(\hat{w}^*) = \alpha$, as in bilaterally efficient models. These considerations are unique to our environment.

Part 3 expresses unemployed workers' job-finding rate (18) and the flow opportunity cost of employment (19) as functions of the worker's surplus share and the expected discounted match duration.

Part 4 shows that the continuation set of the worker and that of the firm in (20) follow threshold rules in the log-wage-to-productivity ratio $\hat{w}$. Workers do not quit as long as $\hat{w} > \hat{w}^-$, while firms refrain from firing the worker as long as $\hat{w} < \hat{w}^+$. Thus, the continuation set for the match is given by $\hat{\mathcal{Z}}^{h*} \cap \hat{\mathcal{Z}}^{j*} = (\hat{w}^-, \hat{w}^+)$. These thresholds satisfy $\hat{w}^- \leq \log(\hat{\rho}\hat{U})$ and $\hat{w}^+ \geq 0$, reflecting both parties' willingness to accept flow payoffs below that from their respective outside option. Finally, the smooth pasting conditions apply at the worker's quit threshold $\hat{w}^-$ and at the firm's firing threshold $\hat{w}^+$, reflecting the optimality of agents' continuation thresholds.

Finally, it is worth highlighting that the optimal entry wage (see Part 2) will be set at an optimal distance from both separation thresholds (see Part 4). To convey the intuition, consider a wage-to-productivity ratio $\hat{w}$ close to the quit threshold $\hat{w}^-$. The worker's and firm's value functions are increasing for $\hat{w}$ sufficiently close to $\hat{w}^-$ since both values are zero when $\hat{w} < \hat{w}^-$ and positive when $\hat{w} > \hat{w}^-$.[13] Therefore, around the quit threshold, raising wages is Pareto improving, as it results in a higher flow payoff for the worker and at the same time a lower quit probability, which extends the expected match duration and increases the firm's value. Following a symmetric argument, lowering wages is Pareto improving near the layoff threshold.

2.4 Understanding the Economic Mechanisms

We now characterize the mechanisms that drive workers' and firms' equilibrium behavior.

Static Considerations. It is instructive to first consider equilibrium policies under fixed productivity.

Proposition 3. *If $\gamma = \sigma = 0$, then optimal policies are given by*

$$(\hat{w}^-, \hat{w}^*, \hat{w}^+) = \log(\hat{\rho}\hat{U}, \alpha + (1-\alpha)\hat{\rho}\hat{U}, 1),$$

with $\eta(\hat{w}^*) = \alpha$ and $\mathcal{T}(\hat{w}^*, \hat{\rho}) = 1/(\hat{\rho} + \delta)$. No smooth pasting conditions apply in this case.

Proof. See Online Appendix A.3. □

Note that $\hat{w}^- < \hat{w}^* < \hat{w}^+$ and $\hat{w} = \hat{w}^*$ for the duration of the match, absent productivity fluctuations, so there are no endogenous job separations. From this result, we see that lack of commitment and wage

[13]Figure II1 in Supplementary Material II.1 plots the worker's and the firm's values as functions of $\hat{w}$.

rigidity by themselves do not generate inefficient job separations. Absent productivity fluctuations, agents' behavior is bilaterally efficient, in that it maximizes the joint match surplus.

In addition to the static forces outlined above, two dynamic considerations guide workers' and firms' choices: the *option value effect* and the *anticipatory effect*.

Dynamic Consideration I: The Option Value Effect. To understand the role of productivity fluctuations in creating the option value effect, we temporarily abstract from the drift in worker productivity.

Proposition 4. *If $\hat{\gamma} = 0$ and $\alpha = 1/2$, then, to a first-order approximation, the optimal entry wage is given by $\hat{w}^* = \log\left((1+\hat{\rho}\hat{U})/2\right)$ and job-separation thresholds satisfy $\hat{w}^{\pm} = \hat{w}^* \pm h(\varphi, \Phi)$ for some function $h(\varphi, \Phi)$ with $\varphi := \sqrt{2(\hat{\rho}+\delta)}/\sigma$ and $\Phi := (1-\hat{\rho}\hat{U})/(1+\hat{\rho}\hat{U})$. The following properties apply:*

1. *$h(\varphi, \Phi)$ is decreasing in φ and increasing in Φ.*
2. *$\lim_{\varphi \to 0} h(\varphi, \Phi) = 3\Phi$ and $\lim_{\varphi \to \infty} h(\varphi, \Phi) = \Phi$.*
3. *$\varphi h(\varphi, \Phi)$ is increasing in φ.*

Furthermore, the equilibrium surplus share is $\eta(\hat{w}^*) = \alpha = 1/2$ and the expected discounted match duration,

$$\mathcal{T}(\hat{w}^*, \hat{\rho}) = \frac{1 - 2\left(e^{\varphi h(\varphi,\Phi)} + e^{-\varphi h(\varphi,\Phi)}\right)^{-1}}{\hat{\rho}+\delta}, \tag{21}$$

is increasing in φ and Φ and satisfies $\mathcal{T}'_{\hat{w}}(\hat{w}^*, \hat{\rho}) = 0$.

Proof. See Online Appendix A.3. □

Proposition 4 demonstrates that idiosyncratic volatility, by itself, does not affect the split of the match surplus between the worker and the firm. Such an economy is symmetric around the entry wage, which implies $\mathcal{T}'_{\hat{w}}(\hat{w}^*, \rho) = 0$ and $\eta(\hat{w}^*) = \alpha$. Thus, a larger $\hat{w}^*$ reduces the match duration by increasing the likelihood of a layoff but increases the match duration by reducing the likelihood of a quit. Weighing both forces, $\mathcal{T}(\cdot, \rho)$ is maximized at $\hat{w}^* = (1+\hat{\rho}\hat{U})/2$ and $\eta(\hat{w}^*) = 1/2$.

This result provides a tight characterization of the worker's and the firm's optimal policy functions, which yield the continuation region of the match $(\hat{w}^-,\ \hat{w}^+)$ being symmetrically centered around the optimal entry wage $\hat{w}^*$. Second, the width of the continuation region is increasing in volatility σ and decreasing in $\hat{\rho}\hat{U}$ (Part 1). The width of the inaction region increases with σ due to the option value effect: Though the worker's productivity might fall below the wage, the firm is willing to wait before firing the worker because productivity may increase in the future. The width of the inaction region decreases with $\hat{\rho}\hat{U}$, a higher value of which decreases match surplus and makes it more costly to wait.

The option value effect naturally arises in models of inaction. However, our model features a departure from canonical models of inaction (e.g., Barro, 1972; Bernanke, 1983; Alvarez *et al.*, 2011). In those models, the width of the continuation region typically grows unboundedly with the volatility σ. Instead, in our model, the width of the continuation region has an upper bound (Part 2). To see the intuition behind this result, consider the problem of a firm that finds itself in a match with negative flow profits—the worker case is exactly analogous. The marginal benefit from remaining in a currently unprofitable match is that, with some probability in the future, productivity increases enough to make the match profitable by rendering the wage-to-productivity ratio less than unity. At the same time, inaction on the part of the firm is risky: Productivity may increase by a large enough amount for the worker to choose to quit. Given the two job-separation thresholds, as the volatility goes to infinity, the probability of remaining in the profitable part of the inaction region approaches zero. Thus, the two-sided lack of commitment imposes an upper bound on the option value associated with remaining in a match.

The inefficiency generated by the lack of commitment also manifests itself in the expected duration of the match given by equation (21). Since the separation thresholds, indexed by $h(\varphi, \Phi)$, remain bounded as $\sigma \to \infty$, the expected match duration decreases as the volatility of productivity shocks increases (Part 3).

Dynamic Consideration II: The Anticipatory Effect. To understand the role of the productivity drift in generating an anticipatory effect, we temporarily abstract from volatility in worker productivity (i.e., $\sigma = 0$) and focus on the case with weakly positive drift (i.e., $\hat{\gamma} \geq 0$), with other cases being analogous.

Proposition 5. *If $\sigma = 0$ and $\hat{\gamma} \geq 0$, then the quit threshold is $\hat{w}^- = \log\left(\hat{\rho}\hat{U}\right)$ and the entry wage is*

$$w^* = \hat{w}^- + \tilde{T}\left(\frac{\alpha + (1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}, \frac{\hat{\rho}+\delta}{\hat{\gamma}}, \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\right),$$

where $\tilde{T}(\cdot)$, defined in equation (A.25) of Online Appendix A.3, is increasing in its first argument and decreasing in its second argument. Moreover:

1. *As $\hat{\gamma} \to 0$, then $\left(\tilde{T}(\cdot), \mathcal{T}(\hat{w}^*, \hat{\rho}), \eta(\hat{w}^*)\right) \to \left(\log\left(\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}\right), \frac{1}{\hat{\rho}+\delta}, \alpha\right)$.*
2. *As $\hat{\gamma} \to \infty$, then $\left(\tilde{T}(\cdot), \mathcal{T}(\hat{w}^*, \hat{\rho}), \eta(\hat{w}^*)\right) \to \left(\tilde{T}^{limit}, 0, \eta^{limit}\right)$, where $\tilde{T}^{limit}$ and η^{limit} are defined as*

$$\frac{\alpha + (1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}} = \frac{e^{\tilde{T}^{limit}} - 1 - \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\left(1 - \frac{\tilde{T}^{limit}}{e^{\tilde{T}^{limit}}-1}\right)}{\tilde{T}^{limit}},$$

$$\eta^{limit} = \alpha + \frac{1-\alpha}{\tilde{T}^{limit}} \frac{(1-\hat{\rho}\hat{U})\eta^{limit}}{\eta^{limit} + \hat{\rho}\hat{U}(1-\eta^{limit})}. \quad (22)$$

Proof. See Online Appendix A.3. □

When productivity grows at a constant rate, the job-separation threshold $\hat{w}^-$ equals the static opportunity cost of employment since workers benefit from remaining matched up to that point and workers have no incentive to delay separation beyond that point. The fact that $\hat{w}^-$ is insensitive to the drift differs from the canonical result in Sheshinski and Weiss (1977) who studied the problem of a firm setting prices subject to menu costs with positive trend inflation. Their main result is that, in order to economize on menu costs associated with price changes, firms both decrease the lower threshold of the inaction region for real prices and increase the nominal reset price in response to higher trend inflation. Here, the quit threshold $\hat{w}^-$ is independent of the drift due to limited commitment—the firm has no control over worker quits. From Proposition 5, the entry wage $\hat{w}^*$ is increasing in both the weighted sum of opportunity costs $(\alpha + (1-\alpha)\hat{\rho}\hat{U})$ and the drift $(\hat{\gamma})$. We refer to the latter as the anticipatory effect: Workers anticipate higher future productivity and modify their search strategy accordingly. The following two cases illustrate this point by exploring two limiting behaviors of the anticipatory effect.

As $\hat{\gamma} \to 0$ (Part 1), the equilibrium entry wage $\hat{w}^*$ is the same as in the case without drift; thus, $\eta(\hat{w}^*) = \alpha$. As the drift increases, workers optimally search for a job with a higher entry wage. Therefore, the average wage in the economy increases above the weighted sum of opportunity costs; recall that $\hat{w}^-$ remains fixed. This results from the worker internalizing the trade-off whereby a higher wage implies (i) a reduced job-finding rate and (ii) a lower frequency of inefficient job separations and, thus, a longer expected match duration. As $\hat{\gamma} \to \infty$ (Part 2), the entry wage w^* becomes unresponsive to the drift because the job-finding rate becomes so small that it dominates the trade-off. Thus, the effect of the drift on the entry wage is bounded, in contrast to the reset price in Sheshinski and Weiss (1977). Finally, as seen in (22), the anticipatory effect gives workers a higher surplus share when $\hat{\gamma} \to \infty$ compared to $\hat{\gamma} \to 0$.

Workers' lack of commitment gives them the option to quit, which implies the invariance of $\hat{w}^-$ to $\hat{\gamma}$ and a decreased value of searching for a job. To see this, suppose a worker commits to some $\hat{w}^-$ as $\delta \to 0$. Then, the worker chooses a single instrument, namely the entry wage w^* to balance two objectives. On one hand, the worker chooses w^* to steer the rate of inefficient separations, which occur at a tenure of $(w^* - \hat{w}^-)/\hat{\gamma}$, as captured by the surplus channel. On the other hand, the worker chooses w^* close to the weighted sum of opportunity costs, as captured by the share channel. Since these objectives are conflicting, lack of commitment distorts both the expected match duration and job-finding rates in equilibrium.

2.5 Discussion of Model Assumptions

For expositional clarity, we imposed certain assumptions that are not essential for our theory of labor markets with inefficient turnover: (i) homotheticity of the home production technology and vacancy costs;

(ii) no on-the-job search; and (iii) time dependence of wage setting.

Regarding (i), shocks to worker productivity Z_t affect agents' choices through the relative flow value of employment (W_t/Z_t), home production ($\tilde{B}Z_t$), and vacancy costs ($\tilde{K}Z_t$). In order to focus on the novel first margin, we abstract from the other two by assuming that home production and vacancy costs are homothetic in worker productivity. This assumption implies that all workers face the same job-finding rate and entry wage per efficiency unit and also rules out any efficient endogenous job separations. As a result, it allows us to focus on our economic mechanisms within—rather than between—worker types. It is straight-forward to relax these homotheticity assumptions in numerical simulations.

Regarding (ii), workers can reset their wages by undergoing a costly unemployment spell, similar to models with costly on-the-job search. Even allowing for on-the-job search, inefficient separations into unemployment would occur for analogous reasons. Qualitatively, on-the-job search would widen the inaction region since now employment yields an option value of receiving outside employment offers. However, a fully specified model of on-the-job search under wage rigidity would need to take a stance on the wage renegotiation protocol. Blanco and Drenik (2023) take a step in this direction.

Finally, regarding (iii), time-dependent wage setting à la Calvo (1983) is common in macroeconomic modeling (e.g., Erceg *et al.*, 2000; Christiano *et al.*, 2005; Gertler and Trigari, 2009; Broer *et al.*, 2023) and a tractable stand-in for transaction costs (Barro, 1977), wage norms (Hall, 2005), fairness concerns (Akerlof, 1982), or information asymmetries (Hall and Lazear, 1984). Wages have been empirically documented to be reset at certain intervals (Taylor, 1979), synchronized within firms (Grigsby *et al.*, 2021), and subject to staggered institutional contracts (Adamopoulou *et al.*, 2022). While necessarily parsimonious, the current model of wage setting is motivated by these empirical regularities. Our assumptions allow for sharp analytical results, the essence of which we expect to carry over to alternative models of state-dependent wage setting that require numerical solution methods (cf. Alvarez *et al.*, 2016a,b; Auclert *et al.*, 2023).[14]

3 Aggregate Shocks in Labor Markets with Inefficient Turnover

How does inefficient turnover affect the transmission of aggregate shocks in the labor market? To answer this question, we extend our model to encompass shocks to aggregate productivity and monetary policy.

[14]For example, Alvarez *et al.* (2016a) conclude that "for small aggregate shocks the [multiproduct pricing] models behave similarly irrespective of the nature of the sticky price friction" (p. 2850).

3.1 An Economy with Aggregate Shocks

To characterize the labor market response to a broad set of aggregate shocks, we modify the baseline model by introducing shocks to economy-wide TFPR, defined as $TFPR_t := A_t P_t$, where A_t denotes aggregate productivity and P_t denotes the aggregate price level. We assume that the logarithm of TFPR follows a Brownian motion with drift χ and volatility ζ:

$$\mathrm{d}\log TFPR_t = \chi\,\mathrm{d}t + \zeta\,\mathrm{d}\mathcal{W}_t^{\mathrm{TFP}},$$

where $\mathcal{W}_t^{\mathrm{TFP}}$ is a Wiener process. Studying shocks to TFPR has two benefits. On one hand, it allows us to study shocks to aggregate productivity, which are the predominant source of exogenous fluctuations studied in the quantitative macro-labor literature (Shimer, 2005a; Hall, 2005). On the other hand, it allows us to study shocks to the aggregate price level, which is endogenously determined in a monetary economy. We provide two alternative microfoundations for the aggregate price level when monetary policy is conducted either via money supply (Supplementary Material III.1) or an interest rate-based Taylor rule (Supplementary Material III.2). In both models, monetary policy moves the aggregate price level P_t and thus $TFPR_t$.[15] We assume that the vacancy posting cost $\tilde{K}Z_t$ and the value of home production $\tilde{B}Z_t$ are linear in TFPR. This assumption arises naturally when the TFPR shock is due to price movements as long as $\tilde{K}Z_t$ and $\tilde{B}Z_t$ are denominated in real terms. Under the interpretation of the TFPR shock being due to productivity movements, this assumption can be justified by appealing to recruiting expenses incurred in the process of workers operating a recruiting technology (cf. Shimer, 2010).

The introduction of aggregate shocks requires minor adjustments to our framework. Given fluctuations in TFPR, the relevant state variable becomes the *real wage-to-productivity ratio* $\hat{w} := w - z - \log TFPR$, which equals the worker's nominal wage w minus *worker productivity* $z + \log TFPR$. All policies $(\hat{w}^+, \hat{w}^*, \hat{w}^-)$ are then expressed in TFPR-adjusted terms. In addition, it will be useful to keep track of the negative of the cumulative shocks to $z + \log TFPR$ since the beginning of a spell of employment or unemployment, denoted $\Delta z := \hat{w} - \hat{w}^*$, which evolves as

$$\mathrm{d}\Delta z = -(\gamma + \chi)\,\mathrm{d}t + \sigma\,\mathrm{d}\mathcal{W}_t^z + \zeta\,\mathrm{d}\mathcal{W}_t^{\mathrm{TFP}}.$$

Let $G_h(\Delta z)$ and $g^h(\Delta z)$ respectively denote the *cumulative distribution function* (CDF) and *probability density function* (PDF) of cumulative worker productivity shocks within a spell in steady state. Note that the

[15] By studying the labor market effects of monetary policy through the aggregate price level, we abstract from other important monetary policy channels (e.g., Hall, 2017; Kehoe *et al.*, 2019, 2022). Our goal is to highlight how monetary policy "greases the wheels of the labor market" (Tobin, 1972) by moving real wages, thereby redistributing surplus between workers and firms.

support of this distribution is given by $[-\Delta^-, \Delta^+]$, where $\Delta^- := \hat{w}^* - \hat{w}^-$ and $\Delta^+ := \hat{w}^+ - \hat{w}^*$. For any integer $k \in \mathbb{N}$, we define the k^{th} moment of this distribution as $\mathbb{E}_h(\Delta z^k) := \int_{\Delta z} \Delta z^k \, \mathrm{d}G_h(\Delta z)$.

Our model implies a set of observable steady-state statistics. First, agents transition from employment to unemployment at rate s, from unemployment to employment at rate $f(\hat{\theta}(\hat{w}^*))$, and total employment is $\mathcal{E}$. Second, the model implies a distribution of log nominal wage changes between consecutive job spells Δw and distributions of employment durations τ^m and unemployment durations τ^u.[16] We use subscript $\mathcal{D}$ to denote moments of these distributions observed in the microdata—e.g., $\mathbb{E}_{\mathcal{D}}[\cdot]$ and $\mathbb{V}ar_{\mathcal{D}}[\cdot]$ denote the mean and the variance of this distribution, respectively. These moments will be useful to define sufficient statistics for the effects of aggregate shocks on labor market outcomes.

3.2 Sufficient Statistics for Aggregate Employment and Real Wages

Starting from the steady state without aggregate shocks, we consider a small, unanticipated shock $\zeta > 0$ to TFPR at time $t = 0$, so that $\log(TFPR_0) = \lim_{t\uparrow 0} \log(TFPR_t) + \zeta$. We are interested in the economy's IRF and CIR of aggregate employment and TFPR-adjusted wages to such an aggregate shock.[17]

An Illustration. Figure 2 shows the evolution of key variables after an unanticipated one-off increase in TFPR—i.e., the price level or physical productivity. The distribution of real wage-to-productivity ratios $\hat{w}$ shifts to the left (Panel A), resulting in lower TFPR-adjusted wages per capita $\overline{w}_t := \int_0^1 \mathbb{1}[E_{it} = h] w_{it} \, \mathrm{d}i$ (Panel B), movements in the job-separation rate s_t (Panel C), the relative shares of quits and layoffs (Panel D), the job-finding rate f_t (Panel E), and aggregate employment $\mathcal{E}_t$ (Panel F).

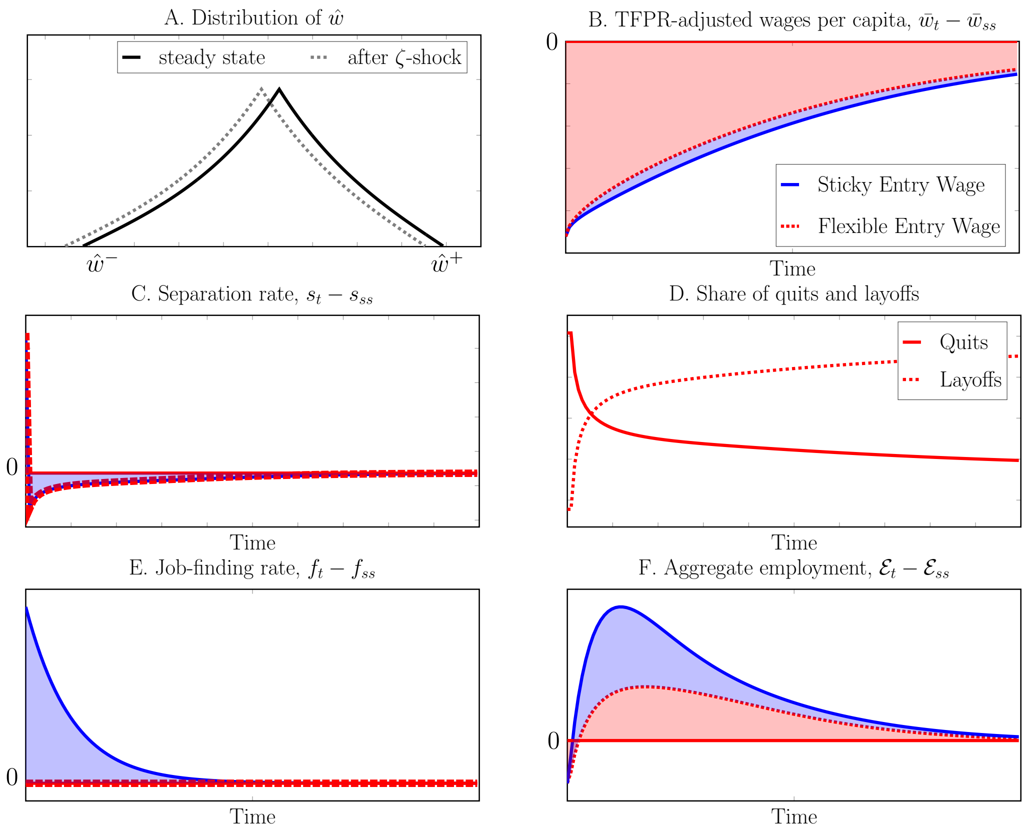
While the wages of employed workers are assumed to be rigid, we allow for two polar cases when modeling the wages of new matches, which are commonly considered a key determinant of the job-finding rate (Pissarides, 2009). In the first case, we model *flexible entry wages* by assuming that unemployed workers adjust their search behavior to the new TFPR level, so $\hat{w}^*$ remains at its steady-state level. Consequently, firms' TFPR-adjusted value of a hire is unaffected, so job-filling and job-finding rates remain unchanged (dashed line in Panel E). The only effect of the TFPR shock is to shift $\hat{w}$ in the inaction region, which affects the time path of endogenous job separations in the form of quits and layoffs (Panel D). In summary, employment dynamics under flexible entry wages are driven only by job-separation rates.

In the second case, we model *sticky entry wages* by assuming that in period $t = 0$ unemployed workers are unaware of the realization of the shock and learn about it only after becoming employed. Given this

[16] While worker wages and productivities do not have a stationary distribution in *levels*, the distribution of wage *changes* across jobs is stationary. Although not necessary for our purposes, the former could be rendered stationary by assuming, for example, that workers permanently leave the labor force at a constant hazard rate.

[17] By the certainty equivalence principle, the IRF following an aggregate shock that departs from the steady state with steady-state policies is equivalent to the solution based on a first-order perturbation of the model with business cycle fluctuations.

FIGURE 2. IMPULSE RESPONSE FUNCTIONS OF LABOR MARKET VARIABLES



Notes: Panel A shows the distribution of real wage-to-productivity ratios $\hat{w} := w_{it} - z_{it} - \log TFPR_t$ in steady state and after a TFPR shock of size ζ. Panels B–F show the IRFs of the average log TFPR-adjusted per-capita wage $\overline{w}_t$, the job-separation rate s_t, the shares of quits and layoffs, the job-finding rate f_t, and aggregate employment $\mathcal{E}_t$, respectively. *Source:* Model simulations.

lack of information, unemployed workers do not adjust their search behavior to the higher TFPR and keep searching for jobs that pay the same nominal wage as in the steady state. Once workers find a job, their future search strategies incorporate their knowledge about the shock and search for jobs that pay the same steady state wage $\hat{w}^*$.[18] Consequently, temporarily lower entry wages induce firms to post more vacancies and the job-finding rate increases (solid line in Panel E). In summary, employment dynamics under sticky entry wages are driven by both job-separation and job-finding rates.

The case of sticky entry wages is motivated by the empirical evidence in Grigsby *et al.* (2021), who document that new-hire wages evolve similarly to incumbent workers within a firm at business cycle frequencies, and Hazell and Taska (2022), who show that wages for new hires rarely change between successive vacancies at the same job. Microfounding this assumption is outside the scope of this paper.[19]

[18]Firms are fully informed about the shock realization, which affects workers' job-finding rate through the free-entry condition.
[19]Since the steady-state entry wage is constrained efficient, any perturbation around that level has a second-order welfare effect

Impulse Responses. Henceforth, our goal is to characterize the effects of a TFPR shock on aggregate employment $\mathcal{E}$ and TFPR-adjusted wages per capita $\overline{w}$. To this end, we define $IRF_x(\zeta, t) := x_t - x_{ss}$ as the value of variable x at time t relative to its steady-state value x_{ss}, following an unanticipated one-off TFPR shock ζ at time 0. Following Alvarez *et al.* (2016a), we define the CIR of variable x to a TFPR shock ζ as

$$CIR_x(\zeta) = \int_0^\infty IRF_x(\zeta, t)\, \mathrm{d}t,$$

which is simply the area under the IRF for all $t > 0$. The CIR summarizes in a single scalar the full path—i.e., the on-impact response and dynamics—of the labor market response to the TFPR shock. Therefore, the CIR can be interpreted as a *TFPR multiplier*. To illustrate the logic behind the CIR, suppose that there are no nominal rigidities so that the nominal wages of both newly hired and incumbent workers respond one-for-one to the shock. In this case, $IRF_x(\zeta, t) = 0$ for all t and thus $CIR_x(\zeta) = 0$ for $x \in \{\mathcal{E}, \overline{w}\}$, which reflects the fact that given our assumptions there are no consequences of TFPR shocks. With nominal rigidities, a TFPR shock affects both employment and wages, the magnitude of which is given by the CIR.

Next, we relate the economy's CIR to moments that can be computed with conventional labor market microdata. A key insight is that the CIR can be characterized only in terms of cross-sectional steady-state moments. Intuitively, changes in a worker's idiosyncratic productivity and changes in TFPR symmetrically affect the log-real-wage-to-productivity ratio $W_{it}/(Z_{it}TFPR_t)$, so the response of a match to idiosyncratic worker productivity changes in steady state can inform the aggregate effects of shocks to TFPR.

For ease of exposition, we assume $\gamma + \chi = 0$ for the remainder of the main text. However, all results and their proofs in Online Appendix B refer to the general case with $\gamma + \chi \gtreqless 0$. At the end of this section, we discuss the differences with the general case.

CIR of Employment with Flexible Entry Wages. To facilitate the exposition of the analysis, we first present the case with flexible entry wages. Proposition 6 characterizes the CIR up to a first order.[20]

Proposition 6. *Up to first order, the CIR of employment under flexible entry wages is given by*

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = -(1 - \mathcal{E}_{ss})\frac{\mathbb{E}_h[\Delta z]}{\sigma^2} + o(\zeta) \tag{23}$$

$$= \underbrace{\frac{1}{f(\hat{\theta}(\hat{w}^*))}}_{\text{average unemployment duration}} \times \underbrace{\frac{1}{\mathbb{V}ar_{\mathcal{D}}[\Delta w]}}_{\text{inverse dispersion}} \times \underbrace{\frac{1}{3}\mathbb{E}_{\mathcal{D}}\left[\Delta w \frac{\Delta w^2}{\mathbb{E}_{\mathcal{D}}[\Delta w^2]}\right]}_{\text{asymmetries}} + o(\zeta). \tag{24}$$

on workers. Thus, the assumption of sticky entry wages could be replaced by any first-order cost of entry wage adjustments arising from imperfect knowledge about aggregate shocks, as in models of sticky information (Mankiw and Reis, 2002; Alvarez *et al.*, 2021), rational inattention (Woodford, 2009; Maćkowiak and Wiederholt, 2009), dispersed knowledge (Hellwig *et al.*, 2014), and level-k thinking (Farhi and Werning, 2019). For notable models of rigid entry wages, see Fukui (2020) and Menzio (2022).

[20] That is, $CIR_x(\zeta) = CIR_x(0) + (CIR_x)'(0)\zeta + o(\zeta^2)$, where $CIR_x(0) = 0$.

Proof. See Online Appendix B.1. □

Let us begin by inspecting the result in equation (23) of Proposition 6, which expresses the CIR in terms of model objects. To build intuition, we consider two cases in which aggregate employment has a zero response to a TFPR shock. In the first case, all job separations are exogenous, so the IRF of the job-separation rate identically equals zero. In the second case, all job separations are endogenous but the mass of workers quitting exactly equals the mass of workers saved from layoffs along the entire IRF. In both cases, equation (23) features $\mathbb{E}_h[\Delta z] = 0$. As a third case, consider an economy with $\mathbb{E}_h[\Delta z] < 0$. Such an economy features a larger share of layoffs than quits, so a shock-induced reduction in TFPR-adjusted wages reduces the separation rate and increases total employment. Finally, notice that the CIR is scaled by the steady-state unemployment rate, $1 - \mathcal{E}_{ss}$, which is informative of the steady-state job-finding rate $f(\hat{\theta}(\hat{w}^*))$ and thus the speed of (re-)matching.

Next, we inspect the result in equation (24), which expresses the CIR in terms of a sufficient statistic that depends only on the observed distributions of wage changes across jobs and unemployment duration. This sufficient statistic is composed of three terms: (i) the average unemployment duration; (ii) the inverse of the dispersion of wage changes; and (iii) a measure of the asymmetries of the wage change distribution. Note that these moments summarize the entire distribution of workers over the inaction band, not just the mass of workers at the separation thresholds. Each of the three terms in the CIR plays an intuitive role. First, the steady-state unemployment rate scales the aggregate employment response. Second, a larger dispersion of wage changes indicates a wider inaction region or matches that are more resilient to shocks, which is inversely related to the share of endogenous separations and responsiveness of aggregate employment to a given impulse. Third, the measure of asymmetries reflects the relative distances of the separation thresholds $\hat{w}^-$ and $\hat{w}^+$ from the entry wage $\hat{w}^*$ and thus the relative incidence of quits versus layoffs. For example, consider a distribution of nominal wage changes that is positively skewed—i.e., featuring a large mass of workers who experience small wage cuts due to a relatively high layoff risk. In this example, a positive shock to TFPR reduces the relative cost of wages, leading firms to reduce layoffs and thereby increasing aggregate employment.

Proposition 6 also shed new light on the conventional wisdom whereby fluctuations in the job separation rate are not the primary driver of aggregate employment dynamics (e.g., Shimer, 2005b). In the context of a TFPR shock, equation (23) points to conditions under which aggregate employment fluctuations due to endogenous job separations are either small or large.[21] Moreover, it allows us to verify

[21]For example, all else equal, the rate of inefficient job separations is more responsive to TFPR shocks for larger TFPR trends χ. Alternatively, following a sequence of negative productivity shocks, an inflationary shock reduces the incidence of inefficient job separations due to firings (see Blanco *et al.*, 2022c, for empirical evidence consistent with this theoretical result).

those conditions in the data. Given the conventional wisdom, one might be tempted to conclude that sticky wages cannot lead to significant inefficiencies at the micro and macro level. However, equation (23) shows that the CIR of aggregate employment can be small despite the presence of inefficient separations at the micro level. Thus, time-series data on aggregate job separations cannot be used to assess the incidence of inefficient turnover. Instead, in to do so, labor market microdata is needed.

CIR of Employment with Sticky Entry Wages. Having characterized the aggregate employment response under flexible entry wages, we now move to the case of sticky entry wages in Proposition 7.

Proposition 7. *Up to first order, the CIR of employment under sticky entry wages is given by*

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = -(1-\mathcal{E}_{ss})\left[\frac{\mathbb{E}_h[\Delta z]}{\sigma^2} - \frac{1}{f(\hat{\theta}(\hat{w}^*))+s}\left[\underbrace{\frac{1-\alpha}{\alpha}\left[\frac{\eta'(\hat{w}^*)}{(1-\eta(\hat{w}^*))} - \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}\right]}_{\text{job-finding effect}} - \underbrace{\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}}_{\substack{\text{new-hires'}\\ \text{separation}\\ \text{effect}}}\right]\right] + o(\zeta) \tag{25}$$

$$= -(1-\mathcal{E}_{ss})\left[\frac{\mathbb{E}_h[\Delta z]}{\sigma^2} - \frac{1}{f(\hat{\theta}(\hat{w}^*))+s}\left[\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} + \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} - \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}\right]\right] + o(\zeta). \tag{26}$$

Proof. See Online Appendix B.2. □

Focusing first on equation (25) of Proposition 7, the first term in brackets reflects the same forces at play in the CIR under flexible entry wages. The remaining terms in brackets capture two new mechanisms at play when entry wages are sticky. First, the *job-finding effect* captures the fact that lower TFPR-adjusted entry wages increase the firm's surplus share (i.e., $\eta'(\hat{w}^*)/(1-\eta(\hat{w}^*))$) but also could affect the expected match duration (i.e., $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})/\mathcal{T}(\hat{w}^*,\hat{\rho})$) and the match surplus, both of which shape firms' incentives to post vacancies. Second, the *new hires' separation effect* captures the fact that lower TFPR-adjusted entry wages directly affect the separation rate of initially unemployed workers (i.e., $\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)/\mathcal{T}(\hat{w}^*,0)$).

Next, we move to equation (26), which is obtained from combining (25) with the optimality condition for $\hat{w}^*$ in (17). The goal of this step is to take advantage of the fact that workers internalize the effect of entry wages on net job creation. To shed further light on the two key elasticities appearing in equation (26), we first show that $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})/\mathcal{T}(\hat{w}^*,\hat{\rho}) - \mathcal{T}'_{\hat{w}}(\hat{w}^*,0)/\mathcal{T}(\hat{w}^*,0) \approx 0$. While this property trivially holds when $\hat{\rho} \downarrow 0$, the following lemma shows that the elasticity of the expected match duration to the entry wage is independent of the discount factor $\hat{\rho}$ up to second order.

Lemma 2. *Up to a second-order approximation of the match duration $\mathcal{T}(\hat{w}, \hat{\rho})$ around $\hat{w} = \hat{w}^*$ and for all $\hat{\rho}$,*

$$\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})} = \frac{\Delta^+ - \Delta^-}{\Delta^+ \Delta^-}.$$

Proof. See Online Appendix B.3. □

Lemma 2 shows that the elasticity of match duration is a function of the quit and layoff thresholds expressed in terms of cumulative shocks to worker productivity, Δ^- and Δ^+. Thus, the key sufficient statistic for the effect of lower entry wages on job creation in equation (26) is $\eta'(\hat{w}^*)/\eta(\hat{w}^*)$. From this, one may be inclined to conclude that the prevalence of inefficient separations cannot be an important determinant of aggregate job creation. However, we find that this is not generally the case. The following result shows this by characterizing the elasticity of the worker's share to changes in the entry wage.

Proposition 8. *The rent-sharing elasticity $\eta'(\hat{w}^*)/\eta(\hat{w}^*)$ satisfies the following properties:*

1. *If $\Delta^-, \Delta^+ \to \infty$, then*

$$\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} = \frac{\alpha + (1-\alpha)\hat{\rho}\hat{U}}{\alpha(1 - \hat{\rho}\hat{U})}. \tag{27}$$

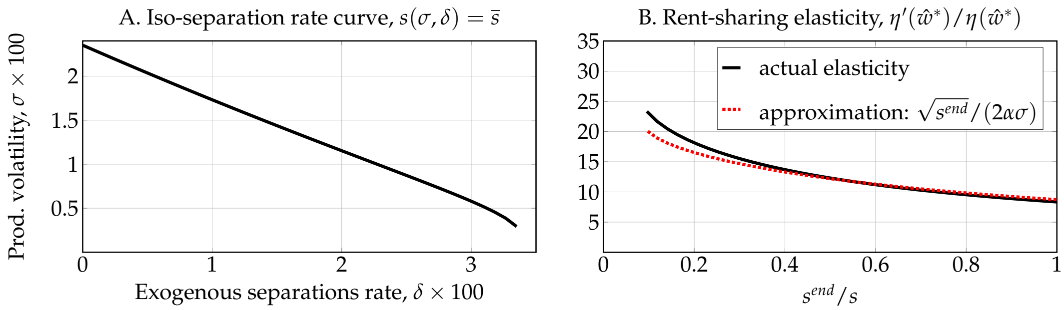
2. *If $\Delta^- = \Delta^+$ and Δ^+ is small enough, then*

$$\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} = \frac{\sqrt{s^{end}}}{2\alpha\sigma}. \tag{28}$$

Proof. See Online Appendix B.4. □

Proposition 8 characterizes the rent-sharing elasticity $\eta'(\hat{w}^*)/\eta(\hat{w}^*)$ under two polar cases, namely as the inaction region grows infinitely wide (Part 1) and for a symmetric and narrow enough inaction region (Part 2). The two results are best explained with the aid of Figure 3, which we construct in two steps. First, we set $\delta = 0$ and calibrate the model to match the U.S. economy's job-finding rate $\bar{f}$ and separation rate $\bar{s}$ with a replacement ratio $\tilde{B}$ of 0.29. We purposely choose α so that $\Delta^+ = \Delta^-$ and thus $\mathcal{T}'_{\hat{w}}(\hat{w}^*, \hat{\rho}) = 0$. Second, for different levels of the exogenous separation rate δ, we find the productivity volatility σ as a function of δ that keeps the total separation rate constant. The objective of this exercise is to vary the fraction of endogenous job separations s^{end}/s from 0 to 100 percent while keeping the opportunity cost $\hat{\rho}\hat{U}$ and the total separation rate fixed by construction. Panel A of the figure shows combinations of δ and σ that constitute the "iso-separation rate curve" defined by $s(\delta, \sigma) = \bar{s}$, while Panel B plots the rent-sharing elasticity $\eta'(\hat{w}^*)/\eta(\hat{w}^*)$ as a function of the share of endogenous job separations s^{end}/s.

FIGURE 3. ISO-SEPARATION RATE CURVE AND THE ELASTICITY OF RENT SHARING



Notes: Panel A shows the iso-separation rate curve defined by $s(\delta, \sigma) = \bar{s}$. Panel B shows the rent-sharing elasticity as a function of the share of endogenous separations (black solid line) and compares it to an approximation of the rent-sharing elasticity given by $\sqrt{s^{end}}/(2\alpha\sigma)$ based on equation (28). Note that the productivity volatility σ is a function of δ derived from the iso-separation rate curve. The parameter values for $\delta = 0$ are $(\gamma + \chi, \sigma, \rho, \alpha, \tilde{K}, \delta, \tilde{B}) = (0, 0.0235, 0.0048, 0.452, 1.87, 0, 0.29)$. The steady-state targets for this calibration are $(f(\theta(\hat{w}^*)), s) = (0.55, 0.034)$ with $\Delta^+ = \Delta^-$. *Source:* Model simulations.

Consider the limiting case as $\delta \to \bar{s}$ (i.e., $s^{end}/s \to 0$), so that all separations are exogenous, as in Part 1 of Proposition 8. Then, a marginal increase in the entry wage increases workers' surplus share according to equation (27), reflecting the well-known result that, absent inefficient turnover, the rent-sharing elasticity is inversely proportional to the flow surplus $1 - \hat{\rho}\hat{U}$ (Shimer, 2005a). As the share of inefficient separations (i.e., s^{end}/s) increases in Panel B of Figure 3, the rent-sharing elasticity (black solid line) decreases due to a novel mechanism in our framework with sticky entry wages. A higher entry wage increases the layoff probability and decreases the quit probability. By construction, the expected duration of the match does not change, so match surplus is constant. As workers make optimal quit decisions, a marginally lower quit probability leaves their value unchanged due to an envelope condition (i.e., $\hat{W}'(\hat{w}^-) = 0$). But a marginal increase in the layoff probability reduces the worker's value, since layoff decisions are made by the firm. Therefore, the rent-sharing elasticity decreases in the share of endogenous job separations, which the following Section shows how to measure using conventional labor market microdata.

4 Model Extensions

In this section, we show that our baseline model can be extended in two important dimensions. These extensions generate new insights, which are of interest in their own right, but also demonstrate that our framework can speak to many phenomena beyond the restrictions imposed in the main text.

4.1 Staggered Wage Renegotiations

As a first extension, Supplementary Material IV.1 extends our model to feature staggered wage renegotiations, which we assume to follow a Nash bargaining protocol with worker weight α and to occur at rate $\delta^r \geq 0$ à la Calvo (1983). The generalized model nests as a special case the economy with fully rigid wages ($\delta^r \to 0$) presented in the main text and also the polar opposite case with fully flexible wages ($\delta^r \to \infty$). By convexifying between these two cases, the generalized model allows us to replicate empirical frequencies of wage changes in employment. The generalized model with staggered wage renegotiations yields several results but our main conclusion is that all of our key insights extend to an environment with $0 < \delta^r < \infty$ subject to minor modifications. For example, we extend Proposition 2 with a simple change in the definition of the match duration $\mathcal{T}(\hat{w}, \hat{\rho})$ to reference the duration of a wage spell. In our baseline model, match surplus is related to the expected discounted duration of a match. With wage renegotiations, the surplus is proportional to the expected discounted duration of the current wage spell. Second, we show that an increase in the frequency of wage renegotiations widens the inaction region. Importantly, we extend all sufficient statistics for the CIRs to the case of staggered wage renegotiations. For example, the CIR of aggregate employment under flexible entry wages and staggered wage renegotiations is

$$\frac{\text{CIR}_{\mathcal{E}}(\zeta)}{\zeta} = \frac{1}{3f(\hat{\theta}(\hat{w}^*))} \frac{\left(1 - \frac{\mathcal{F}^{\Delta w}}{s+\mathcal{F}^{\Delta w}}\right) \mathbb{E}_{\mathcal{D}}\left[\Delta w_{EUE}^3\right] + \frac{\mathcal{F}^{\Delta w}}{s+\mathcal{F}^{\Delta w}} \overline{\mathbb{E}}[\Delta w_B^3]}{\left[\left(1 - \frac{\mathcal{F}^{\Delta w}}{s+\mathcal{F}^{\Delta w}}\right) \mathbb{E}_{\mathcal{D}}\left[\Delta w_{EUE}^2\right] + \frac{\mathcal{F}^{\Delta w}}{s+\mathcal{F}^{\Delta w}} \overline{\mathbb{E}}[\Delta w_B^2]\right]^2} + o(\zeta),$$

where $\mathcal{F}^{\Delta w}$ is the frequency of wage changes, Δw_{EUE} is the wage change across matches, and Δw_B is the wage change within a match. As $\mathcal{F}^{\Delta w} \to 0$, we recover the CIR of aggregate employment in Proposition 6. More generally, the generalized sufficient statistics follow similar intuitions to those in our baseline model but incorporating a convex combination of wage changes across and within matches.

4.2 Partial Commitment

As a second extension, Supplementary Material IV.2 extends our model to feature *partial commitment* by introducing worker- and firm-specific costs of unilateral separations, $\phi^h e^z$ and $\phi^j e^z$. This nests as a special case the economy with two-sided lack of commitment ($\phi^h, \phi^j \to 0$) and also the case with full commitment ($\phi^h, \phi^j \to \infty$). By allowing for arbitrary degrees of commitment on behalf of workers and firms, the generalized model with partial commitment can capture real-world costs from quitting for workers (e.g., gaps on workers' resumes) or from layoffs for firms (e.g., experience rating for UI). While partial commitment generates additional asymmetries in the incidence of quits and layoffs, our main insights remain qualitatively unchanged. Here, we highlight three main results. First, as expected, separation costs increase the width of the continuation region, leading to a lower rate of inefficient separations. Second, and more surprisingly, separation costs increase job creation by providing a form of commitment to agents, which increases ex-ante match surplus through an extended match duration. Third, separation costs do not affect the sufficient statistic formulas with flexible entry wages in Proposition 6 but they do affect their equilibrium values. With sticky entry wages, the sufficient statistic in Proposition 7 is extended with an additional term, which captures the effect of costly future separations on the job-finding rate.

5 Identification Based on Labor Market Microdata

We prove formal identification of our model in two steps. First, we recover the unobserved distribution of cumulative productivity changes using labor market microdata on wage changes between jobs. Second, we link the prevalence of inefficient job separations to the distribution of cumulative productivity changes. For simplicity, we focus on the case with $\gamma + \chi = 0$. Supplementary Material V presents the general case.

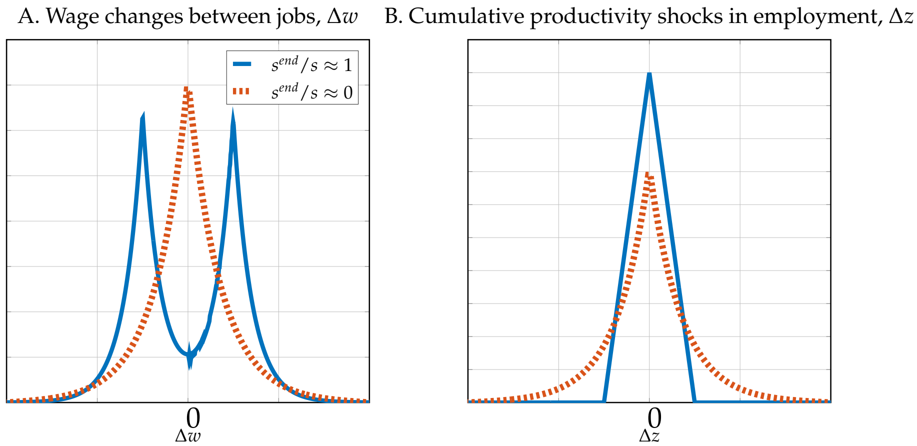
5.1 Intuition Behind the Identification Argument

Intuitively, how are observed wage changes between jobs informative about the prevalence of inefficient job separations? Figure 4 illustrates their distribution, $l^w(\Delta w)$ (Panel A), and that of cumulative productivity shocks in employment, $g^h(\Delta z)$ (Panel B), for each of two extreme cases.

In the first case, when most job separations are endogenous (solid blue line), then most separated workers experienced cumulative productivity shocks in employment of either $-\Delta^-$ or Δ^+. As a result, the distribution of wage changes of laid-off workers is concentrated around $-\Delta^-$, while that of workers who quit is concentrated around Δ^+. This results in a bimodal distribution of wage changes between jobs, with dispersion around the two modes caused by productivity shocks during unemployment.

In the second case, when most job separations are exogenous (dashed red line), then most separated workers experienced cumulative productivity shocks in employment close to zero—i.e., away from the two endogenous separation thresholds. With a constant job-finding probability during unemployment, the distribution of wage changes between jobs mimics the distribution of cumulative productivity shocks in employment, which is symmetric and single peaked at zero.

FIGURE 4. DISTRIBUTIONS OF WAGE CHANGES AND CUMULATIVE PRODUCTIVITY SHOCKS



Notes: The figure plots the distribution of wage changes between jobs $l^w(\Delta w)$ and the distribution of cumulative productivity shocks in employment $g^h(\Delta z)$ for two calibrations. In the first calibration, we set $(\Delta^-, \Delta^+, \gamma + \chi, \sigma, \delta, f(\theta(\hat{w}^*))) = (0.05, 0.05, 0, 0.02, 0, 0.5)$ so that $s^{end}/s \approx 1$ (blue solid line). In the second calibration, we set $(\Delta^-, \Delta^+, \gamma + \chi, \sigma, \delta, f(\theta(\hat{w}^*))) = (0.2, 0.2, 0, 0.1, 0.04, 0.5)$ so that $s^{end}/s \approx 0$ (red dashed line). *Source:* Model simulations.

5.2 Formal Identification Results

We provide equilibrium conditions for the steady-state distributions of cumulative productivity shocks $g^h(\Delta z)$ and $g^u(\Delta z)$ in Supplementary Material V.1. The following result states our formal identification result for the distribution of cumulative productivity shocks in employment, $g^h(\Delta z)$, which our analysis relies on.

Proposition 9. *The distribution of cumulative productivity shocks in employment, $g^h(\Delta z)$, can be recovered as:*

1. *The volatility of workers' productivity shocks is recovered from*

$$\sigma^2 = \frac{\mathbb{E}_{\mathcal{D}}[(\Delta w)^2]}{\mathbb{E}_{\mathcal{D}}[\tau^m + \tau^u]}. \tag{29}$$

2. *The distribution of workers' cumulative productivity shocks is recovered from*

$$g^h(\Delta z) = s\mathcal{E}\left[\int_{-\Delta^-}^{\Delta z} \frac{2(\Delta z - y)}{\sigma^2}\overline{g}^h(y)\,\mathrm{d}y + \overline{G}^h(-\Delta^-)\frac{2(\Delta z + \Delta^-)}{\sigma^2}\right], \tag{30}$$

where $\overline{G}^h(\Delta z)$ is the distribution of Δz conditional on a job separation, which is given by

$$\overline{G}^h(\Delta z) = \frac{\sigma^2}{2f(\hat{\theta}(\hat{w}^*))}\frac{\mathrm{d}l^w(-\Delta z)}{\mathrm{d}z} - [1 - L^w(-\Delta z)], \tag{31}$$

where $L^w(\Delta w)$ denotes the CDF corresponding to the PDF of wage changes between jobs, $l^w(\Delta w)$.

Proof. See Supplementary Material V.2. □

Equation (29) in Part 1 of Proposition 9 shows that the volatility of productivity σ equals the dispersion of wage changes between jobs, $\mathbb{E}_{\mathcal{D}}[(\Delta w)^2]$, divided by the average time elapsed between the starting dates of two consecutive jobs $\mathbb{E}_{\mathcal{D}}[\tau^m + \tau^u]$. Next, in order to identify the distribution of Δz conditional on a job separation, we consider the dynamics of h-to-u and u-to-h worker flows. Consider a worker who at time t_0 starts a job with wage w_{t_0}, at time $t_0 + \tau^m$ separates, and at time $t_0 + \tau^m + \tau^u$ finds a new job with wage $w_{t_0+\tau^m+\tau^u}$. This worker's wage change between jobs is given by

$$\Delta w = w_{t_0+\tau^m+\tau^u} - w_{t_0}, \tag{32}$$

$$= \underbrace{(w_{t_0+\tau^m+\tau^u} - z_{t_0+\tau^m+\tau^u})}_{=\hat{w}^*} - \underbrace{(w_{t_0} - z_{t_0})}_{=\hat{w}^*} + \underbrace{z_{t_0+\tau^m+\tau^u} - z_{t_0},}_{=\Delta z \text{ after } h\text{-}u\text{-}h \text{ transition}} \tag{33}$$

$$= \underbrace{\hat{w}^* - \hat{w}^*}_{=0} + \underbrace{z_{t_0+\tau^m} - z_{t_0}}_{\Delta z | h\text{-}u \text{ transition starting from } z_{t_0}} + \underbrace{z_{t_0+\tau^m+\tau^u} - z_{t_0+\tau^m}.}_{\Delta z | u\text{-}h \text{ transition starting from } z_{t_0+\tau^m}} \tag{34}$$

Equation (32) gives the definition of Δw. Next, equation (33) adds and subtracts $z_{t_0+\tau^h+\tau^u} - z_{t_0}$ before grouping terms into the wage-to-productivity ratio in the old job, the wage-to-productivity ratio in the new job, and the cumulative productivity shocks between the starting dates of the two jobs. Then, equation (34) adds and subtracts $z_{t_0+\tau^m}$ before applying the definition of $\hat{w}^*$ and that of Δz. In summary, the wage change across jobs equals the sum of three random variables: (i) the difference of entry wage-to-productivity ratios across jobs, which is identically zero; (ii) Δz conditional on a job separation starting from z_{t_0}; and (iii) Δz conditional on finding a new job, which is independent of productivity z_t for $t \in (t_0 + \tau^m, t_0 + \tau^m + \tau^u)$. Exploiting this independence, we can use data on Δw to infer the distribution of the second term, which is given by (31). Finally, the distribution of cumulative productivity shocks in (30) can be derived from (31) by exploiting ergodicity—i.e., the cross-sectional distribution of cumulative shocks can be deduced from the distribution of shocks experienced during completed job spells.

The following result derives the share of inefficient job separations in terms of known objects:

Proposition 10. *The share of inefficient job separations, s^{end}/s, is given by*

$$\frac{s^{end}}{s} = \frac{\frac{\sigma^2}{2\mathcal{E}}\left[\lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+}(g^h)'(\Delta z)\right]}{s}. \tag{35}$$

Proof. The proof directly follows from the equilibrium conditions derived in Supplementary Material V.1. □

Equation (35) in Proposition 10 expresses the share of inefficient job separations, s^{end}/s, in terms of the distribution of cumulative productivity shocks, $g^h(\Delta z)$, already recovered in Proposition 9 above.

5.3 Discussion of Identification Assumptions

We conclude with a discussion of some assumptions underlying our identification result, which could be relaxed: (i) the threshold nature of policies; (ii) the absence of other sources of wage changes; (iii) specifics of the productivity process; and (iv) the absence of alternative sources of heterogeneity.

Regarding (i), that the job-separation rate is δ for $\Delta z_t \in (-\Delta^-, \Delta^+)$ and ∞ for $\Delta z_t \in \{-\Delta^-, \Delta^+\}$ is not crucial and can be replaced with a general job-separation hazard, as in Alvarez *et al.* (2021).

Regarding (ii), we have ignored other sources of wage changes such as those arising from job-to-job transitions. This assumption could be relaxed following the methodology in Baley and Blanco (2022).

Regarding (iii), our assumption of a particular stochastic process for Δz_t can be empirically tested and generalized, as in Baley and Blanco (2021). For example, it would be straightforward to let the parameters of the productivity process depend on a worker's employment state. What is critical is that the data contain enough information to recover productivity changes during unemployment. In our model, lack of selection in job finding and the identified productivity process in employment together yield this result.

Finally, regarding (iv), we abstract from firm and match productivity shocks. This simplification is motivated by empirical evidence suggesting that worker heterogeneity explains the largest share of wage dynamics (Guiso *et al.*, 2005; Friedrich *et al.*, 2021; Engbom *et al.*, 2023). Additionally, a benefit of focusing on worker heterogeneity is that it allows our model to parsimoniously speak to both worker quits and firm layoffs—both of which are empirically salient (e.g. Elsby *et al.*, 2010). Conversely, a model with only firm- or match-specific heterogeneity would predict no worker quits because workers' flow value of employment and flow value of nonemployment would both be constant within a wage segment, even in the presence of staggered renegotiations. Adding other sources of heterogeneity would require different data (e.g., linked employer-employee records), different model ingredients (e.g., a multi-worker firm wage-setting protocol), and different identification (e.g., exploiting synchrony in coworker outcomes). Future work could calibrate richer models with multiple dimensions of empirically disciplined heterogeneity.

6 Exploring the Quantitative Implications

This section takes a first pass at confronting our theory with the data. To this end, we feed the empirical distribution of wage changes and employment transitions into our model, which we use to summarize the economy's dynamic response to an unanticipated aggregate shock through our CIR sufficient statistic.

6.1 Data

Data Source. To calibrate our model, we use the 2008–2012 panel of the SIPP, which is a multistage, stratified, representative sample of the U.S. population. Each individual is followed for a period of up to 48 months. For each month, the data contain the respondent's employment status and wages.

Sample Selection. We impose standard sample selection criteria, restricting the data to male household reference persons aged 25–54. We keep only workers who were employed for at least 12 months over the panel and job spells that last for a minimum of two months. Lastly, in an attempt to minimize measurement error, we focus on workers who directly report their hourly wages. Because the distribution of (filtered) wage changes is characterized by a few outliers, we trim observations in the top and bottom five percent of the distribution of wage changes. Finally, we restrict ourselves to job transitions within industries and occupations with at least one month of unemployment between jobs.

Variable Construction. To filter out transitory wage fluctuations due to measurement error, we implement the wage filter developed in Blanco *et al.* (2022a) based on the break test of Stevens (2020), which is similar to that by Barattieri *et al.* (2014). Our filter divides employment spells into segments and performs a nonparametric test for whether wages across segments belong to the same distribution. This procedure requires specifying the threshold to reject the null hypothesis of no change in the distribution, which we calibrate to match the within-job quarterly wage change frequency reported by Barattieri *et al.* (2014).[22] Finally, we use the time-aggregation-corrected job finding rate based on the method in Shimer (2012) based on the *Current Population Survey* (CPS) data for the same period.

6.2 Empirical Sufficient Statistics

Next, we estimate the sufficient statistic for the CIR of aggregate employment presented in equation (24). To do so, we compute moments of the observed distribution of wage changes between jobs, Δw, and the duration between filtered-wage changes, τ. Table 2 shows the results of this estimation. The measured CIR for employment is 3.13, indicating that shocks to TFPQ or the aggregate price level have expansionary effects. In other words, the model points toward layoffs being the most important source of unemployment risk, consistent with empirical evidence by Elsby *et al.* (2010).

[22]Specifically, we match a quarterly wage-change frequency of 16.3 percent for incumbent workers from Barattieri *et al.* (2014).

TABLE 2. SUFFICIENT STATISTICS FOR THE CIR OF AGGREGATE EMPLOYMENT

$\frac{1}{f(\hat{\theta}(\tilde{w}^*))}$	$\frac{1}{\mathbb{V}ar_{\mathcal{D}}[\Delta w]}$	$\frac{1}{3}\left[\mathbb{E}_{\mathcal{D}}\left[\Delta w \frac{\Delta w^2}{\mathbb{E}_{\mathcal{D}}[\Delta w^2]}\right]\right]$	$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta}$
4.14	33.31	0.022	3.13

Notes: This table shows the CIR of aggregate employment (rightmost column) that is equal to the product of its three components (first three columns) based on equation (24). *Source:* SIPP 2008–2012 and CPS 2008–2012.

7 Conclusion

There is mounting empirical evidence that not all job separations can be rationalized using bilaterally efficient models. To understand the sources and consequences of inefficient turnover, we developed a theory of labor markets with four features: search frictions, productivity fluctuations, wage rigidity due to staggered renegotiations, and two-sided lack of commitment to remaining in a match. A defining feature of our theory was the distinction between quits and layoffs as two separate equilibrium outcomes following a voluntary-involuntary interpretation. Inefficient turnover manifested itself not only in job separations but also in job creation and wage determination. We first characterized the unique BRE of this model. We then derived sufficient statistics for the labor market response to aggregate shocks (cf. Alvarez *et al.*, 2016a) based on conventional labor market microdata on wage changes between jobs.

While the parsimony of this framework is useful in delineating several novel theoretical insights, an empirically grounded quantification may require several extensions. Adding nonhomotheticities (e.g., in home production), alternative sources of heterogeneity (e.g., match productivity shocks), and additional quit motives (e.g., on-the-job search) could yield efficient endogenous separations. Blanco and Drenik (2023) take a step in this direction. Furthermore, adding asymmetric renegotiation costs would add a state-dependent motive for wage adjustments and allow the model to match asymmetries in the empirical distribution of wage changes (e.g., Blanco *et al.*, 2022b). We expect that many of our insights will carry over to such richer environments. Incorporating these and other features into a unified framework with empirical discipline will allow future work to assess the implications of inefficient turnover in the labor market for issues including monetary policy (e.g., state dependent employment effects), fiscal policy (e.g., UI), and labor market regulations (e.g., severance pay).

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A Theory of Labor Markets with Inefficient Turnover

Online Appendix

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Contents of the Online Appendix

A Proofs for Section 2: A Model of Labor Markets with Inefficient Turnover

A.1 Proof of Lemma 1

For the next proof, it will be useful to define the equilibrium conditions and the normalized equilibrium conditions. The equilibrium conditions are:

$$\rho u(z) = \tilde{B}e^z + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 u(z)}{\partial z^2} + \max_w f(\theta(z;w))[h(z;w) - u(z)], \tag{A.1}$$

$$0 = \min\left\{\tilde{K}e^z - q(\theta(z;w))j(z;w), \theta(z;w)\right\}, \tag{A.2}$$

$$\rho h(z;w) = \begin{cases} \max\left\{e^w + \gamma\frac{\partial h(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 h(z;w)}{\partial z^2} + \delta\left[u(z) - h(z;w)\right], \rho u(z)\right\} & \forall z \in \mathcal{Z}^{j*}(w), \\ \rho u(z) & \forall z \in (\mathcal{Z}^{j*}(w))^c, \end{cases} \tag{A.3}$$

$$\rho j(z;w) = \begin{cases} \max\left\{e^z - e^w + \gamma\frac{\partial j(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 j(z;w)}{\partial z^2} - \delta j(z;w), 0\right\} & \forall z \in \mathcal{Z}^{h*}(w), \\ 0 & \forall z \in (\mathcal{Z}^{h*})(w)^c, \end{cases} \tag{A.4}$$

$$\mathcal{Z}^{j*}(w) = int\left\{z \in \mathbb{R} : j(z;w) > 0 \text{ or } e^z - e^w > 0\right\}, \tag{A.5}$$

$$\mathcal{Z}^{h*}(w) = int\left\{z \in \mathbb{R} : h(z;w) > u(z) \text{ or } 0 < e^w - \rho u(z) + \gamma\frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 u(z)}{\partial z^2}\right\}, \tag{A.6}$$

$$j(\cdot;w) \in \mathbb{C}^1(\mathcal{Z}^{h*}(w)) \cap \mathbb{C}(\mathbb{R}), \;\; h(\cdot;w) \in \mathbb{C}^1(\mathcal{Z}^{j*}(w)) \cap \mathbb{C}(\mathbb{R}). \tag{A.7}$$

The equilibrium conditions in the normalized state space $\hat{w}$ are:

$$\hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}} f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w}), \tag{A.8}$$

$$0 = \min\left\{\tilde{K} - q(\hat{\theta}(\hat{w}))\hat{J}(\hat{w}), \hat{\theta}(\hat{w})\right\}, \tag{A.9}$$

$$\hat{\rho}\hat{W}(\hat{w}) = \begin{cases} \max\{0, e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\frac{\partial \hat{W}(\hat{w})}{\partial \hat{w}} + \frac{\sigma^2}{2}\frac{\partial^2 \hat{W}(\hat{w})}{\partial \hat{w}^2} - \delta\hat{W}(\hat{w})\} & \forall \hat{w} \in \hat{\mathcal{Z}}^{j*} \\ 0 & \forall \hat{w} \in (\hat{\mathcal{Z}}^{j*})^c \end{cases} \tag{A.10}$$

$$\hat{\rho}\hat{J}(\hat{w}) = \begin{cases} \max\{0, 1 - e^{\hat{w}} - \hat{\gamma}\frac{\partial \hat{J}(\hat{w})}{\partial \hat{w}} + \frac{\sigma^2}{2}\frac{\partial^2 \hat{J}(\hat{w})}{\partial \hat{w}^2} - \delta\hat{J}(\hat{w})\} & \forall \hat{w} \in \hat{\mathcal{Z}}^{h*} \\ 0 & \forall \hat{w} \in (\hat{\mathcal{Z}}^{h*})^c \end{cases} \tag{A.11}$$

$$\hat{\mathcal{Z}}^{h*} := int\left\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) > 0 \; or \; (e^{\hat{w}} - \hat{\rho}\hat{U}) > 0\right\}, \tag{A.12}$$

$$\hat{\mathcal{Z}}^{j*} := int\left\{\hat{w} \in \mathbb{R} : \hat{J}(\hat{w}) > 0 \; or \; \left(1 - e^{\hat{w}}\right) > 0\right\}, \tag{A.13}$$

$$\hat{J} \in \mathbb{C}^1(\hat{\mathcal{Z}}^{h*}) \cap \mathbb{C}(\mathbb{R}), \;\; \hat{W} \in \mathbb{C}^1(\hat{\mathcal{Z}}^{j*}) \cap \mathbb{C}(\mathbb{R}), \tag{A.14}$$

where $\hat{w} = w - z$, $\hat{\rho} = \rho - \gamma - \sigma^2/2$ and $\hat{\gamma} = \gamma + \sigma^2$.

Lemma 1. *Assume that values* $(u(z), h(z;w), j(z;w), \theta(z;w))$ *and policies* $(w^*(z), \mathcal{Z}^{j*}(w), \mathcal{Z}^{h*}(w))$ *are a recursive equilibrium—i.e., they satisfy conditions* (A.1), (A.2), (A.3), (A.4), (A.5), (A.6), *and* (A.7)—, *then*

$$(\hat{U}, \hat{J}(w-z), \hat{W}(w-z), \hat{\theta}(w-z), \hat{w}^*) = \left(\frac{u(z)}{e^z}, \frac{j(z;w)}{e^z}, \frac{h(z;w)-u(z)}{e^z}, \theta(z;w), w^*(z) - z\right).$$

satisfy (A.8), (A.9), (A.10), (A.11), *and* (A.14) *with continuation sets* $\hat{\mathcal{Z}}^{h*}$ *and* $\hat{\mathcal{Z}}^{j*}$ *given by* (A.12) *and* (A.13). *Moreover, if* $(\hat{U}, \hat{J}(\hat{w}), \hat{W}(\hat{w}), \hat{\theta}(\hat{w}))$ *and policies* $(\hat{w}^*, \hat{\mathcal{Z}}^{j*}, \hat{\mathcal{Z}}^{h*})$ *satisfy* (A.8)-(A.14), *then*

$$(u(z), j(z;w), h(z;w), \theta(z;w), w^*(z)) = (\hat{U}e^z, \hat{J}(w-z)e^z, (\hat{W}(w-z) + \hat{U})e^z, \hat{\theta}(w-z), \hat{w}^* + z)$$

satisfy (A.1), (A.2), (A.3), (A.4), *and* (A.7) *with continuation sets* $\mathcal{Z}^{h*}(w)$ *and* $\mathcal{Z}^{j*}(w)$ *given by* (A.5) *and* (A.6).

Proof. The general idea for the proof is to use a guess-and-verify strategy for each equilibrium condition.

Condition (A.1) holds iff. (A.8) is satisfied: Using $\hat{U} = \frac{u(z)}{e^z}$, we have $\hat{U}e^z = u'(z)$ and $\hat{U}e^z = u''(z)$. Using this result and the fact that $\theta(z;w) = \hat{\theta}(w-z)$, and $\hat{W}(w-z) = \frac{h(z;w)-u(z)}{e^z}$,

$$\rho u(z) = \tilde{B}e^z + \gamma u'(z) + \frac{\sigma^2}{2}u''(z) + \max_w f(\theta(z;w))[h(z;w) - u(z)] \iff$$
$$\rho \hat{U}e^z = \tilde{B}e^z + \gamma \hat{U}e^z + \frac{\sigma^2}{2}\hat{U}e^z + \max_w f(\hat{\theta}(w-z))[\hat{W}(w-z)e^z] \iff$$
$$\underbrace{\left(\rho - \gamma - \frac{\sigma^2}{2}\right)}_{=\hat{\rho}} \hat{U}e^z = \tilde{B}e^z + e^z \max_{\hat{w}} f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w}) \iff \hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}} f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w}).$$

Condition (A.2) holds iff. (A.9) is satisfied: Using $\hat{J}(w-z) = \frac{j(z;w)}{e^z}$ and $\theta(z;w) = \hat{\theta}(w-z)$, we have

$$0 = \min\left\{\tilde{K}e^z - q(\theta(z;w))j(z;w), \theta(z;w)\right\} \iff 0 = \min\left\{\tilde{K}e^z - q(\hat{\theta}(w-z))\hat{J}(w-z)e^z, \hat{\theta}(w-z)\right\}$$
$$\iff 0 = \min\left\{\tilde{K} - q(\hat{\theta}(\hat{w}))\hat{J}(\hat{w}), \frac{\hat{\theta}(\hat{w})}{e^z}\right\}.$$

Now, we show that $0 = \min\left\{\tilde{K} - q(\hat{\theta}(\hat{w}))\hat{J}(\hat{w}), \frac{\hat{\theta}(\hat{w})}{e^z}\right\}$ iff. $0 = \min\left\{\tilde{K} - q(\hat{\theta}(\hat{w}))\hat{J}(\hat{w}), \hat{\theta}(\hat{w})\right\}$. Assume that $\tilde{K} - q(\hat{\theta}(\hat{w}))\hat{J}(\hat{w}) = 0$. Then $\frac{\hat{\theta}(\hat{w})}{e^z} \geq 0$ which implies that $\hat{\theta}(\hat{w}) \geq 0$. Now, assume that $\hat{\theta}(\hat{w}) = 0$, then $\frac{\hat{\theta}(\hat{w})}{e^z} = 0$.

Condition (A.3) holds iff. (A.10) is satisfied: Assume $h(z;w)$ satisfies (A.3) and $z \in \mathcal{Z}^j(w)$. Then,

$$0 = \max\{u(z) - h(z;w), -\rho h(z;w) + \gamma\frac{\partial h(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 h(z;w)}{\partial z^2} + \delta(u(z) - h(z;w)) + e^w\}.$$

Using that $\hat{U} = \frac{u(z)}{e^z}$ and $\hat{W}(w-z) = \frac{h(z;w)-u(z)}{e^z}$, we have $h(z;w) = \hat{W}(w-z)e^z + \hat{U}e^z$, $\frac{\partial h(z;w)}{\partial z} = \hat{W}(w-z)e^z - \hat{W}'(w-z)e^z + \hat{U}e^z$, and $\frac{\partial^2 h(z;w)}{\partial z^2} = \hat{W}(w-z)e^z - 2\hat{W}'(w-z)e^z + \hat{W}''(w-z)e^z + \hat{U}e^z$. Thus, since $e^z > 0$

$$
\begin{aligned}
0 &= \max\left\{u(z) - h(z;w), -\rho h(z;w) + \gamma\frac{\partial h(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 h(z;w)}{\partial z^2} + \delta(u(z) - h(z;w)) + e^w\right\} \\
&= \max\{-\hat{W}(w-z), -(\underbrace{\rho - \gamma - \sigma^2/2}_{=\hat{\rho}} + \delta)\hat{W}(w-z) - \underbrace{\gamma + \sigma^2}_{=\hat{\gamma}}\hat{W}'(w-z) + \frac{\sigma^2}{2}\hat{W}''(w-z) - \underbrace{\rho - \gamma - \sigma^2/2}_{=\hat{\rho}}\hat{U} + e^{w-z}\}, \\
&= \max\left\{-\hat{W}(\hat{w}), -(\hat{\rho} + \delta)\hat{W}(\hat{w}) - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}) - \hat{\rho}\hat{U} + e^{\hat{w}}\right\},
\end{aligned}
$$

or

$$
\hat{\rho}\hat{W}(\hat{w}) = \max\left\{0, e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}) - \delta\hat{W}(\hat{w})\right\}.
$$

Now, assume that $z \in (\mathcal{Z}^j(w))^c$.

$$
u(z) - h(z;w) = 0 \iff \frac{u(z) - h(z;w)}{e^z} = 0 \iff \hat{W}(\hat{w}) = 0.
$$

The equivalence between (A.4) and (A.11) can be established following similar steps.

Condition (A.5) holds iff. (A.12) is satisfied: Assume $z \in \mathcal{Z}^h(w)$. Then,

$$
h(z;w) > u(z) \text{ or } -\rho u(z) + \gamma u'(z) + \frac{\sigma^2}{2}u''(z) + e^w > 0
$$

Using that $\hat{U} = \frac{u(z)}{e^z}$ and $\hat{W}(w-z) = \frac{h(z;w)-u(z)}{e^z}$, with $e^z > 0$

$$
\begin{aligned}
&\hat{W}(w-z) > 0 \text{ or } -\rho\hat{U}e^z + \gamma\hat{U}e^z + \frac{\sigma^2}{2}\hat{U}e^z + e^w > 0 \iff \\
&\hat{W}(w-z) > 0 \text{ or } e^{w-z} - (\rho - \gamma - \sigma^2/2)\hat{U} > 0 \iff \hat{W}(\hat{w}) > 0 \text{ or } e^{\hat{w}} - \hat{\rho}\hat{U} > 0.
\end{aligned}
$$

Thus, $z \in \mathcal{Z}^h(w)$ iff. $w - z \in \hat{\mathcal{Z}}^h$. The equivalence between (A.6) and (A.13) follows from similar steps.

Remaining conditions: The equivalence between equations (A.7) and (A.14) is trivially established.

□

A.2 Proof of Proposition 2

Proof. We prove each equilibrium property separately.

1. Using the recursive definition of the value function, we have

$$\hat{W}(\hat{w}) = \mathbb{E}\left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t}(e^{\hat{w}_t} - \hat{\rho}\hat{U})\,\mathrm{d}t | \hat{w}_0 = \hat{w}\right]$$
$$\hat{J}(\hat{w}) = \mathbb{E}\left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t}(1 - e^{\hat{w}_t})\,\mathrm{d}t | \hat{w}_0 = \hat{w}\right]$$

where τ^{m*} is the nontrivial Nash equilibrium of the game between the firm and the worker. Summing up the previous two equations, we have

$$\hat{S}(\hat{w}) = \hat{W}(\hat{w}) + \hat{J}(\hat{w}) = \mathbb{E}_{\hat{w}}\left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t}(1-\hat{\rho}\hat{U})\,\mathrm{d}t\right] = (1-\hat{\rho}\hat{U})\mathbb{E}_{\hat{w}}\left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t}\,\mathrm{d}t\right] = (1-\hat{\rho}\hat{U})\mathcal{T}(\hat{w},\hat{\rho}).$$

Now, we show that $1 > \hat{\rho}\hat{U} > \tilde{B}$ by contradiction. Assume that $\hat{\rho}\hat{U} \leq \tilde{B} < 1$. Using the free entry condition and worker optimality, we have that $\hat{\theta}(\hat{w}) \geq 0$ and $\hat{W}(\hat{w}) \geq 0$ for all $\hat{w}$; thus, the product is also nonnegative at $\hat{w}^*$ and

$$\hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}} f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w}) \geq \tilde{B} \quad \Longrightarrow \quad \hat{\rho}U \geq \tilde{B},$$

So, we have that $\hat{\rho}\hat{U} = \tilde{B} < 1$. Then, we have that $\max_{\hat{w}} f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w}) = 0$ and, therefore, $f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w}) = 0\ \forall\hat{w}$. By weakly dominated strategies, we have that $(\log(\hat{\rho}\hat{U}), 0) = (\log(\tilde{B}), 0) \subset \mathcal{Z}^j \cap \mathcal{Z}^h$. Thus, for any $\hat{w} \in (\log(\tilde{B}), 0)$, we have that $(\hat{J}(\hat{w}), \hat{W}(\hat{w})) > (0,0)$ and using the free entry condition $f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w}) > 0$. Thus, a contradiction. Assume instead that $\hat{\rho}\hat{U} \geq 1$. Then, $\mathcal{T}(\hat{w}^*, \hat{\rho}) = 0$ for all $\hat{w}$ since $\hat{S}(\hat{w})$ is nonnegative and $0 = \hat{S}(\hat{w}) \geq (\hat{J}(\hat{w}), \hat{W}(\hat{w})) \geq 0\ \forall\hat{w}$ and $\max_{\hat{w}} f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w}) = 0$ with the free entry condition. With these argument, we have that $\hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}} f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w}) = \tilde{B} < 1$, and we have the contradiction.

2. To show this property, first we show that $\hat{J}(\hat{w}) > 0$ for all $\hat{w} \in (\log(\hat{\rho}\hat{U}), 0)$. Define

$$\tau_{(w^-,0)} = \inf_t\{t : \hat{w}_t \notin (\log(\hat{\rho}\hat{U}), 0)\}.$$

By optimality of the firm,

$$\hat{J}(\hat{w}) = \mathbb{E}_{\hat{w}}\left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t}(1-e^{\hat{w}_t})\,\mathrm{d}t\right] \geq \mathbb{E}_{\hat{w}}\left[\int_0^{\min\{\tau_{(\log(\hat{\rho}\hat{U}),0)},\tau^{m*}\}} e^{-\hat{\rho}t}(1-e^{\hat{w}_t})\,\mathrm{d}t\right] > 0.$$

Thus, there is an open set around the optimally chosen starting wage $\hat{w}$ that lies entirely within the continuation region s.t. $\hat{J}(\hat{w}) > 0$, $\hat{\theta}(\hat{w}) > 0$, and $\hat{J}(\hat{w}) - \hat{K}\hat{\theta}(\hat{w})^\alpha = 0$. Therefore,

$$\arg\max_{\hat{w}} \left\{ f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w}) \right\} = \arg\max_{\hat{w}} \left\{ \left(\frac{\hat{J}(\hat{w})}{\tilde{K}} \right)^{\frac{1-\alpha}{\alpha}} \hat{W}(\hat{w}) \right\} = \arg\max_{\hat{w}} \left\{ \hat{J}(\hat{w})^{1-\alpha} \hat{W}(\hat{w})^\alpha \right\}.$$

Since $\hat{W}(\hat{w}) = \eta(\hat{w})\hat{S}(\hat{w})$ and $\hat{J}(\hat{w}) = (1-\eta(\hat{w}))\hat{S}(\hat{w})$ and $\hat{S}(\hat{w}) = (1-\hat{\rho}\hat{U})\mathcal{T}(\hat{w},\hat{\rho})$,

$$\arg\max_{\hat{w}} \left\{ f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w}) \right\} = \arg\max_{\hat{w}} \left\{ \hat{J}(\hat{w})^{1-\alpha} \hat{W}(\hat{w})^\alpha \right\} = \arg\max_{\hat{w}} \left\{ (1-\eta(\hat{w}))^{1-\alpha} \eta(\hat{w})^\alpha \mathcal{T}(\hat{w},\hat{\rho}) \right\}.$$

Taking first-order conditions, $\eta'(\hat{w}^*) \left(\frac{\alpha}{\eta(\hat{w}^*)} - \frac{1-\alpha}{1-\eta(\hat{w}^*)} \right) = -\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}$. We now show the following claim: There exists a unique solution to

$$\max_{\hat{w}} \hat{W}(\hat{w})^\alpha \hat{J}(\hat{w})^{1-\alpha}. \tag{A.15}$$

We divide this proof into 4 steps.

• **The following result holds:**

$$\arg\max_{\hat{w}} \hat{W}(\hat{w})^\alpha \hat{J}(\hat{w})^{1-\alpha} = \arg \max_{\hat{w} \in [\hat{w}^-, \hat{w}^+]} \alpha \log\left(\hat{W}(\hat{w})\right) + (1-\alpha)\log\left(\hat{J}(\hat{w})\right).$$

As we show below, for $\sigma^2 > 0$ we have $-\infty < \hat{w}^- < \log(\hat{\rho}\hat{U}) < 0 < \hat{w}^+ < \infty$. Now, we show that there exists a $\hat{w} \in (\hat{w}^-, \hat{w}^+)$ such that $\hat{W}(\hat{w}) > 0$ and $\hat{J}(\hat{w}) > 0$ by constradiction. Assume the opposite inequalities hold. Then, since the values satisfy $\hat{W}(\hat{w}) \geq 0$ and $\hat{J}(\hat{w}) \geq 0$, it must be the case that $\hat{W}(\hat{w}) = \hat{J}(\hat{w}) = 0$. Replacing these equalities into the definition of the values, we obtain

$$(\rho+\delta)\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{\rho}\hat{U}, \;\; (\rho+\delta)\hat{J}(\hat{w}) = 1 - e^{\hat{w}}, \tag{A.16}$$

which results in a contradiction since the values that satisfy (A.16) are positive for any $\hat{w} \in (\log(\hat{\rho}\hat{U}), 0)$. Thus, we can restrict the domain of $\hat{w}$ to $[\hat{w}^-, \hat{w}^+]$ in problem (A.15).

• **Problem (A.15) attains a maximum.** This result follows from the Weierstrass Theorem since the set $[\hat{w}^-, \hat{w}^+]$ is compact and the objective function is the composition and sum of two continuous value functions.

• **The functions $\hat{J}(\hat{w})$ and $\hat{W}(\hat{w})$ have a unique global maximum—i.e., there exist unique $\hat{w}^{*j}$ and $\hat{w}^{*h}$ such that**

$$\hat{w}^{*j} = \arg\max_{\hat{w}} \hat{J}(\hat{w}), \;\; \hat{w}^{*h} = \arg\max_{\hat{w}} \hat{W}(\hat{w})$$

with $\hat{w}^{*j} < \hat{w}^{*h}$. We will show that $\hat{w}^{*j} = \arg\max_{\hat{w}} \hat{J}(\hat{w})$ is unique. The proof for $\hat{W}(\hat{w})$ is similar. Assume, by contradiction, that there exist at least two global maxima at $\hat{w}^{*j} < \hat{w}^{**j}$ (from the argument above, we conclude that these maxima cannot occur at the boundary of the game's continuation set). Without loss of generality, assume they are consecutive. The HJB equation within the game's continuation set is given by

$$(\hat{\rho}+\delta)\hat{J}(\hat{w}) = 1 - e^{\hat{w}} - \hat{\gamma}\hat{J}'(\hat{w}) + \frac{\sigma^2}{2}\hat{J}''(\hat{w}).$$

Since the function is smooth, at the two optima, we have

$$(\hat{\rho}+\delta)\hat{J}(\hat{w}^{*j}) + e^{\hat{w}^{*j}} - 1 = \frac{\sigma^2}{2}\hat{J}''(\hat{w}^{*j}),$$
$$(\hat{\rho}+\delta)\hat{J}(\hat{w}^{**j}) + e^{\hat{w}^{**j}} - 1 = \frac{\sigma^2}{2}\hat{J}''(\hat{w}^{**j}),$$

with $\hat{J}(\hat{w}) \leq \hat{J}(\hat{w}^{*j})$ for all $\hat{w} \in (\hat{w}^{*j}, \hat{w}^{**j})$. There are two cases to consider. First, $\hat{J}(\hat{w}) = \hat{J}(\hat{w}^{*j})$ for all $\hat{w} \in (\hat{w}^{*j}, \hat{w}^{**j})$. Here, we have a contradiction since $\hat{J}(\hat{w})$ is constant in the interval, thus $\hat{J}'(\hat{w}) = \hat{J}''(\hat{w}) = 0$ for all $\hat{w} \in (\hat{w}^{*j}, \hat{w}^{**j})$ and

$$(\hat{\rho}+\delta)\hat{J}(\hat{w}^{*j}) + e^{\hat{w}} - 1 = 0, \ \forall \hat{w} \in (\hat{w}^{*j}, \hat{w}^{**j}),$$

which is not constant. Next, assume that the function is not constant. Then, since $\hat{J}(\hat{w})$ is continuous and the set $[\hat{w}^{*j}, \hat{w}^{**j}]$ is compact, the function has a minimum at some $\hat{w}^{\min j} < \hat{w}^{**j}$ satisfying $\hat{J}(\hat{w}^{\min j}) < \hat{J}(\hat{w}^{**j})$ and $e^{\hat{w}^{\min j}} - 1 < e^{\hat{w}^{**j}} - 1$. By definition of minimum, $\hat{J}''(\hat{w}^{\min j}) \geq 0$. Therefore, combining the previous inequalities, we have

$$0 \leq \frac{\sigma^2}{2}\hat{J}''(\hat{w}^{\min j}) = (\hat{\rho}+\delta)\hat{J}(\hat{w}^{\min j}) + e^{\hat{w}^{\min j}} - 1 < (\hat{\rho}+\delta)\hat{J}(\hat{w}^{**j}) + e^{\hat{w}^{**j}} - 1 = \frac{\sigma^2}{2}\hat{J}''(\hat{w}^{**j}).$$

Since the function is concave near a maximum, we have a contradiction. We can follow similar steps to rule multiple local maxima. Finally, it is easy to show that $\hat{w}^{*j} < \hat{w}^{*h}$.

• **There exists a unique** $\arg\max_{\hat{w}\in[\hat{w}^{*j},\hat{w}^{*h}]} \alpha \log\left(\hat{W}(\hat{w})\right) + (1-\alpha)\log\left(\hat{J}(\hat{w})\right)$**.** We first show that $\hat{W}(\hat{w})$ is strictly log-concave $\forall \hat{w} \in (\hat{w}^{-}, \hat{w}^{*h})$. The proof that shows that $\hat{J}(\hat{w})$ is log-concave is similar. Applying L'Hôpital's rule, we have that $\lim_{\hat{w}\downarrow\hat{w}^{-}} \frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})} = \lim_{\hat{w}\downarrow\hat{w}^{-}} \frac{\hat{W}''(\hat{w})}{\hat{W}'(\hat{w})}$. Recall that $(\delta + \hat{\rho})\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w})$. Taking the limit $\hat{w} \downarrow \hat{w}^{-}$ and using the border conditions $\hat{W}(\hat{w}^{-}) = \hat{W}'(\hat{w}^{-}) = 0$, we have that $0 < \hat{\rho}\hat{U} - e^{\hat{w}^{-}} = \frac{\sigma^2}{2}\hat{W}''(\hat{w})$. Therefore, $\lim_{\hat{w}\downarrow\hat{w}^{-}} \frac{\hat{W}''(\hat{w})}{\hat{W}'(\hat{w})} = \infty$.

It is easy to check that $\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})}$ has a vertical asymptote when $\hat{w} \downarrow \hat{w}^-$ and, therefore, it must be decreasing near $\hat{w}^-$ from the right. Let $\underline{\hat{w}}$ be a wage-to-productivity ratio close to $\hat{w}^-$ such that $\frac{\hat{W}'(\underline{\hat{w}})}{\hat{W}(\underline{\hat{w}})} > 0$ and $\left(\frac{\hat{W}'(\underline{\hat{w}})}{\hat{W}(\underline{\hat{w}})}\right)' < 0$. Since $\hat{W}(\hat{w})$ has a single maximum $\hat{w}^{*h}$, $\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})}$ is positive for all $\hat{w} \in [\underline{\hat{w}}, \hat{w}^{*h})$ and $\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})} = 0$ when evaluated at $\hat{w} = \hat{w}^{*h}$. Now, we show that $\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})}$ is decreasing for all $\hat{w} \in [\underline{\hat{w}}, \hat{w}^{*h})$. Using the worker's HJB equation and the corresponding smooth-pasting and value-matching conditions, we have

$$\hat{W}'(\hat{w}) = \frac{2}{\sigma^2} \int_{\hat{w}^-}^{\hat{w}} \left[(\delta + \hat{\rho})\hat{W}(x) - (e^x - \hat{\rho}\hat{U})\right] \mathrm{d}x + \frac{2\hat{\gamma}}{\sigma^2}\hat{W}(\hat{w}).$$

Dividing both sides by $\hat{W}(\hat{w})$,

$$\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})} = \frac{2}{\sigma^2} \frac{\int_{\hat{w}^-}^{\hat{w}} \left[(\delta + \hat{\rho})\hat{W}(x) - (e^x - \hat{\rho}\hat{U})\right] \mathrm{d}x}{\hat{W}(\hat{w})} + \frac{2\hat{\gamma}}{\sigma^2}.$$

Taking the derivative w.r.t. $\hat{w}$, we obtain

$$\begin{aligned}
\left(\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})}\right)' &= \frac{2}{\sigma^2}\left[(\hat{\rho} + \delta) - \frac{(e^{\hat{w}} - \hat{\rho}\hat{U})}{\hat{W}(\hat{w})} - \frac{\int_{\hat{w}^-}^{\hat{w}} \left[(\delta + \hat{\rho})\hat{W}(x) - (e^x - \hat{\rho}\hat{U})\right] \mathrm{d}x}{\hat{W}(\hat{w})^2}\hat{W}'(\hat{w})\right] \\
&= \frac{2}{\sigma^2}\left[(\hat{\rho} + \delta) - \frac{(e^{\hat{w}} - \hat{\rho}\hat{U})}{\hat{W}(\hat{w})}\right] + \frac{2\hat{\gamma}}{\sigma^2}\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})} - \left(\frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})}\right)^2.
\end{aligned}$$

Define the following function $\phi(\hat{w} - \underline{\hat{w}}) \equiv \frac{\hat{W}'(\hat{w})}{\hat{W}(\hat{w})}$. Then,

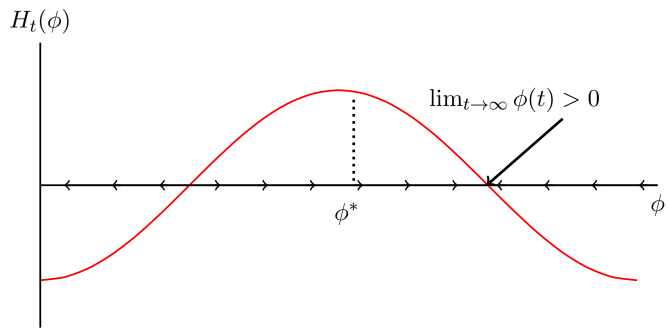
$$\phi'(\hat{w} - \underline{\hat{w}}) = \frac{2}{\sigma^2}\left[(\hat{\rho} + \delta) - \frac{(e^{\hat{w}} - \hat{\rho}\hat{U})}{\hat{W}(\hat{w})}\right] + \frac{2\hat{\gamma}}{\sigma^2}\phi(\hat{w} - \underline{\hat{w}}) - \phi(\hat{w} - \underline{\hat{w}})^2.$$

Given this transformation, the goal is now to show what $\phi'(\hat{w} - \underline{\hat{w}}) < 0$. Let $t \equiv \hat{w} - \underline{\hat{w}}$, then

$$\phi'(t) = \frac{2}{\sigma^2}\left[(\hat{\rho} + \delta) - \frac{(e^{t+\underline{\hat{w}}} - \hat{\rho}\hat{U})}{\hat{W}(t + \underline{\hat{w}})}\right] + \frac{2\hat{\gamma}}{\sigma^2}\phi(t) - \phi(t)^2.$$

Next, we define $F(t) \equiv \frac{2}{\sigma^2}\left[(\hat{\rho} + \delta) - \frac{(e^{t+\underline{\hat{w}}} - \hat{\rho}\hat{U})}{\hat{W}(t+\underline{\hat{w}})}\right]$. Thus, we have that the derivative of the log of the worker's value function satisfies the Ricatti equation $\phi'(t) = F(t) + \frac{2\hat{\gamma}}{\sigma^2}\phi(t) - \phi(t)^2$, with initial condition $\phi(0) > 0$, $\phi'(0) < 0$. Define $T = \hat{w}^{*h} - \underline{\hat{w}}$, then $\phi(T) = 0$ (which follows from $\hat{w}^{*h}$ being an interior maximum). Now, we show that $\phi'(t) < 0$ for all $t \in (0, T)$. Assume that this is not the case and there exists a $t^* \in (0, T)$ s.t. $\phi'(t^*) \geq 0$. Without loss of generality, let t^* be inside the first interval s.t. $\phi'(t) \geq 0$. Then, if we plot $H_t(\phi) := F(t) + \frac{2\hat{\gamma}}{\sigma^2}\phi(t) - \phi(t)^2$, there exists a

FIGURE A1. PHASE LINE FOR $\phi(t)$



$(t^*, \phi^*) > (0,0)$ s.t. $H_{t^*}(\phi^*) \geq 0$. From Figure A1, since $\phi'(0) < 0$ with $\phi(0)$ arbitrary large, we can see that $\lim_{t\to\infty}\phi(t) \geq \phi^* > 0$ and, therefore, $\phi(T) > 0$, which contradicts the terminal condition $\phi(T) = 0$. Thus, $\phi'(t) < 0$ for all $t \in (0,T)$ and $\log(\hat{W}(\hat{w}))$ is a concave function $\forall \hat{w} \in (\hat{w}^{*j}, \hat{w}^{*h})$.

Since $\log\left(\hat{W}(\hat{w})\right)$ and $\log\left(\hat{J}(\hat{w})\right)$ are strictly concave functions $\forall \hat{w} \in [\hat{w}^{*j}, \hat{w}^{*h}]$ and the sum of strictly concave functions is strictly concave, we have that $\arg\max_{\hat{w}\in[\hat{w}^{*j},\hat{w}^{*h}]} \alpha \log\left(\hat{W}(\hat{w})\right) + (1-\alpha)\log\left(\hat{J}(\hat{w})\right)$ exists and is unique.

3. This step follows directly from workers' and firms' optimality conditions.

4. To show that $\hat{\mathcal{Z}}^h$ and $\hat{\mathcal{Z}}^j$ are connected, assume they are not. Without loss of generality, assume that $\hat{\mathcal{Z}}^h = \{\hat{w} : \hat{w} > \hat{w}^-\} \cup (a,b)$ with $a < b < w^-$. Then, since $\hat{w}^- \leq \hat{\rho}\hat{U}$, it must be the case that for all $\hat{w} \in (a,b)$, we have $(e^{\hat{w}} - \hat{\rho}\hat{U}) < 0$ for all $\hat{w} \in (a,b)$, and $\hat{W}(\hat{w}) = \mathbb{E}_{\hat{w}}\left[\int_0^{\tau_{\hat{\mathcal{Z}}^h \cap \hat{\mathcal{Z}}^j}} e^{-(\hat{\rho}+\delta)t}(e^{w_t} - \hat{\rho}\hat{U})\,\mathrm{d}t\right] < 0$ for all $\hat{w} \in (a,b)$ due to continuity of Brownian motions. Since $\hat{W}(\hat{w}) \geq 0$, we have a contradiction. A similar argument holds for the firm's continuation set.

   We prove that $-\infty < \hat{w}^-$ by contradiction. Assume that $-\infty = \hat{w}^-$, then

$$\hat{W}(\hat{w}, \hat{w}^+) := \mathbb{E}\left[\int_0^{\tau_{(-\infty,\hat{w}^+)} \wedge \tau^\delta} e^{-\hat{\rho}t}\left(e^{\hat{w}_t} - \hat{\rho}\hat{U}\right)\mathrm{d}t \middle| \hat{w}_0 = \hat{w}\right].$$

   Then, since $\hat{\rho}\hat{U} < e^{\hat{w}^+}$, it is easy to show

$$\begin{aligned}\hat{W}(\hat{w}, \hat{w}^+) &= \mathbb{E}\left[\int_0^{\tau_{(-\infty,\hat{w}^+)} \wedge \tau^\delta} e^{-\hat{\rho}t}\left(e^{\hat{w}_t} - \hat{\rho}\hat{U}\right)\mathrm{d}t \middle| \hat{w}_0 = \hat{w}\right] \\ &\leq \mathbb{E}\left[\int_0^{\infty} e^{-(\hat{\rho}+\delta)t}\left(e^{\hat{w}_t} - \hat{\rho}\hat{U}\right)\mathrm{d}t \middle| \hat{w}_0 = \hat{w}\right]\end{aligned}$$

$$= \frac{e^w}{\hat{\rho} + \delta + \hat{\gamma} - \sigma^2/2} - \frac{\hat{\rho}\hat{U}}{\hat{\rho} + \delta}$$
$$= \frac{e^w}{\rho + \delta} - \frac{\hat{\rho}\hat{U}}{\rho - \gamma - \sigma^2/2 + \delta}$$

Thus, there exists a small enough $\hat{w}$ s.t. $\hat{W}(\hat{w}, \hat{w}^+) < 0$, and we have a contradiction. A similar argument holds for the firm's separation threshold. Finally, the smooth pasting conditions are necessary and sufficient for the optimal stopping times (see Brekke and Øksendal, 1990).

□

A.3 Proof of Propositions 3, 4, and 5

Define $\hat{\mathcal{Z}} = (\hat{w}^-, \hat{w}^+)$. From Proposition 1, when $\hat{\gamma} > 0$ or $\sigma > 0$, we can work with the HJB conditions

$$(\hat{\rho} + \delta)\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}) \quad \forall \hat{w} \in \hat{\mathcal{Z}} \tag{A.17}$$
$$(\hat{\rho} + \delta)\hat{J}(\hat{w}) = 1 - e^{\hat{w}} - \hat{\gamma}\hat{J}'(\hat{w}) + \frac{\sigma^2}{2}\hat{J}''(\hat{w}) \quad \forall \hat{w} \in \hat{\mathcal{Z}} \tag{A.18}$$
$$\hat{\rho}\hat{U} = \tilde{B} + \tilde{K}^{1-\alpha^{-1}}\hat{J}(\hat{w}^*)^{\frac{1-\alpha}{\alpha}}\hat{W}(\hat{w}^*)$$
$$(1-\alpha)\frac{\mathrm{d}\log \hat{J}(\hat{w}^*)}{\mathrm{d}\hat{w}} = -\alpha\frac{\mathrm{d}\log \hat{W}(\hat{w}^*)}{\mathrm{d}\hat{w}},$$

with the value-matching conditions

$$\hat{W}(\hat{w}^-) = \hat{J}(\hat{w}^-) = \hat{W}(\hat{w}^+) = \hat{J}(\hat{w}^+) = 0$$

and smooth-pasting conditions

$$\hat{W}'(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0.$$

Proof of Proposition 3. If $\hat{\gamma} = \sigma = 0$, conditions (A.17) and (A.18) imply $\hat{W}(\hat{w}) = \frac{e^{\hat{w}} - \hat{\rho}\hat{U}}{\hat{\rho}+\delta}$ and $\hat{J}(\hat{w}) = \frac{1-e^{\hat{w}}}{\hat{\rho}+\delta}$. The variation inequalities imply

$$(\hat{\rho} + \delta)\hat{W}(\hat{w}) = \max\{0, e^{\hat{w}} - \hat{\rho}\hat{U}\}, \ \forall \hat{w} \in \mathbb{R},$$
$$(\hat{\rho} + \delta)\hat{J}(\hat{w}) = \max\{0, 1 - e^{\hat{w}}\}, \quad \forall \hat{w} \in \mathbb{R}.$$

Thus, $\hat{W}(\hat{w}^-) = \hat{J}(\hat{w}^+) = 0$, $\hat{w}^+ = 0$ and $\hat{w}^- = \log\left(\hat{\rho}\hat{U}\right)$. Since

$$\mathcal{T}(\hat{w}, \hat{\rho}) = \begin{cases} (\hat{\rho}+\delta)^{-1} & \text{if } \hat{w} \in [\hat{w}^-, \hat{w}^+] \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathcal{T}_{\hat{w}}(\hat{w}^*, \hat{\rho}) = 0$, we have that the worker's share of the surplus $\eta(\hat{w}^*) = \alpha$. □

Proof of Proposition 4. Let us guess and verify the following solution $\hat{w}^* = \log\left(\frac{1+\hat{\rho}\hat{U}}{2}\right)$ and $\hat{w}^- = \hat{w}^* - h$ and $\hat{w}^+ = \hat{w}^* + h$ for a given h. Using a Taylor approximation of the flow profits around $\hat{w}^*$

$$e^{\hat{w}} - \hat{\rho}\hat{U} \approx e^{\hat{w}^*}(1 + (\hat{w} - \hat{w}^*)) - \hat{\rho}\hat{U} = \frac{1-\hat{\rho}\hat{U}}{2} + e^{\hat{w}^*}(\hat{w} - \hat{w}^*),$$
$$1 - e^{\hat{w}} \approx 1 - e^{\hat{w}^*}(1 + (\hat{w} - \hat{w}^*)) = \frac{1-\hat{\rho}\hat{U}}{2} - e^{\hat{w}^*}(\hat{w} - \hat{w}^*).$$

We can write the optimality conditions as

$$(\hat{\rho}+\delta)\hat{W}(\hat{w}) = \frac{1-\hat{\rho}\hat{U}}{2} + e^{\hat{w}^*}(\hat{w} - \hat{w}^*) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}), \quad \forall \hat{w} \in (w^* - h, w^* + h)$$
$$(\hat{\rho}+\delta)\hat{J}(\hat{w}) = \frac{1-\hat{\rho}\hat{U}}{2} - e^{\hat{w}^*}(\hat{w} - \hat{w}^*) + \frac{\sigma^2}{2}\hat{J}''(\hat{w}), \quad \forall \hat{w} \in (w^* - h, w^* + h)$$

with the border conditions

$$\hat{W}(\hat{w}^* - h) = \hat{J}(\hat{w}^* - h) = \hat{W}(\hat{w}^* + h) = \hat{J}(\hat{w}^* + h) = 0,$$
$$\hat{W}'(\hat{w}^* - h) = \hat{J}'(\hat{w}^* + h) = 0.$$

Now, we show that we can transform $J(x) = \frac{\hat{J}(x+\hat{w}^*) - \frac{1-\hat{\rho}\hat{U}}{2(\hat{\rho}+\delta)}}{e^{\hat{w}^*}}$. A similar argument applies to the value function of the worker. Making the following transformation $J(x) = \frac{\hat{J}(x+\hat{w}^*) - \frac{1-\hat{\rho}\hat{U}}{2(\hat{\rho}+\delta)}}{e^{\hat{w}^*}}$, and using (A.18)

$$(\hat{\rho}+\delta)J(x) = (\hat{\rho}+\delta)\left(\frac{\hat{J}(x+\hat{w}^*) - \frac{1-\hat{\rho}\hat{U}}{2(\hat{\rho}+\delta)}}{e^{\hat{w}^*}}\right),$$
$$= -x + \frac{\sigma^2}{2}\frac{1}{e^{\hat{w}^*}}\hat{J}''(x+\hat{w}^*),$$
$$= -x + \frac{\sigma^2}{2}J''(x).$$

Thus,

$$(\hat{\rho}+\delta)W(x) = x + \frac{\sigma^2}{2}W''(x) \quad \forall x \in (-h, h)$$
$$(\hat{\rho}+\delta)J(x) = -x + \frac{\sigma^2}{2}J''(x) \quad \forall x \in (-h, h)$$

Defining $\Phi = \frac{\frac{1-\hat{\rho}\hat{U}}{2}}{e^{\hat{w}^*}} = \frac{1-\hat{\rho}\hat{U}}{1+\hat{\rho}\hat{U}} > 0$, it is easy to show that $W(h) = J(h) = W(-h) = J(-h) = -\frac{\Phi}{\hat{\rho}+\delta}$ and $W'(-h) = J'(h) = 0$. Thus, $W(x) = J(-x)$. Given that this problem is symmetric, we verify the guess of symmetry of the Ss bands and $\frac{1}{2}W'(0) = -\frac{1}{2}J'(-0)$. The latter property implies that w^* satisfies the proposed Nash bargaining solution.

Now, we show that $h = \omega(\varphi)\Phi$ with $\varphi = \sqrt{2(\hat{\rho}+\delta)}/\sigma$. Note that $W(x) = J(-x)$. Thus, we can only focus on $W(x)$ using the smooth pasting condition evaluated at $-h$. The solution to this system of differential equations is given by

$$W(x) = Ae^{\varphi x} + Be^{-\varphi x} + \frac{x}{\hat{\rho}+\delta}$$
$$W(h) = W(-h) = -\frac{\Phi}{\hat{\rho}+\delta} \ \ \textit{and} \ \ W'(-h) = 0$$

with $\varphi = \sqrt{2(\hat{\rho}+\delta)}/\sigma$. Writing the value-matching conditions

$$Ae^{\varphi h} + Be^{-\varphi h} + \frac{h}{\hat{\rho}+\delta} = -\frac{\Phi}{\hat{\rho}+\delta}; \ \ Ae^{-\varphi h} + Be^{\varphi h} - \frac{h}{\hat{\rho}+\delta} = -\frac{\Phi}{\hat{\rho}+\delta}$$

Solving for A and B,

$$A = -\frac{1}{\hat{\rho}+\delta}\frac{e^{-\varphi h}(-\Phi+h) + e^{\varphi h}(h+\Phi)}{e^{2\varphi h} - e^{-2\varphi h}}; \ \ B = \frac{1}{\hat{\rho}+\delta}\frac{e^{\varphi h}(-\Phi+h) + e^{-\varphi h}(h+\Phi)}{e^{2\varphi h} - e^{-2\varphi h}}$$

Therefore

$$W(x) = -\frac{1}{\hat{\rho}+\delta}\frac{e^{-\varphi h}(-\Phi+h) + e^{\varphi h}(h+\Phi)}{e^{2\varphi h} - e^{-2\varphi h}}e^{\varphi x} + \frac{1}{\hat{\rho}+\delta}\frac{e^{\varphi h}(-\Phi+h) + e^{-\varphi h}(h+\Phi)}{e^{2\varphi h} - e^{-2\varphi h}}e^{-\varphi x} + \frac{x}{\hat{\rho}+\delta}$$

Taking the derivative and evaluating in $x = -h$

$$W'(-h) = -\frac{1}{\hat{\rho}+\delta}\frac{e^{-\varphi h}(-\Phi+h) + e^{\varphi h}(h+\Phi)}{e^{2\varphi h} - e^{-2\varphi h}}\varphi e^{-\varphi h} - \frac{1}{\hat{\rho}+\delta}\frac{e^{\varphi h}(-\Phi+h) + e^{-\varphi h}(h+\Phi)}{e^{2\varphi h} - e^{-2\varphi h}}\varphi e^{\varphi h} + \frac{1}{\hat{\rho}+\delta} = 0$$

or equivalently

$$-\Phi(e^{-2\varphi h}+e^{2\varphi h}-2)=\frac{1}{\varphi}(e^{2\varphi h}-e^{-2\varphi h})-\frac{1}{2\varphi}2\varphi h\left(e^{2\varphi h}+e^{-2\varphi h}+2\right). \tag{A.19}$$

It would be useful to express equation (A.19) using $\sinh(x)=\frac{e^x-e^{-x}}{2}$ and $\cosh(x)=\frac{e^x+e^{-x}}{2}$. Using the hyperbolic functions,

$$-\Phi 2\varphi\left(\cosh(2\varphi h)-1\right)=2\sinh(2\varphi h)-\varphi 2h\left(\cosh(2\varphi h)+1\right).$$

Next, we change variables with $q\equiv 2\varphi h$ and define q as the implicit solution of

$$-2\Phi\varphi\left(\cosh(q)-1\right)+x\left(\cosh(q)+1\right)=2\sinh(q).$$

Thus, $h=\frac{q(2\Phi\varphi)}{2\varphi}$. Let $b=2\Phi\varphi>0$, then we can express the function $x(\cdot)$ as the solution of

$$b=-\frac{2\sinh(q(b))-q(b)\left(\cosh(q(b))+1\right)}{\left(\cosh(q(b))-1\right)}.$$

Notice that if we define

$$f(q)=-\frac{2\sinh(q)-q\left(\cosh(q)+1\right)}{\left(\cosh(q)-1\right)},$$

the following properties about $f(q)$ hold:

1. $\lim_{q\to 0}f(q)=0$ and $\lim_{q\to\infty}f(q)=\infty$.
2. $f(q)$ is increasing and convex, with $\lim_{q\to 0}f'(q)=1/3$ and $\lim_{q\to\infty}f'(q)=1$.
3. $\frac{\mathrm{d}\log(f(q))}{\mathrm{d}\log(q)}>1$.

Given these properties, we can write $h(\varphi,\Phi)=\frac{f^{-1}(2\varphi\Phi)}{2\varphi}$ and show the following properties of $h(\varphi,\Phi)$

1. $h(\varphi,\Phi)$ is increasing in Φ: Since $f^{-1}(\cdot)$ is increasing, we have the result.
2. $h(\varphi,\Phi)$ is decreasing in φ: Taking the derivative of $h(\varphi,\Phi)=\frac{f^{-1}(2\varphi\Phi)}{2\varphi}$ with respect to φ and operating

$$\frac{\partial h(\varphi,\Phi)}{\partial\varphi}=\left.\frac{\mathrm{d}f^{-1}(q)}{\mathrm{d}q}\right|_{q=2\varphi\Phi}\frac{2\Phi}{2\varphi}-\frac{f^{-1}(2\varphi\Phi)}{2\varphi^2}=\frac{f^{-1}(2\varphi\Phi)}{2\varphi^2}\left[\left.\frac{\mathrm{d}f^{-1}(q)}{\mathrm{d}q}\right|_{q=2\varphi\Phi}\frac{2\varphi\Phi}{f^{-1}(2\varphi\Phi)}-1\right]$$
$$=\frac{f^{-1}(2\varphi\Phi)}{2\varphi^2}\left[\left.\frac{\mathrm{d}\log(q)}{\mathrm{d}\log(f(q))}\right|_{x=2\varphi\Phi}\frac{2\varphi\Phi}{f^{-1}(2\varphi\Phi)}-1\right]<0.$$

3. $\lim_{\varphi\downarrow 0} h(\varphi,\Phi) = 3\Phi$ and $\lim_{\varphi\to\infty} h(\varphi,\Phi) = \Phi$: Applying L'Hopital's rule and using properties of the derivative of the inverse,

$$\lim_{\varphi\to\infty} h(\varphi,\Phi) = \lim_{\varphi\to\infty} \frac{f^{-1}(2\varphi\Phi)}{2\varphi} = \lim_{\varphi\to\infty} \frac{1}{f'(2\varphi\Phi)}\Phi = \Phi$$
$$\lim_{\varphi\downarrow 0} h(\varphi,\Phi) = \lim_{\varphi\downarrow 0} \frac{f^{-1}(2\varphi\Phi)}{2\varphi} = \lim_{\varphi\downarrow 0} \frac{1}{f'(2\varphi\Phi)}\Phi = 3\Phi$$

4. $h(\varphi,\Phi) = \omega(2\varphi\Phi)\Phi$: Define $\omega(z) = \frac{f^{-1}(z)}{z}$, then it is easy to see that $h(\varphi,\Phi) = \omega(2\varphi\Phi)\Phi$. Moreover, from property 2 and 3, $\omega(z)$ is decreasing with $\lim_{z\downarrow 0}\omega(z) = 3$ and $\lim_{z\to\infty}\omega(z) = 1$. Moreover, it is easy to show with similar arguments that $\omega(2\varphi\Phi)\Phi$ is increasing in Φ and $\omega(2\varphi\Phi)\varphi$ is increasing in φ.

Now, we can compute $\eta(\hat{w}^*)$ and $\mathcal{T}(\hat{w}^*,\hat{\rho})$. Note that we can define $T(x) = \mathcal{T}(x+\hat{w}^*,\hat{\rho})$, which solves $(\hat{\rho}+\delta)T(x) = 1 + \frac{\sigma^2}{2}T''(x)$, with $T(\pm h(\varphi,\Phi)) = 0$. The solution to this differential equation is given by

$$T(x) = \frac{1 - \frac{e^{\varphi x}+e^{-\varphi x}}{e^{\varphi h}+e^{-\varphi h}}}{\hat{\rho}+\delta}.$$

Thus, $T'(0) = 0$ and $\eta(\hat{w}^*) = \alpha$. Finally, using the property that $sech(x) = \frac{2}{e^x+e^{-x}}$, we have

$$\mathcal{T}(\hat{w}^*,\hat{\rho}) = \frac{1 - sech(\varphi\omega(2\varphi\Phi)\Phi)}{\hat{\rho}+\delta}.$$

□

Proof of Proposition 5. Now, we take the limit $\sigma \downarrow 0$. The equilibrium conditions in this case are

$$(\hat{\rho}+\delta)\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\hat{W}'(\hat{w}) \quad \forall \hat{w} \in \hat{\mathcal{Z}}^j \cap \mathcal{Z}^h$$
$$(\hat{\rho}+\delta)\hat{J}(\hat{w}) = 1 - e^{\hat{w}} - \hat{\gamma}\hat{J}'(\hat{w}) \quad \forall \hat{w} \in \hat{\mathcal{Z}}^j \cap \mathcal{Z}^h$$
$$(1-\alpha)\frac{\mathrm{d}\log \hat{J}(\hat{w}^*)}{\mathrm{d}\hat{w}} = -\alpha\frac{\mathrm{d}\log \hat{W}(\hat{w}^*)}{\mathrm{d}\hat{w}}$$

with the value matching and smooth pasting conditions $\hat{W}(\hat{w}^-) = \hat{J}(\hat{w}^-) = \hat{W}(\hat{w}^+) = \hat{J}(\hat{w}^+) = 0$ and $\hat{W}'(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0$. Without idiosyncratic shocks and $\gamma > 0$ the upper Ss band is not active. Thus, we discard the optimality condition for $\hat{w}^+$. In this case, the stopping time is a deterministic function; hence, it is easier to work with the sequential formulation.

$$\hat{W}(\hat{w}) = \max_{T} \int_0^T e^{-(\hat{\rho}+\delta)s} \left(e^{\hat{w}-\hat{\gamma}s} - \hat{\rho}\hat{U}\right) \mathrm{d}s \tag{A.20}$$

$$\hat{J}(\hat{w}) = \int_0^{T(\hat{w})} e^{-(\hat{\rho}+\delta)s} \left(1 - e^{\hat{w}-\hat{\gamma}s}\right) \mathrm{d}s. \tag{A.21}$$

In equation (A.21), $T(\hat{w})$ is the optimal policy of the worker. Taking the first order conditions with respect to $T(\hat{w})$, $e^{\hat{w}-\hat{\gamma}T(\hat{w})} = \hat{\rho}\hat{U}$. Solving this equation, $T(\hat{w}) = \frac{\hat{w}-\log(\hat{\rho}\hat{U})}{\hat{\gamma}}$. Thus, if $\hat{w} = \hat{w}^*$, we have that $\hat{w}^- = \hat{w}^* - \hat{\gamma}T(\hat{w}^*)$ satisfies $\hat{w}^- = \log(\hat{\rho}\hat{U})$. Taking the derivatives of $\hat{W}(\hat{w})$ and $\hat{J}(\hat{w})$, and using the envelope condition for $\hat{W}'(\hat{w})$, we have

$$\hat{W}'(\hat{w}) = \int_0^{T(w)} e^{-(\hat{\rho}+\delta)s} \left(e^{\hat{w}-\hat{\gamma}s}\right) \mathrm{d}s, \tag{A.22}$$

$$\hat{J}'(\hat{w}) = -\int_0^{T(w)} e^{-(\hat{\rho}+\delta)s} \left(e^{\hat{w}-\hat{\gamma}s}\right) \mathrm{d}s + e^{-(\hat{\rho}+\delta)T(\hat{w})} \left(1 - e^{\hat{w}-\hat{\gamma}T(\hat{w})}\right) \underbrace{T'(\hat{w})}_{=1/\hat{\gamma}}. \tag{A.23}$$

From equations (A.22) and (A.23), we get the Nash bargaining solution

$$-\alpha \frac{\int_0^{T^*} e^{-(\hat{\rho}+\delta)s} \left(e^{\hat{w}^*-\hat{\gamma}s}\right) \mathrm{d}s}{\int_0^{T^*} e^{-(\hat{\rho}+\delta)s} \left(e^{\hat{w}^*-\hat{\gamma}s} - \hat{\rho}\hat{U}\right) \mathrm{d}s} = (1-\alpha) \frac{\left[-\int_0^{T^*} e^{-(\hat{\rho}+\delta)s} \left(e^{\hat{w}^*-\hat{\gamma}s}\right) \mathrm{d}s + e^{-(\hat{\rho}+\delta)T^*} \frac{(1-\hat{\rho}\hat{U})}{\hat{\gamma}}\right]}{\int_0^{T^*} e^{-(\hat{\rho}+\delta)s} \left(1 - e^{\hat{w}^*-\hat{\gamma}s}\right) \mathrm{d}s} \tag{A.24}$$

Define $\Omega(a, T^*) := \frac{1-e^{-aT^*}}{a}$. Operating,

$$\alpha \int_0^{T^*} e^{-(\hat{\rho}+\delta)s} \left(1 - e^{\hat{w}^*-\hat{\gamma}s}\right) \mathrm{d}s = (1-\alpha) \int_0^{T^*} e^{-(\hat{\rho}+\delta)s} \left(e^{\hat{w}^*-\hat{\gamma}s} - \hat{\rho}\hat{U}\right) \mathrm{d}s \left[1 - \frac{e^{-(\hat{\rho}+\delta)T^*}(1-\hat{\rho}\hat{U})}{\hat{\gamma}\int_0^{T^*} e^{-(\hat{\rho}+\delta)s} \left(e^{\hat{w}^*-\hat{\gamma}s}\right) \mathrm{d}s}\right] \iff$$

$$(\alpha + (1-\alpha)\hat{\rho}\hat{U})\, \Omega(\hat{\rho}+\delta, T^*) = e^{\hat{w}^*}\Omega(\hat{\rho}+\delta+\hat{\gamma}, T^*) - \frac{(1-\alpha)e^{-(\hat{\rho}+\delta)T^*}(1-\hat{\rho}\hat{U})}{\hat{\gamma}} \left[1 - \hat{\rho}\hat{U}\frac{\Omega(\hat{\rho}+\delta, T^*)}{e^{\hat{w}^*}\Omega(\hat{\rho}+\delta+\hat{\gamma}, T^*)}\right]$$

Define $\tilde{T} = \hat{\gamma}T^*$ and $\Omega(a, T^*) := \frac{1-e^{-aT^*}}{a} = \hat{\gamma}^{-1}\Omega(\frac{a}{\hat{\gamma}}, \tilde{T})$. Then, the policy $(T^*, \hat{w}^*)$ solves

$$e^{\hat{w}^*-\tilde{T}} = \hat{\rho}\hat{U}$$

$$(\alpha + (1-\alpha)\hat{\rho}\hat{U})\, \hat{\gamma}^{-1}\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}, \tilde{T}\right) = e^{\hat{w}^*}\hat{\gamma}^{-1}\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}+1, \tilde{T}\right) - \frac{(1-\alpha)e^{-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\tilde{T}}(1-\hat{\rho}\hat{U})}{\hat{\gamma}} \left[1 - \hat{\rho}\hat{U}\frac{\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}, \tilde{T}\right)}{e^{\hat{w}^*}\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}+1, \tilde{T}\right)}\right].$$

Therefore, the optimal stopping is given by

$$\frac{\alpha + (1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}} = e^{\tilde{T}}\frac{\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}+1, \tilde{T}\right)}{\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}, \tilde{T}\right)} - \frac{(1-\alpha)(1-\hat{\rho}\hat{U})\left[1 - \frac{\hat{\rho}+\delta}{\hat{\gamma}}\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}, \tilde{T}\right)\right]}{\hat{\rho}\hat{U}\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}, \tilde{T}\right)} \left[1 - \frac{\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}, \tilde{T}\right)}{e^{\tilde{T}}\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}+1, \tilde{T}\right)}\right] \tag{A.25}$$

Now, we show the properties satisfied by $\tilde{T}\left(\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}, \frac{\hat{\rho}+\delta}{\hat{\gamma}}, \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\right)$. Let us define the function

$$f(a,b,c) := e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - cb \frac{e^{-ba}}{1-e^{-ba}} \left[1 - \frac{b+1}{b} \frac{1-e^{-ba}}{e^a - e^{-ba}}\right].$$

Observe that with this function:

$$\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}} = f\left(\tilde{T}\left(\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}, \frac{\hat{\rho}+\delta}{\hat{\gamma}}, \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\right), \frac{\hat{\rho}+\delta}{\hat{\gamma}}, \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\right).$$

The following properties are easy to show:

1. $f(a,b,c)$ is increasing in a.
2. If $a,c>0$, $b\to\infty$, then $f(a,b,c)\to e^a$: To see this property, taking the limit

$$\lim_{a>0,b\to\infty,c\propto b}\left[e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - cb \frac{e^{-ba}}{1-e^{-ba}} \left[1 - \frac{b+1}{b} \frac{1-e^{-ba}}{e^a - e^{-ba}}\right]\right]$$
$$= e^a \underbrace{\lim_{a>0,b\to\infty} \frac{1-e^{-(1+b)a}}{1-e^{-ba}}}_{=1} \underbrace{\lim_{a>0,b\to\infty} \frac{b}{b+1}}_{=1} - \underbrace{\lim_{a>0,b\to\infty} cb\frac{e^{-ba}}{1-e^{-ba}}}_{=0} \left[1 - \underbrace{\lim_{b\to\infty}\frac{b+1}{b}}_{=1} \underbrace{\lim_{a>0,b\to\infty} \frac{1-e^{-ba}}{e^a-e^{-ba}}}_{=e^{-a}}\right] = e^a.$$

3. If $a,c>0$ and $b\to 0$ then $f(a,b,c)\to \frac{e^a-1-c\left(1-\frac{a}{e^a-1}\right)}{a}$: To see this property, taking the limit

$$\lim_{a>0,b\to 0}\left[e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - cb \frac{e^{-ba}}{1-e^{-ba}} \left[1 - \frac{b+1}{b} \frac{1-e^{-ba}}{e^a - e^{-ba}}\right]\right]$$
$$= e^a(1-e^{-a}) \underbrace{\lim_{a>0,b\to 0} \frac{b}{1-e^{-ba}}}_{=1/a} - c \underbrace{\lim_{a>0,b\to 0} \frac{b}{1-e^{-ba}}}_{=1/a} \left[1 - \frac{1}{e^a-1} \underbrace{\lim_{b\to\infty} \frac{1-e^{-ba}}{b}}_{=a}\right] = \frac{e^a-1-c\left(1-\frac{a}{e^a-1}\right)}{a}.$$

4. $e^a \geq f(a,b,c) \geq \frac{e^a-1-c\left(1-\frac{a}{e^a-1}\right)}{a}$ where the upper bound is reached when $b\to\infty$ and the lower bound when $b\downarrow 0$.
5. Duration of the match: It is easy to show that $\mathcal{T}(\hat{w}^*,\hat{\rho}) = \frac{1-e^{-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\tilde{T}(\cdot)}}{\hat{\rho}+\delta}$.
6. The worker's share is given by

$$\eta(\hat{w}^*) = \frac{e^{\hat{\gamma}T^*(\cdot)+\log(\hat{\rho}\hat{U})}\int_0^{T^*} e^{-(\hat{\rho}+\delta+\hat{\gamma})t}\,\mathrm{d}t - \hat{\rho}\hat{U}\int_0^{T^*} e^{-(\hat{\rho}+\delta)t}\,\mathrm{d}t}{(1-\hat{\rho}\hat{U})\int_0^{T^*} e^{-(\hat{\rho}+\delta)t}\,\mathrm{d}t} = \frac{e^{\tilde{T}(\cdot)} \frac{1-e^{-\left(1+\frac{\hat{\rho}+\delta}{\hat{\gamma}}\right)\tilde{T}(\cdot)}}{1-e^{-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\tilde{T}(\cdot)}} \frac{\hat{\rho}+\delta}{\hat{\rho}+\delta+\hat{\gamma}} - 1}{1-\hat{\rho}\hat{U}}\hat{\rho}\hat{U} \quad \text{(A.26)}$$

With these properties, we can characterize the equilibrium policies:

1. $\tilde{T}\left(\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}, \frac{\hat{\rho}+\delta}{\hat{\gamma}}, \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\gamma}\hat{\rho}\hat{U}}\right)$ is increasing in the first argument.

2. If $\hat{\gamma} \to 0$, then $\frac{\hat{\rho}+\delta}{\hat{\gamma}} \to \infty$ and $\lim_{(\hat{\rho}+\delta)/\hat{\gamma}\to\infty} \tilde{T}(\cdot) = \log\left(\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}\right)$. The expected discounted duration in the limit is equal to $\lim_{\hat{\gamma}\to 0} \mathcal{T}(\hat{w}^*, \hat{\rho}) = \frac{1}{\hat{\rho}+\delta}$. The worker's share in the limit is equal to

$$\eta(\hat{w}^*) = \frac{e^{\tilde{T}(\cdot)} \frac{1-e^{-\left(1+\frac{\hat{\rho}+\delta}{\hat{\gamma}}\right)\tilde{T}(\cdot)}}{1-e^{-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\tilde{T}(\cdot)}} \frac{\hat{\rho}+\delta}{\hat{\rho}+\delta+\hat{\gamma}} - 1}{1-\hat{\rho}\hat{U}} \hat{\rho}\hat{U} = \frac{e^{\tilde{T}(\cdot)}-1}{1-\hat{\rho}\hat{U}}\hat{\rho}\hat{U} = \frac{\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}-1}{1-\hat{\rho}\hat{U}}\hat{\rho}\hat{U} = \alpha$$

3. If $\hat{\gamma} \to \infty$, then $\frac{\hat{\rho}+\delta}{\hat{\gamma}} \to 0$, which provides the same $\tilde{T}(\cdot)$ as $\hat{\rho}+\delta \to 0$. As we have shown before, under this limit, $\tilde{T}(\cdot)$ converges to the implicit solution given by

$$\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}} = \frac{e^{\tilde{T}(\cdot)} - 1 - \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\left(1-\frac{\tilde{T}(\cdot)}{e^{\tilde{T}(\cdot)}-1}\right)}{\tilde{T}(\cdot)}.$$

Given the convergence, we now show the limit for $\eta(\hat{w}^*)$ since clearly $\mathcal{T}(\hat{w}^*, \rho) \to 0$. Let us depart from equation (A.24)

$$-\alpha\frac{\int_0^{T^*} e^{-(\hat{\rho}+\delta)s}\left(e^{\hat{w}^*-\hat{\gamma}s}\right)\mathrm{d}s}{\int_0^{T^*} e^{-(\hat{\rho}+\delta)s}\left(e^{\hat{w}^*-\hat{\gamma}s}-\hat{\rho}\hat{U}\right)\mathrm{d}s} = (1-\alpha)\frac{\left[-\int_0^{T^*} e^{-(\hat{\rho}+\delta)s}\left(e^{\hat{w}^*-\hat{\gamma}s}\right)\mathrm{d}s + e^{-(\hat{\rho}+\delta)T^*}\frac{(1-\hat{\rho}\hat{U})}{\hat{\gamma}}\right]}{\int_0^{T^*} e^{-(\hat{\rho}+\delta)s}\left(1-e^{\hat{w}^*-\hat{\gamma}s}\right)\mathrm{d}s}$$

Taking the limit as $\hat{\rho}+\delta \to 0$

$$\alpha\int_0^{T^*}\left(1-e^{w_t}\right)\mathrm{d}t = (1-\alpha)\int_0^{T^*}\left(e^{w_t}-\hat{\rho}\hat{U}\right)\mathrm{d}t - \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\gamma}}\frac{\int_0^{T^*}\left(e^{w_t}-\hat{\rho}\hat{U}\right)\mathrm{d}t}{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}.$$

Operating and using the occupancy measure

$$\alpha+(1-\alpha)\hat{\rho}\hat{U} + \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\gamma}T^*}\frac{\frac{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}{T^*}-\hat{\rho}\hat{U}}{\frac{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}{T^*}} = \frac{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}{T^*}$$

It is easy to check that

$$\alpha+(1-\alpha)\hat{\rho}\hat{U} + \frac{1-\alpha}{\hat{\gamma}T^*}\frac{\mathbb{E}[e^{\hat{w}}]-\hat{\rho}\hat{U}}{\mathbb{E}[e^{\hat{w}}]}(1-\hat{\rho}\hat{U}) = \mathbb{E}[e^{\hat{w}}].$$

From (A.26), since $\hat{\rho}+\delta \to 0$, we have that $\eta(\hat{w}^*) = \frac{\mathbb{E}[e^{\hat{w}}]-\hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}}$. Combining these results

$$\alpha+(1-\alpha)\hat{\rho}\hat{U} + \frac{1-\alpha}{\hat{\gamma}T^*}\frac{\mathbb{E}[e^{\hat{w}}]-\hat{\rho}\hat{U}}{\mathbb{E}[e^{\hat{w}}]}(1-\hat{\rho}\hat{U}) = \mathbb{E}[e^{\hat{w}}] \iff \eta(\hat{w}^*) = \alpha + \frac{1-\alpha}{\tilde{T}}\frac{(1-\hat{\rho}\hat{U})\eta(\hat{w}^*)}{\eta(\hat{w}^*)+\hat{\rho}\hat{U}(1-\eta(\hat{w}^*))}.$$

□

B Proofs for Section 3: Agg. Shocks in Labor Markets with Ineff. Turnover

B.1 Proof of Proposition 6: CIR of Employment with Flexible Entry Wage

We divide the proof of Proposition 6 into three propositions. Proposition B.1 relates the CIR to a perturbation of two Bellman equations describing future employment fluctuations for initially employed and unemployed workers. This proposition covers both the case with flexible and sticky entry wages. Proposition B.2 relates steady-state moments of the perturbed Bellman equations to steady-state moments of the distribution of Δz. Finally, Proposition B.3 related the steady-state moments of Δz to observable moments in the steady-state.

Let $g^h(\Delta z)$ and $g^u(\Delta z)$ be the distributions of Δz across employed and unemployed workers, respectively. The support of $g^h(\Delta z)$ is given by $[-\Delta^-, \Delta^+]$, where $\Delta^- := \hat{w}^* - \hat{w}^-$ and $\Delta^+ := \hat{w}^+ - \hat{w}^*$. We denote by $\mathbb{E}_h[\cdot]$ and $\mathbb{E}_u[\cdot]$ the expectation operators under the distributions $g^h(\Delta z)$ and $g^u(\Delta z)$, respectively.

Proposition B.1. *Given steady-state policies $(\hat{w}^-, \hat{w}^*, \hat{w}^+)$ and distributions $(g^h(\Delta z), g^u(\Delta z))$, the CIR is given by*

$$CIR_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z + \zeta) \,\mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z, \zeta) g^u(\Delta z + \zeta) \,\mathrm{d}\Delta z,$$

where the value functions $m_{\mathcal{E},h}(\Delta z)$ and $m_{\mathcal{E},u}(\Delta z, \zeta)$ are defined as:

$$m_{\mathcal{E},h}(\Delta z) = \mathbb{E}\left[\int_0^{\tau^m} (1 - \mathcal{E}_{ss}) \,\mathrm{d}t + m_{\mathcal{E},u}(0,0) \,\middle|\, \Delta z_0 = \Delta z \right], \tag{B.1}$$

$$m_{\mathcal{E},u}(\Delta z, \zeta) = \mathbb{E}\left[\int_0^{\tau^u(\zeta)} (-\mathcal{E}_{ss}) \,\mathrm{d}t + m_{\mathcal{E},h}(-\zeta) \,\middle|\, \Delta z_0 = \Delta z \right]. \tag{B.2}$$

$$0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) \,\mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z, 0) g^u(\Delta z) \,\mathrm{d}\Delta z.$$

with $\tau^u(\zeta)$ being distributed according to a Poisson process with arrival rate $f(\hat{\theta}(\hat{w}^* - \zeta))$.

Proof. We define the cumulative impulse response of aggregate employment to an aggregate TFPR shock as

$$\mathrm{CIR}_{\mathcal{E}}(\zeta) = \int_0^{\infty} \int_{-\infty}^{\infty} \left(g^h(\Delta z, \zeta, t) - g^h(\Delta z) \right) \mathrm{d}\Delta z \,\mathrm{d}t.$$

Note that $\mathcal{E}_t = \int_{-\infty}^{\infty} g^h(\Delta z, \zeta, t) \,\mathrm{d}\Delta z$ is a function of ζ since aggregate shocks affect net flows into employment. The proof proceeds in three steps. Step 1 rewrites the CIR as the integral over time of two value functions, one for employed and unemployed workers, up to a finite time $\mathcal{T}$. Step 2 expresses the CIR as $\mathcal{T} \to \infty$. Step 3 uses the equivalence of the combined Dirichlet-Poisson problem (i.e., the mapping between the sequential problem and the corresponding HJB equations and boundary conditions).

Step 1. Here, we follow a recursive representation for the CIR. The CIR satisfies

$$\text{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} \lim_{\mathcal{T}\to\infty} \left[m_{\mathcal{E},h}(\Delta z, \mathcal{T}) g^h(\Delta z + \zeta) + m_{\mathcal{E},u}(\Delta z, \mathcal{T}) g^u(\Delta z + \zeta) \right] \mathrm{d}\Delta z,$$

where we defined

$$m_{\mathcal{E},h}(\Delta z_0, \mathcal{T}) := \int_0^{\mathcal{T}} \left[\int_{-\infty}^{\infty} \left[(1 - \mathcal{E}_{ss})\, g^h(\Delta z, t | \Delta z_0, h) + (-\mathcal{E}_{ss}) g^u(\Delta z, t | \Delta z_0, h) \right] \mathrm{d}\Delta z\, \mathrm{d}t \right], \tag{B.3}$$

$$m_{\mathcal{E},u}(\Delta z_0, \zeta, \mathcal{T}) := \int_0^{\mathcal{T}} \left[\int_{-\infty}^{\infty} \left[(1 - \mathcal{E}_{ss})\, g^h(\Delta z, \zeta, t | \Delta z_0, u) + (-\mathcal{E}_{ss}) g^u(\Delta z, \zeta, t | \Delta z_0, u) \right] \mathrm{d}\Delta z\, \mathrm{d}t \right]. \tag{B.4}$$

Proof of Step 1. Following Baley and Blanco (2022), it can be shown that

$$\begin{aligned} \text{CIR}_{\mathcal{E}}(\zeta) &= \int_0^{\infty} \int_{-\infty}^{\infty} \left(g^h(\Delta z, \zeta, t) - g^h(\Delta z) \right) \mathrm{d}\Delta z\, \mathrm{d}t \\ &= \int_{-\infty}^{\infty} \lim_{\mathcal{T}\to\infty} m_{\mathcal{E},h}(\Delta z, \mathcal{T}) g^h(\Delta z + \zeta)\, \mathrm{d}\Delta z + \int_{-\infty}^{\infty} \lim_{\mathcal{T}\to\infty} m_{\mathcal{E},u}(\Delta z, \zeta, \mathcal{T}) g^u(\Delta z + \zeta)\, \mathrm{d}\Delta z \end{aligned} \tag{B.5}$$

where we define

$$m_{\mathcal{E},h}(\Delta z_0, \mathcal{T}) \equiv \int_0^{\mathcal{T}} \left[\int_{-\infty}^{\infty} \left[(1 - \mathcal{E}_{ss})\, g^h(\Delta z, t | \Delta z_0, h) + (-\mathcal{E}_{ss}) g^u(\Delta z, t | \Delta z_0, h) \right] \mathrm{d}\Delta z\, \mathrm{d}t \right]$$

$$m_{\mathcal{E},u}(\Delta z_0, \zeta, \mathcal{T}) \equiv \int_0^{\mathcal{T}} \left[\int_{-\infty}^{\infty} \left[(1 - \mathcal{E}_{ss})\, g^h(\Delta z, \zeta, t | \Delta z_0, u) + (-\mathcal{E}_{ss}) g^u(\Delta z, \zeta, t | \Delta z_0, u) \right] \mathrm{d}\Delta z\, \mathrm{d}t \right].$$

Step 2. The CIR satisfies

$$\text{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z + \zeta)\, \mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z, \zeta) g^u(\Delta z + \zeta)\, \mathrm{d}\Delta z$$

and the value functions $m_{\mathcal{E},h}(\Delta z_0)$ and $m_{\mathcal{E},u}(\Delta z_0, \zeta)$ satisfy the following HJB and border conditions:

$$0 = 1 - \mathcal{E}_{ss} - (\gamma + \chi) \frac{\mathrm{d}m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2} \frac{d^2 m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z^2} + \delta(m_{\mathcal{E},u}(0,0) - m_{\mathcal{E},h}(\Delta z)), \tag{B.6}$$

$$0 = -\mathcal{E}_{ss} - (\gamma + \chi) \frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2} \frac{d^2 m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z^2} + f(\hat{\theta}(\hat{w}^* - \zeta))(m_{\mathcal{E},h}(-\zeta) - m_{\mathcal{E},u}(\Delta z, \zeta)) \tag{B.7}$$

$$0 = m_{\mathcal{E},u}(0,0) - m_{\mathcal{E},h}(\Delta z), \text{ for all } \Delta z \notin (-\Delta^-, \Delta^+) \tag{B.8}$$

$$0 = \lim_{\Delta z \to -\infty} \frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z} = \lim_{\Delta z \to \infty} \frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z} \tag{B.9}$$

$$0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z, 0) g^u(\Delta z)\,\mathrm{d}\Delta z. \tag{B.10}$$

Proof of Step 2. We divide this proof in steps *a* to *d*.

a. We show that $\lim_{\mathcal{T}\to\infty} m_{\mathcal{E},h}(\Delta z, \mathcal{T}) = m_{\mathcal{E},h}(\Delta z)$ and $\lim_{\mathcal{T}\to\infty} m_{\mathcal{E},u}(\Delta z, \zeta, \mathcal{T}) = m_{\mathcal{E},u}(\Delta z, \zeta)$: This property holds due to the convergence of the distribution of Δz over time to its ergodic distribution for any initial condition (Stokey, 1989).

b. We show that $0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z, \mathcal{T}) g^h(\Delta z)\,\mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z, 0, \mathcal{T}) g^u(\Delta z)\,\mathrm{d}\Delta z$: The logic of the proof is to repeat the steps behind (B.5) in the reverse order. Departing from the definition,

$$\begin{aligned}
&\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z_0, \mathcal{T}) g^h(\Delta z_0)\,\mathrm{d}\Delta z_0 + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z_0, 0, \mathcal{T}) g^u(\Delta z_0)\,\mathrm{d}\Delta z_0 \\
&=^{(1)} \int_{-\infty}^{\infty} \int_0^{\mathcal{T}} \left[\int_{-\infty}^{\infty} \left[(1-\mathcal{E}_{ss})\, g^h(\Delta z, t|\Delta z_0, h) + (-\mathcal{E}_{ss}) g^u(\Delta z, t|\Delta z_0, h) \right] \mathrm{d}\Delta z\,\mathrm{d}t \right] g^h(\Delta z_0)\,\mathrm{d}\Delta z_0 \\
&\cdots + \int_{-\infty}^{\infty} \int_0^{\mathcal{T}} \left[\int_{-\infty}^{\infty} \left[(1-\mathcal{E}_{ss})\, g^h(\Delta z, 0, t|\Delta z_0, u) + (-\mathcal{E}_{ss}) g^u(\Delta z, 0, t|\Delta z_0, u) \right] \mathrm{d}\Delta z\,\mathrm{d}t \right] g^u(\Delta z_0)\,\mathrm{d}\Delta z_0 \\
&=^{(2)} \int_0^{\mathcal{T}} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left[(1-\mathcal{E}_{ss})\, g^h(\Delta z, t|\Delta z_0, h) + (-\mathcal{E}_{ss}) g^u(\Delta z, t|\Delta z_0, h) \right] g^h(\Delta z_0)\,\mathrm{d}z_0\,\mathrm{d}\Delta z \right] \mathrm{d}t \\
&\cdots + \int_0^{\mathcal{T}} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left[(1-\mathcal{E}_{ss})\, g^h(\Delta z, 0, t|\Delta z_0, u) + (-\mathcal{E}_{ss}) g^u(\Delta z, 0, t|\Delta z_0, u) \right] g^u(\Delta z_0)\ \mathrm{d}\Delta z_0\,\mathrm{d}\Delta z \right] \mathrm{d}t \\
&=^{(3)} \int_0^{\mathcal{T}} \int_{-\infty}^{\infty} \left[(1-\mathcal{E}_{ss}) \underbrace{\int_{-\infty}^{\infty} g^h(\Delta z, 0, t|\Delta z_0) g(\Delta z_0)\,\mathrm{d}\Delta z_0}_{=g^h(\Delta z)} \mathrm{d}\Delta z + (-\mathcal{E}_{ss}) \underbrace{\int_{-\infty}^{\infty} g^u(\Delta z, 0, t|\Delta z_0) g(\Delta z_0)\,\mathrm{d}\Delta z_0\,\mathrm{d}\Delta z}_{=g^u(\Delta z)} \right] \mathrm{d}t \\
&=^{(4)} \int_0^{\mathcal{T}} (1-\mathcal{E}_{ss})\,\mathcal{E}_{ss}\,\mathrm{d}t + \int_0^{\mathcal{T}} (-\mathcal{E}_{ss})\,(1-\mathcal{E}_{ss})\,\mathrm{d}t = 0
\end{aligned}$$

In (1), we apply the definitions (B.3) and (B.4); (2) applies Fubini's theorem; (3) uses the steady-state conditions for $g^h(\cdot)$ and $g^u(\cdot)$, and the definition $g(\Delta z) = g^h(\Delta z) + g^u(\Delta z)$; and (4) computes the integral using the definitions of aggregate employment and unemployment.

c. To show that $0 = \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},u}(\Delta z, 0) g^u(\Delta z)\,\mathrm{d}\Delta z$, see Baley and Blanco (2022).

d. We show that the CIR satisfies (B.5) with $m_{\mathcal{E},h}(\Delta z_0)$ and $m_{\mathcal{E},u}(\Delta z_0, \zeta)$ satisfying (B.6)–(B.10): Writing the HJB for $m_{\mathcal{E},h}(\Delta z_0, \mathcal{T})$ and $m_{\mathcal{E},u}(\Delta z_0, \zeta, \mathcal{T})$, we have that

$$\begin{aligned}
0 = 1 - \mathcal{E}_{ss} - \frac{\mathrm{d}m_{\mathcal{E},h}(\Delta z, \mathcal{T})}{\mathrm{d}\mathcal{T}} - (\gamma + \chi)\frac{\mathrm{d}m_{\mathcal{E},h}(\Delta z, \mathcal{T})}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},h}(\Delta z, \mathcal{T})}{\mathrm{d}\Delta z^2} \\
+ \delta(m_{\mathcal{E},u}(0, 0, \mathcal{T}) - m_{\mathcal{E},h}(\Delta z, \mathcal{T})),
\end{aligned}$$

$$
\begin{aligned}
0 &= -\mathcal{E}_{ss} - \frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})}{\mathrm{d}\mathcal{T}} - (\gamma+\chi)\frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})}{\mathrm{d}\Delta z^2} \\
&\quad + f(\hat{\theta}(\hat{w}^* - \zeta))(m_{\mathcal{E},h}(-\zeta,\mathcal{T}) - m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})) \\
0 &= m_{\mathcal{E},u}(0,0,\mathcal{T}) - m_{\mathcal{E},h}(\Delta z,\mathcal{T}), \text{ for all } \Delta z \notin (-\Delta^-,\Delta^+) \\
0 &= \lim_{\Delta z\to-\infty} \frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})}{\mathrm{d}\Delta z} = \lim_{\Delta z\to\infty} \frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})}{\mathrm{d}\Delta z} \\
0 &= \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z,\mathcal{T}) g^h(\Delta z)\,\mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},u}(\Delta z,0,\mathcal{T}) g^u(\Delta z)\,\mathrm{d}\Delta z.
\end{aligned}
$$

The border condition for $m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})$ is implied from the fact that the job-finding rate $f(\hat{\theta}(\hat{w}^*))$ is independent of Δz, so the function $m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})$ is constant in the entire domain. Taking the limit $\mathcal{T}\to\infty$ and using point-wise convergence of $m_{\mathcal{E},h}(\Delta z_0,\mathcal{T})$ and $m_{\mathcal{E},u}(\Delta z_0,\zeta,\mathcal{T})$, we have the result.

Step 3. The solution of the differential equations (B.6) to (B.9) satisfy (B.1) and (B.2).

Proof of Step 3. This is just an application of Øksendal (2007), Chapter 9.

□

Before starting the next step of the proof, we summarize the conditions that characterize the distributions of Δz.

Steady-State Cross-Sectional Distribution Δz**.** Below we describe the Kolmogorov Forward Equations (KFE) for $g^h(\Delta z)$ and $g^u(\Delta z)$.

$$
\delta g^h(\Delta z) = (\gamma+\chi)(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \quad \text{for all } \Delta z \in (-\Delta^-,\Delta^+)/\{0\} \tag{B.11}
$$

$$
f(\hat{\theta}(\hat{w}^*))g^u(\Delta z) = (\gamma+\chi)(g^u)'(\Delta z) + \frac{\sigma^2}{2}(g^u)''(\Delta z) \quad \text{for all } \Delta z \in (-\infty,\infty)/\{0\} \tag{B.12}
$$

$$
g^h(\Delta z) = 0, \text{ for all } \Delta z \notin (-\Delta^-,\Delta^+) \tag{B.13}
$$

$$
\lim_{\Delta z\to-\infty} g^u(\Delta z) = \lim_{\Delta z\to\infty} g^u(\Delta z) = 0. \tag{B.14}
$$

$$
1 = \int_{-\infty}^{\infty} g^u(\Delta z)\,\mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\,\mathrm{d}\Delta z, \tag{B.15}
$$

$$
f(\hat{\theta}(\hat{w}^*))(1-\mathcal{E}) = \delta\mathcal{E} + \frac{\sigma^2}{2}\left[\lim_{\Delta z\downarrow-\Delta^-}(g^h)'(\Delta z) - \lim_{\Delta z\uparrow\Delta^+}(g^h)'(\Delta z)\right], \tag{B.16}
$$

$$
g^h(\Delta z), g^u(\Delta z) \in \mathbb{C},\ g^u(\Delta z) \in \mathbb{C}^2((-\infty,\infty)/\{0\}),\ g^h(\Delta z) \in \mathbb{C}^2((-\Delta^-,\Delta^+)/\{0\})
$$

Proposition B.2. *Assume flexible entry wages. Up to first order, the CIR of employment is given by:*

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = -(1-\mathcal{E}_{ss})\frac{(\gamma+\chi)\mathbb{E}_h[a]+\mathbb{E}_h[\Delta z]}{\sigma^2} + o(\zeta).$$

Proof. The proof proceeds in three steps. Step 1 computes the value function for an unemployed worker $m_{\mathcal{E},u}(\Delta z)$ (when entry wages are flexible, the job-finding rate and this value function are independent of the shock ζ, so we omit this argument). Step 2 computes the value for the employed worker at $\Delta z = 0$—i.e., $m_{\mathcal{E},h}(0)$. Step 3 characterizes the CIR as a function of steady-state aggregate variables and moments.

Step 1. The CIR is given by

$$\text{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z+\zeta)\, \mathrm{d}\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}),$$

with

$$\begin{aligned}
0 &= 1-\mathcal{E}_{ss} - (\gamma+\chi)\frac{\mathrm{d}m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z^2} + \delta\left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0) - m_{\mathcal{E},h}(\Delta z)\right),\\
0 &= -\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0) - m_{\mathcal{E},h}(\Delta z), \text{ for all } \Delta z \notin (-\Delta^-, \Delta^+)\\
0 &= \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}).
\end{aligned} \tag{B.17}$$

Proof of Step 1. To show this result, note that the solution to (B.7) and (B.9) is $m_{\mathcal{E},u}(\Delta z) = m_{\mathcal{E},u}(0)$, for all Δz. Thus,

$$0 = -\mathcal{E}_{ss} + f(\hat{\theta}(\hat{w}^*))(m_{\mathcal{E},h}(0) - m_{\mathcal{E},u}(0)) \iff m_{\mathcal{E},u}(0) = -\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0). \tag{B.18}$$

Replacing (B.18) into the CIR, we have the result.

Step 2. We show that $m_{\mathcal{E},h}(0) = \frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} - (1-\mathcal{E}_{ss})\mathbb{E}_h[a]$, where $\mathbb{E}_h[a]$ is the cross-sectional expected age of the match or the worker's tenure at the current match.

Proof of Step 2. Observe that $m_{\mathcal{E},h}(\Delta z)$ satisfies the following recursive representation

$$m_{\mathcal{E},h}(\Delta z) = \mathbb{E}\left[\int_0^{\tau^m}(1-\mathcal{E}_{ss})\,\mathrm{d}t + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)\middle| \Delta z_0 = \Delta z\right]. \tag{B.19}$$

Define the following auxiliary function

$$\Psi(\Delta z|\varphi) = \mathbb{E}\left[\int_0^{\tau^m} e^{\varphi t}(1-\mathcal{E}_{ss})\,\mathrm{d}t + e^{\varphi\tau^m}\left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)\middle| \Delta z_0 = \Delta z\right]. \tag{B.20}$$

and note that $\Psi(\Delta z|0) = m_{\mathcal{E},h}(\Delta z)$. The auxiliary function $\Psi(\Delta z|\varphi)$ satisfies the following HJB and border conditions:

$$-\varphi\Psi(\Delta z|\varphi) + \delta\left(\Psi(\Delta z|\varphi) - \left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)\right) = (1-\mathcal{E}_{ss}) - (\gamma+\chi)\frac{\partial\Psi(\Delta z|\varphi)}{\partial\Delta z} + \frac{\sigma^2}{2}\frac{\partial^2\Psi(\Delta z|\varphi)}{\partial\Delta z^2}, \quad \text{(B.21)}$$

$$\Psi(\Delta z,\varphi) = \left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right) \text{ for all } \Delta z \notin (-\Delta^-, \Delta^+).$$

Taking the derivative with respect to φ in (B.21), we have that

$$(\delta-\varphi)\frac{\partial\Psi(\Delta z|\varphi)}{\partial\varphi} - \Psi(\Delta z|\varphi) = -(\gamma+\chi)\frac{\partial^2\Psi(\Delta z,\varphi)}{\partial\Delta z\partial\varphi} + \frac{\sigma^2}{2}\frac{\partial^3\Psi(\Delta z|\varphi)}{\partial\Delta z^2\partial\varphi},$$

$$\frac{\partial\Psi(\Delta z|\varphi)}{\partial\varphi} = 0 \text{ for all } \Delta z \notin (-\Delta^-, \Delta^+).$$

Using the Schwarz theorem to exchange partial derivatives, evaluating at $\varphi = 0$, and using $\Psi(\Delta z|0) = m_{\mathcal{E},h}(\Delta z)$, we obtain

$$\delta\frac{\partial\Psi(\Delta z|0)}{\partial\varphi} - m_{\mathcal{E},h}(\Delta z) = -(\gamma+\chi)\frac{\partial}{\partial\Delta z}\left(\frac{\partial\Psi(\Delta z|0)}{\partial\varphi}\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial\Delta z^2}\left(\frac{\partial\Psi(\Delta z|0)}{\partial\varphi}\right), \quad \text{(B.22)}$$

$$\frac{\partial\Psi(-\Delta^-|0)}{\partial\varphi} = \frac{\partial\Psi(\Delta^+|0)}{\partial\varphi} = 0. \quad \text{(B.23)}$$

Equations (B.22) and (B.23) correspond to the HJB and border conditions of the function $\frac{\partial\Psi(\Delta z|0)}{\partial\varphi} = \mathbb{E}\left[\int_0^{\tau^m} m_{\mathcal{E},h}(\Delta z_t)\,\mathrm{d}t\middle|\Delta z_0 = \Delta z\right]$. Evaluating $\frac{\partial\Psi(\Delta z|0)}{\partial\varphi}$ at $\Delta z = 0$, using the occupancy measure and result (B.17), we write the previous equation as:

$$\frac{\partial\Psi(0|0)}{\partial\varphi} = \mathbb{E}\left[\int_0^{\tau^m} m_{\mathcal{E},h}(\Delta z_t)\,\mathrm{d}t\middle|\Delta z_0 = 0\right] = \mathbb{E}_{\mathcal{D}}[\tau^m]\frac{\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\,\mathrm{d}\Delta z}{\mathcal{E}_{ss}}$$

$$= \mathbb{E}_{\mathcal{D}}[\tau^m]\left(\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} - m_{\mathcal{E},h}(0)\right)\frac{(1-\mathcal{E}_{ss})}{\mathcal{E}_{ss}}, \quad \text{(B.24)}$$

where $\mathbb{E}_{\mathcal{D}}[\tau^m]$ is the mean duration of completed employment spells (the subscript highlights that the moment can be easily computed from the data). From (B.20), we also have that

$$\frac{\partial\Psi(0|0)}{\partial\varphi} = \mathbb{E}\left[\int_0^{\tau^m} s\,(1-\mathcal{E}_{ss})\,\mathrm{d}s + \tau^m\left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)\middle|\Delta z_0 = 0\right]$$

$$= \mathbb{E}_{\mathcal{D}}\left[\tau^m\right]\left[(1-\mathcal{E}_{ss})\frac{\mathbb{E}_h[a]}{\mathcal{E}_{ss}} + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)\right], \quad \text{(B.25)}$$

Combining (B.24) and (B.25), and solving for $m_{\mathcal{E},h}(0)$ we obtain $m_{\mathcal{E},h}(0) = \frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} - (1-\mathcal{E}_{ss})\mathbb{E}_h[a]$.

Step 3. Up to a first-order approximation, the CIR is given by:

$$\mathrm{CIR}_{\mathcal{E}}(\zeta) = -(1-\mathcal{E}_{ss})\frac{(\gamma+\chi)\mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]}{\sigma^2}\zeta + o(\zeta^2).$$

Proof of Step 3. To help the reader, we summarize below the conditions used in this step of the proof.

$$\mathrm{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z+\zeta)\,\mathrm{d}\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}) \tag{B.26}$$

with

$$\delta m_{\mathcal{E},h}(\Delta z) = 1-\mathcal{E}_{ss} - (\gamma+\chi)\frac{\mathrm{d}m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z^2} + \delta m_{\mathcal{E},u}(0), \tag{B.27}$$

$$m_{\mathcal{E},u}(0) = m_{\mathcal{E},h}(\Delta z) \text{ for all } \Delta z \notin (-\Delta^-, \Delta^+) \tag{B.28}$$

$$0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\Delta z + m_{\mathcal{E},u}(0)(1-\mathcal{E}_{ss}). \tag{B.29}$$

1. **Zeroth Order:** If $\zeta = 0$, condition (B.29) implies

$$\mathrm{CIR}_{\mathcal{E}}(0) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}) = 0.$$

2. **First Order:** Taking the derivative of (B.26) we obtain $\mathrm{CIR}'_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z+\zeta)\,\mathrm{d}\Delta z$, which evaluated at $\zeta = 0$ becomes $\mathrm{CIR}'_{\mathcal{E}}(0) = \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z$. Using condition (B.11) to replace $\delta = \frac{(\gamma+\chi)(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)}{g^h(\Delta z)}$ into equation (B.27), we obtain

$$\frac{(\gamma+\chi)(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)}{g^h(\Delta z)} m_{\mathcal{E},h}(\Delta z) = 1-\mathcal{E}_{ss} - (\gamma+\chi)m'_{\mathcal{E},h}(\Delta z) + \frac{\sigma^2}{2}m''_{\mathcal{E},h}(\Delta z) + \frac{(\gamma+\chi)g'(\Delta z) + \frac{\sigma^2}{2}g''(\Delta z)}{g(\Delta z)} m_{\mathcal{E},u}(0).$$

Multiplying both sides by $g^h(\Delta z)\Delta z$ and integrating between $-\Delta^-$ and Δ^+,

$$0 = (1-\mathcal{E}_{ss})\mathbb{E}_h[\Delta z] - (\gamma+\chi)T_1 + \frac{\sigma^2}{2}T_2 + m_{\mathcal{E},u}(0)T_3 \tag{B.30}$$

$$T_1 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z) + m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\Delta z$$

$$T_2 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z) g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\Delta z$$

$$T_3 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left((\gamma + \chi)(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \right) \mathrm{d}\Delta z.$$

Next, we operate on the terms T_1, T_2, and T_3. The term T_1 is equal to

$$\begin{aligned}
T_1 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z) + m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\Delta z \qquad \text{(B.31)} \\
&=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z) + m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \Delta z \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z) + m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\Delta z \\
&=^{(2)} \int_{-\Delta^-}^{0} \Delta z \frac{\mathrm{d}\left(m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) \right)}{\mathrm{d}\Delta z} \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \Delta z \frac{\mathrm{d}\left(m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) \right)}{\mathrm{d}\Delta z} \mathrm{d}\Delta z \\
&=^{(3)} \underbrace{\Delta z m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) \Big|_{-\Delta^-}^{0} + \Delta z m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) \Big|_{0}^{\Delta^+}}_{=0} - \left[\int_{-\Delta^-}^{0} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) \, \mathrm{d}\Delta z + \int_{0}^{\Delta^+} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) \, \mathrm{d}\Delta z \right] \\
&=^{(4)} - \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) \, \mathrm{d}\Delta z =^{(5)} m_{\mathcal{E},u}(0)(1 - \mathcal{E}_{ss}).
\end{aligned}$$

Here, (1) divides the integral at the discontinuity point of $g^h(\Delta z)$; (2) uses the property of the derivative of a product of functions; (3) integrates and uses the border conditions for $g^h(\Delta z)$; (4) uses the continuity of $m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)$; and (5) uses (B.29).

The term T_2 satisfies

$$\begin{aligned}
T_2 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z) g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\Delta z \qquad \text{(B.32)} \\
&=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[m''_{\mathcal{E},h}(\Delta z) g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z) g(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\Delta z \\
&=^{(2)} \Delta z \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \Big|_{-\Delta^-}^{0} + \Delta z \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \Big|_{0}^{\Delta^+} \\
&\cdots - \left[\int_{-\Delta^-}^{0} \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\Delta z \right] \\
&=^{(3)} \underbrace{\Delta z \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \Big|_{\Delta^-}^{\Delta^+}}_{= -m_{\mathcal{E},u}(0) \, \Delta z (g^h)'(\Delta z)|_{\Delta^-}^{\Delta^+}} \\
&\cdots - \left[\int_{-\Delta^-}^{0} \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\Delta z \right] \\
&=^{(4)} -m_{\mathcal{E},u}(0) \; \Delta z (g^h)'(\Delta z) \Big|_{\Delta^-}^{\Delta^+} - \int_{\Delta^-}^{\Delta^+} m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z) \, \mathrm{d}\Delta z + \int_{\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \, \mathrm{d}\Delta z \\
&=^{(5)} -m_{\mathcal{E},u}(0) \; \Delta z (g^h)'(\Delta z) \Big|_{\Delta^-}^{\Delta^+} - \left[\underbrace{m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) \Big|_{\Delta^-}^{\Delta^+}}_{=0} - \int_{\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \, \mathrm{d}\Delta z \right] \\
&\quad + \int_{\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \, \mathrm{d}\Delta z \\
&= -m_{\mathcal{E},u}(0) \; \Delta z (g^h)'(\Delta z) \Big|_{\Delta^-}^{\Delta^+} + 2 \int_{\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z) g'(\Delta z) \, \mathrm{d}\Delta z.
\end{aligned}$$

Here, (1) divides the integral at the discontinuity $g^h(\Delta z)$; (2) uses the equality $m''_{\mathcal{E},h}(\Delta z)g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) = \frac{\mathrm{d}\left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right]}{\mathrm{d}\Delta z}$ and integrates by parts; (3) uses conditions (B.28) and the border conditions of $g^h(\Delta z)$; and (4)-(5) integrate by parts and operate.

Finally, the term T_3 is equal to

$$
\begin{aligned}
T_3 &= \int_{-\Delta-}^{\Delta+} \Delta z \left((\gamma+\chi)(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \right) \mathrm{d}\Delta z \qquad \text{(B.33)} \\
&=^{(1)} (\gamma+\chi) \left[\int_{-\Delta^-}^{0} \Delta z (g^h)'(\Delta z)\, \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \Delta z (g^h)'(\Delta z)\, \mathrm{d}\Delta z \right] \\
&\quad + \frac{\sigma^2}{2} \left[\int_{\Delta^-}^{0} \Delta z (g^h)''(\Delta z)\, \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \Delta z (g^h)''(\Delta z)\, \mathrm{d}\Delta z \right] \\
&=^{(2)} (\gamma+\chi) \left[\underbrace{\Delta z g^h(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z g^h(\Delta z)\Big|_{0}^{\Delta^+}}_{=0} - \underbrace{\int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\, \mathrm{d}\Delta z}_{=\mathcal{E}_{ss}} \right] \\
&\quad \cdots + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z (g^h)'(\Delta z)\Big|_{0}^{\Delta^+} - \int_{-\Delta^-}^{\Delta^+} (g^h)'(\Delta z)\, \mathrm{d}\Delta z \right] \\
&=^{(3)} -(\gamma+\chi)\mathcal{E}_{ss} + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - \underbrace{g^h(\Delta z)\Big|_{\Delta^-}^{\Delta^+}}_{=0} \right] \\
&=^{(4)} -(\gamma+\chi)\mathcal{E}_{ss} + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} \right]
\end{aligned}
$$

Here, (1) divides the integral at the discontinuity point of $g^h(\Delta z)$; (2) integrates by parts; and (3)–(4) use the border conditions of $g^h(\Delta z)$.

Combining results (B.30), (B.31), (B.32), (B.33) and those in Step 2, we obtain

$$
\begin{aligned}
0 &= (1-\mathcal{E}_{ss})\mathbb{E}_h[\Delta z] - (\gamma+\chi)T_1 + \frac{\sigma^2}{2}T_2 + m_{\mathcal{E},u}(0)T_3 \\
&= (1-\mathcal{E}_{ss})\mathbb{E}_h[\Delta z] - (\gamma+\chi)m_{\mathcal{E},u}(0) + \sigma^2 \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\, \mathrm{d}\Delta z,
\end{aligned}
$$

which implies $\int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\, \mathrm{d}\Delta z = -(1-\mathcal{E}_{ss})\frac{[(\gamma+\chi)\mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]]}{\sigma^2}$.

□

Proposition B.3. *Up to first order, the $CIR_{\mathcal{E}}(\zeta)$ can be expressed in terms of data moments as follows:*

(i) If $(\gamma+\chi)=0$, $\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = \underbrace{\frac{1}{f(\hat{\theta}(\hat{w}^*))}}_{avg.\ u\ dur.} \underbrace{\frac{1}{\mathbb{V}ar_{\mathcal{D}}[\Delta w]}}_{dispersion} \left[\underbrace{\frac{1}{3}\mathbb{E}_{\mathcal{D}}\left[\Delta w \frac{\Delta w^2}{\mathbb{E}_{\mathcal{D}}[\Delta w^2]}\right]}_{asymmetries} \right] + o(\zeta).$

(ii) If $(\gamma+\chi)\neq 0$,

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = \underbrace{\frac{1}{f(\hat{\theta}(\hat{w}^*))}}_{avg.\ u\ dur.} \underbrace{\frac{1}{\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]}}_{dispersion}$$

$$\times \left[\underbrace{\mathbb{E}_{\mathcal{D}}[\Delta w]\left(\mathcal{E}_{ss}\left(\mathbb{C}ov_{\mathcal{D}}\left[\widetilde{\Delta w}, \widetilde{\Delta w} - \tilde{\tau}\right] + \frac{\mathbb{V}ar_{\mathcal{D}}[\tilde{\tau}] - \mathcal{E}_{ss}\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\tau^m}\right]}{2}\right) + (1-\mathcal{E}_{ss})\left(\frac{\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1}{2}\right)\right)}_{asymmetries} \right] + o(\zeta).$$

Proof. The goal is to express the sufficient statistics of the CIR, $\mathbb{E}_h[a]$ and $\mathbb{E}_h[\Delta z]$, in terms of moments of the distribution of Δw and (τ^u, τ^m). We do so separately for the case with $(\gamma+\chi)=0$ and $(\gamma+\chi)\neq 0$. Let $\tilde{x} \equiv x/\mathbb{E}_{\mathcal{D}}[x]$ denote random variable x relative to its mean in the data.

Proposition VI.3 expresses moments of the wage distribution as a linear combination of moments of the distribution of productivity changes among completed employment and unemployment spells:

$$\begin{aligned}
\mathbb{E}_{\mathcal{D}}[\Delta w] &= -\left[\bar{\mathbb{E}}_u[\Delta z] + \bar{\mathbb{E}}_h[\Delta z]\right] \\
\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right] &= \left[\bar{\mathbb{E}}_u\left[\Delta z^2\right] + 2\bar{\mathbb{E}}_h[\Delta z]\,\bar{\mathbb{E}}_u[\Delta z] + \bar{\mathbb{E}}_h\left[\Delta z^2\right]\right] \\
\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right] &= -\left[\bar{\mathbb{E}}_u\left[\Delta z^3\right] + 3\bar{\mathbb{E}}_h[\Delta z]\,\bar{\mathbb{E}}_u\left[\Delta z^2\right] + 3\bar{\mathbb{E}}_h\left[\Delta z^2\right]\bar{\mathbb{E}}_u[\Delta z] + \bar{\mathbb{E}}_h\left[\Delta z^3\right]\right],
\end{aligned}$$

where $\bar{\mathbb{E}}_h[\cdot]$ and $\bar{\mathbb{E}}_u[\cdot]$ denote the expectation operators under the distributions $\bar{g}^h(\Delta z)$ and $\bar{g}^u(\Delta z)$, respectively. Using results from the same Proposition, we can express the moments of productivity changes for completed unemployment spells in terms of model parameters:

$$\bar{\mathbb{E}}_u[\Delta z] = \frac{\left(\mathcal{L}_2^{-1} - \mathcal{L}_2\right)}{\mathcal{L}_1}, \quad \bar{\mathbb{E}}_u\left[\Delta z^2\right] = \frac{2\left(\mathcal{L}_2^{-2} + \mathcal{L}_2^2 - 1\right)}{\mathcal{L}_1^2}, \quad \bar{\mathbb{E}}_u\left[\Delta z^3\right] = \frac{6\left(-\mathcal{L}_2^3 + \mathcal{L}_2 - \mathcal{L}_2^{-1} + \mathcal{L}_2^{-3}\right)}{\mathcal{L}_1^3},$$

where

$$\mathcal{L}_1 = \sqrt{\frac{2f(\hat{\theta}(\hat{w}^*))}{\sigma^2}} \text{ and } \mathcal{L}_2 = \sqrt{\frac{(\gamma+\chi) + \sqrt{(\gamma+\chi)^2 + 2\sigma^2 f(\hat{\theta}(\hat{w}^*))}}{-(\gamma+\chi) + \sqrt{(\gamma+\chi)^2 + 2\sigma^2 f(\hat{\theta}(\hat{w}^*))}}}.$$

From these two sets of equations, we solve for the moments of productivity changes for completed employment spells and obtain

$$\bar{\mathbb{E}}_h\left[\Delta z\right] = -\left(\frac{\left(\mathcal{L}_2^{-1} - \mathcal{L}_2\right)}{\mathcal{L}_1}\right) - \mathbb{E}_{\mathcal{D}}\left[\Delta w\right]$$

$$\bar{\mathbb{E}}_h\left[\Delta z^2\right] = \mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right] + 2\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]\left(\frac{\left(\mathcal{L}_2^{-1} - \mathcal{L}_2\right)}{\mathcal{L}_1}\right) - \frac{2}{\mathcal{L}_1^2}$$

$$\bar{\mathbb{E}}_h\left[\Delta z^3\right] = -\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right] - 3\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right]\left(\frac{\left(\mathcal{L}_2^{-1} - \mathcal{L}_2\right)}{\mathcal{L}_1}\right) + \frac{6}{\mathcal{L}_1^2}\mathbb{E}_{\mathcal{D}}\left[\Delta w\right].$$

The remaining steps are case-specific.

Case I: $(\gamma + \chi) = 0$. To obtain $\mathbb{E}_h[\Delta z]$, evaluate (VI.7) at $m = 1$, use the fact that $\mathcal{L}_2 = 1$, $\mathbb{E}_{\mathcal{D}}\left[\Delta w\right] = 0$ and $\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]} = \mathcal{E}_{ss}$, and substitute σ^2 from Lemma V.1: $\mathbb{E}_h\left[\Delta z\right] = -\frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right]}{3\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right]}$. Finally, replace this expression into (23):

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = -\left(1 - \mathcal{E}_{ss}\right)\frac{\mathbb{E}_h\left[\Delta z\right]}{\sigma^2} = \left(1 - \mathcal{E}_{ss}\right)\frac{\frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right]}{3\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right]}}{\frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right]}{\mathbb{E}_{\mathcal{D}}[\tau]}}$$

$$= \frac{1}{f(\hat{\theta}(\hat{w}^*))}\frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right]}{3\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right]^2} = \frac{1}{f(\hat{\theta}(\hat{w}^*))}\frac{1}{\mathbb{V}ar_{\mathcal{D}}\left[\Delta w^2\right]}\frac{1}{3}\mathbb{E}_{\mathcal{D}}\left[\Delta w\frac{\Delta w^2}{\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right]}\right].$$

Case II: $(\gamma + \chi) \neq 0$. To obtain $\mathbb{E}_h[\Delta z]$, evaluate (VI.8) at $m = 1$, use the result that $\left(\mathcal{L}_2^{-1} - \mathcal{L}_2\right)/\mathcal{L}_1 = -(\gamma + \chi)/f(\hat{\theta}(\hat{w}^*))$ and $\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]} = \mathcal{E}_{ss}$, and substitute $(\gamma + \chi)$ and σ^2 from Lemma V.1:

$$\mathbb{E}_h\left[\Delta z\right] = \frac{\mathcal{E}_{ss}}{2}\frac{\bar{\mathbb{E}}_h\left[\Delta z^2\right]}{\bar{\mathbb{E}}_h\left[\Delta z\right]} - \frac{\sigma^2}{2(\gamma + \chi)}$$

$$= \frac{\mathcal{E}_{ss}}{2}\left(\frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right] - 2\frac{\mathbb{E}_{\mathcal{D}}[\Delta w]^2\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]} - \sigma^2\mathbb{E}_{\mathcal{D}}[\tau^u]}{-\mathbb{E}_{\mathcal{D}}[\Delta w]\left(1 - \frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]}\right)}\right) - \frac{\sigma^2}{2(\gamma + \chi)}$$

$$= -\frac{1}{2}\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]\left(\mathbb{E}_{\mathcal{D}}\left[\widetilde{\Delta w}^2\right] - 2\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]}\right) + \frac{1}{2}\frac{\mathbb{E}_{\mathcal{D}}[(\Delta w - (\gamma + \chi)\tau)^2]}{\mathbb{E}_{\mathcal{D}}[\tau]\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]}\left(\mathbb{E}_{\mathcal{D}}[\tau^u] - \mathbb{E}_{\mathcal{D}}[\tau]\right)$$

$$= -\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]\left(\frac{1}{2}\left(\mathbb{E}_{\mathcal{D}}\left[\widetilde{\Delta w}^2\right] - 2\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]}\right) + \frac{1}{2}\mathbb{E}_{\mathcal{D}}[(\widetilde{\Delta w} - \tilde{\tau})^2]\left(1 - \frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]}\right)\right)$$

$$= -\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]\left(\frac{1}{2}\left(var_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right) + \mathcal{E}_{ss}\left(1 + \frac{1}{2}var_{\mathcal{D}}[(\widetilde{\Delta w} - \tilde{\tau})]\right)\right).$$

The average cross-sectional age of a job spell is obtained from the occupancy measure:

$$\mathbb{E}_h[a] = \mathcal{E}_{ss} \frac{\mathbb{E}[\int_0^{\tau^m} t \, \mathrm{d}t | \hat{w}_0 = \hat{w}^*]}{\mathbb{E}[\tau^m | \hat{w}_0 = \hat{w}^*]} = \frac{\mathcal{E}_{ss}}{2} \frac{\mathbb{E}_{\mathcal{D}}[\tau^{m2}]}{\mathbb{E}_{\mathcal{D}}[\tau^m]},$$

where we solve the Riemann integral.

Finally, we substitute these expressions into (23):

$$\begin{aligned}
\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} &= -\left(1-\mathcal{E}_{ss}\right) \frac{\mathbb{E}_h\left[\Delta z\right] + (\gamma+\chi)\mathbb{E}[a]}{\sigma^2} \\
&= -\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]} \frac{\mathbb{E}_{\mathcal{D}}[\tau]}{\mathbb{E}_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)^2\right]} \\
&\quad \times \left(-\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]\left(\frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right]-1\right) + \mathcal{E}_{ss}\left(1+\frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}[\widetilde{\Delta w}-\tilde{\tau}]\right)\right)\right) + (\gamma+\chi)\mathbb{E}[a]\right) \\
&= \frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]}{f(\hat{\theta}(\hat{w}^*))\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]} \\
&\quad \times \left(\frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right]-1\right) + \mathcal{E}_{ss}\left(1+\frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}[\widetilde{\Delta w}-\tilde{\tau}]\right)\right) - \frac{1}{\mathbb{E}_{\mathcal{D}}\left[\tau\right]}\mathbb{E}[a]\right) \\
&= \frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]}{2f(\hat{\theta}(\hat{w}^*))\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]} \\
&\quad \times \left(\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right]-1\right)(1-\mathcal{E}_{ss}) + \mathcal{E}_{ss}\left(1+\left(\mathbb{V}ar_{\mathcal{D}}[\widetilde{\Delta w}-\tilde{\tau}]\right) + \mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right]\right) - \mathcal{E}_{ss}\frac{\mathbb{E}_{\mathcal{D}}[\tau^m]}{\mathbb{E}_{\mathcal{D}}\left[\tau\right]}\mathbb{E}_{\mathcal{D}}[\widetilde{\tau^m}^2]\right) \\
&= \frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]}{f(\hat{\theta}(\hat{w}^*))\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]} \\
&\quad \times \left(\frac{\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right]-1\right)}{2}(1-\mathcal{E}_{ss}) + \mathcal{E}_{ss}\left(\mathbb{C}ov_{\mathcal{D}}\left[\widetilde{\Delta w}, \widetilde{\Delta w}-\tilde{\tau}\right] + \frac{\mathbb{V}ar_{\mathcal{D}}\left[\tilde{\tau}\right] - \mathcal{E}_{ss}\mathbb{V}ar_{\mathcal{D}}[\widetilde{\tau^m}^2]}{2}\right)\right).
\end{aligned}$$

□

B.2 Proof of Proposition 7: CIR of Employment with Sticky Entry Wage

Proposition 7. *Assume sticky entry wages. Up to first order, the CIR of employment is given by:*

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = (1-\mathcal{E}_{ss})\left[-\frac{[\gamma\mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]]}{\sigma^2} + \frac{1}{f(\hat{\theta}(\hat{w}^*)) + s}\left[\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} + \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} - \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}\right]\right] + o(\zeta). \tag{B.34}$$

Proof. We divide the proof in two steps. Step 1 characterizes $m_{\mathcal{E},u}(\Delta z, \zeta)$. Steps 2 uses the equilibrium conditions to show (B.34). The starting point is the CIR for employment, which is given by

$$\text{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z + \zeta)\, \mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z, \zeta) g^u(\Delta z + \zeta)\, \mathrm{d}\Delta z, \tag{B.35}$$

with

$$0 = 1 - \mathcal{E}_{ss} - \gamma \frac{\mathrm{d} m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2} \frac{d^2 m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z^2} + \delta(m_{\mathcal{E},u}(0,0) - m_{\mathcal{E},h}(\Delta z)), \text{ for all } \Delta z \in (-\Delta^-, \Delta^+) \tag{B.36}$$

$$0 = -\mathcal{E}_{ss} - \gamma \frac{\mathrm{d} m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2} \frac{d^2 m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z^2} + f(\hat{\theta}(\hat{w}^* - \zeta))(m_{\mathcal{E},h}(-\zeta) - m_{\mathcal{E},u}(\Delta z, \zeta)) \tag{B.37}$$

$$0 = m_{\mathcal{E},u}(0,0) - m_{\mathcal{E},h}(\Delta z), \text{ for all } \Delta z \notin (-\Delta^-, \Delta^+) \tag{B.38}$$

$$0 = \lim_{\Delta z \to -\infty} \frac{\mathrm{d} m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z} = \lim_{\Delta z \to \infty} \frac{\mathrm{d} m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z} \tag{B.39}$$

$$0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z) g^u(\Delta z)\, \mathrm{d}\Delta z \tag{B.40}$$

The key differences between the CIR with flexible wages and the CIR with sticky wages are found in the HJB equation at the moment of the shock. With sticky entry wages, the job-finding probability is given by $f(\hat{\theta}(\hat{w}^* - \zeta))$, since now the TFPR-adjusted entry wage is lower. As a consequence, we need to evaluate $m_{\mathcal{E},h}(\Delta z)$ at $\Delta z = -\zeta$ because conditional on finding a job, the TFPR-adjusted entry wage is lower. Observe that following the first job separation, the aggregate TFPR shock is fully absorbed (see the term $m_{\mathcal{E},u}(0,0)$ in equation (B.36)).

Step 1. The value function $m_{\mathcal{E},u}(\Delta z, \zeta)$ is independent of Δz and satisfies $m_{\mathcal{E},u}(\Delta z, \zeta) = -\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^* - \zeta))} + m_{\mathcal{E},h}(-\zeta)$.

Proof of Step 1. We guess and verify that $m_{\mathcal{E},u}(\Delta z, \zeta) = m_{\mathcal{E},u}(0, \zeta)$ for all Δz. From the equilibrium conditions (B.37) and (B.39),

$$0 = -\mathcal{E}_{ss} + f(\hat{\theta}(\hat{w}^* - \zeta))(m_{\mathcal{E},h}(-\zeta) - m_{\mathcal{E},u}(0, \zeta)).$$

Thus, $m_{\mathcal{E},u}(0, \zeta) = m_{\mathcal{E},u}(\Delta z, \zeta) = -\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^* - \zeta))} + m_{\mathcal{E},h}(-\zeta)$.

Step 2. Up to a first-order approximation, the CIR is given by:

$$\text{CIR}_{\mathcal{E}}(\zeta) = -(1 - \mathcal{E}_{ss}) \frac{(\gamma + \chi)\mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]}{\sigma^2} \zeta + \frac{(1 - \mathcal{E}_{ss})}{f(\hat{\theta}(\hat{w}^*)) + s} \left(\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} + \frac{\mathcal{T}'(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})} - \frac{\mathcal{T}'(\hat{w}^*, 0)}{\mathcal{T}(\hat{w}^*, 0)} \right) \zeta + o(\zeta^2).$$

Proof of Step 2. From Step 1, we have that

$$\text{CIR}'_{\mathcal{E}}(0) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z + \left(-\frac{\mathcal{E}_{ss} f_{\hat{w}}(\hat{\theta}(\hat{w}^*))}{f(\hat{\theta}(\hat{w}^*))^2} - m'_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}).$$

Since $\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z$ satisfies the same system of functional equations as the CIR with flexible entry wages characterized in Online Appendix B.1,

$$\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z = -(1-\mathcal{E}_{ss})\frac{\gamma \mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]}{\sigma^2}. \tag{B.41}$$

Observe that we can write

$$\begin{aligned} m_{\mathcal{E},h}(\Delta z) &= \mathbb{E}\left[\int_0^{\tau^m} (1-\mathcal{E}_{ss})\,\mathrm{d}t + m_{\mathcal{E},u}(\Delta z, 0)\,\middle|\, \Delta z_0 = \Delta z\right], \\ &= (1-\mathcal{E}_{ss})\mathcal{T}(\hat{w}^* + \Delta z, 0) - \frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0). \end{aligned}$$

Taking the derivative with respect to Δz, evaluating it at $\Delta z = 0$, and using $s = 1/\mathcal{T}(\hat{w}^*, 0)$ from the Renewal Principle, we have that

$$m'_{\mathcal{E},h}(0) = (1-\mathcal{E}_{ss})\mathcal{T}'_{\hat{w}}(\hat{w}^*, 0) = \frac{s}{f(\hat{\theta}(\hat{w}^*)) + s}\mathcal{T}'_{\hat{w}}(\hat{w}^*, 0) = \frac{1}{f(\hat{\theta}(\hat{w}^*)) + s}\frac{\mathcal{T}'(\hat{w}^*, 0)}{\mathcal{T}(\hat{w}^*, 0)}. \tag{B.42}$$

From the free entry condition $f(\hat{\theta}(\hat{w}^*)) = \left(\frac{\hat{J}(\hat{w}^*)}{\tilde{K}}\right)^{\frac{1-\alpha}{\alpha}}$, and the definition $(1-\eta(\hat{w}^*)) = \hat{J}(\hat{w}^*)/\hat{S}(\hat{w}^*)$, we can compute the elasticity of the job finding rate with respect to the entry wage:

$$\frac{f_{\hat{w}}(\hat{\theta}(\hat{w}^*))}{f(\hat{\theta}(\hat{w}^*))} = \frac{\frac{1-\alpha}{\alpha}\left(\frac{\hat{J}(\hat{w}^*)}{\tilde{K}}\right)^{\frac{1-\alpha}{\alpha}-1}\frac{\hat{J}'(\hat{w}^*)}{\tilde{K}}}{\left(\frac{\hat{J}(\hat{w}^*)}{\tilde{K}}\right)^{\frac{1-\alpha}{\alpha}}} = \frac{1-\alpha}{\alpha}\frac{\hat{J}'(\hat{w}^*)}{\hat{J}(\hat{w}^*)} = \frac{1-\alpha}{\alpha}\left[-\frac{\eta'(\hat{w}^*)}{(1-\eta(\hat{w}^*))} + \frac{\mathcal{T}'(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})}\right].$$

Finally, combining this result with the fact that $\mathcal{E}_{ss} = \frac{f(\hat{\theta}(\hat{w}^*))}{f(\hat{\theta}(\hat{w}^*))+s}$, $s = \frac{1}{\mathcal{T}(\hat{w}^*,0)}$, $\eta'(\hat{w}^*)\left(\frac{\alpha}{\eta(\hat{w}^*)} - \frac{1-\alpha}{1-\eta(\hat{w}^*)}\right) = -\frac{\mathcal{T}'(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}$, and operating, we obtain

$$\begin{aligned} -\frac{\mathcal{E}_{ss} f_{\hat{w}}(\hat{\theta}(\hat{w}^*))}{f(\hat{\theta}(\hat{w}^*))^2} &= \frac{1}{f(\hat{\theta}(\hat{w}^*)) + s}\left[-\frac{1-\alpha}{\alpha}\left[-\frac{\eta'(\hat{w}^*)}{(1-\eta(\hat{w}^*))} + \frac{\mathcal{T}'(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})}\right]\right] \\ &= \frac{1}{f(\hat{\theta}(\hat{w}^*)) + s}\left[\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} + \frac{\mathcal{T}'(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})}\right]. \end{aligned} \tag{B.43}$$

Combining results in equations (B.41), (B.42), and (B.43), we obtain the desired result:

$$\mathrm{CIR}'_{\mathcal{E}}(0) = -(1-\mathcal{E}_{ss})\frac{[(\gamma+\chi)\mathbb{E}_h[a]+\mathbb{E}_h[\Delta z]]}{\sigma^2} + \frac{1-\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))+s}\left[\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} + \frac{\mathcal{T}'(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} - \frac{\mathcal{T}'(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}\right].$$

□

B.3 Proof of Lemma 2

The following proposition provides a characterization of $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})/\mathcal{T}(\hat{w}^*,\hat{\rho})$ stated in Lemma 2.

Proposition B.4. *Up to a second-order approximation of $\mathcal{T}(\hat{w},\hat{\rho})$ around $\hat{w}=\hat{w}^*$,*

$$\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} = \frac{\Delta^+-\Delta^-}{\Delta^+\Delta^-}.$$

Proof. To show this property, it is useful to change the state variable in $\mathcal{T}(\hat{w},\hat{\rho})$ from $\hat{w}$ to Δz. Define $\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) := \mathcal{T}(\hat{w}^*+\Delta z,\hat{\rho})$. Then, applying Itô's Lemma, we obtain

$$\delta\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) = 1-\hat{\rho}\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) - (\gamma+\chi)\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,\hat{\rho}) + \frac{\sigma^2}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(\Delta z,\hat{\rho})\ \forall\ \Delta z\in(-\Delta^-,\Delta^+), \tag{B.44}$$

$$\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) = 0\ \forall\ \Delta z\notin(-\Delta^-,\Delta^+). \tag{B.45}$$

Let $(\gamma+\chi)\neq 0$ and $\Delta^+\neq\Delta^-$. In this case, we proceed with a second-order Taylor approximation of $\tilde{\mathcal{T}}(\Delta z,\hat{\rho})$ around $\Delta z=0$,

$$\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) = \tilde{\mathcal{T}}(0,\hat{\rho}) + \tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})\Delta z + \frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})\Delta z^2 + O(\Delta z^3).$$

From the border conditions in (B.45), we obtain (we omit the term $O(\Delta z^3)$ to save on notation)

$$\begin{aligned}\tilde{\mathcal{T}}(0,\hat{\rho}) + \tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})\Delta^+ + \frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})(\Delta^+)^2 &= 0,\\ \tilde{\mathcal{T}}(0,\hat{\rho}) + \tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})(-\Delta^-) + \frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})(\Delta^-)^2 &= 0.\end{aligned} \tag{B.46}$$

Taking the difference

$$\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})(\Delta^++\Delta^-) = -\frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})\left((\Delta^+)^2-(\Delta^-)^2\right) \iff \tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho}) = -\frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})\left(\Delta^+-\Delta^-\right).$$

Replacing this last equation into the HJB equation in (B.44) evaluated at $\Delta z = 0$ and into (B.46), we obtain

$$\tilde{\mathcal{T}}(0,\hat{\rho}) = \frac{1 + \left(\frac{\sigma^2 + (\gamma+\chi)(\Delta^+ - \Delta^-)}{2}\right)\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})}{\hat{\rho}+\delta}$$
$$\tilde{\mathcal{T}}(0,\hat{\rho}) = -\frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})\left((\Delta^+)^2 - \Delta^+\left(\Delta^+ - \Delta^-\right)\right).$$

Combining these equations and solving for $\tilde{\mathcal{T}}(0,\hat{\rho})$ and $\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})$, we have $\frac{\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})}{\tilde{\mathcal{T}}(0,\hat{\rho})} = \frac{\mathcal{T}'_{\hat{w}^*}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} = \frac{\Delta^+ - \Delta^-}{\Delta^+\Delta^-}$. □

B.4 Proof of Proposition 8

We divide Proposition 8 into two propositions. Proposition B.5 "rescales the speed of time" to provide a recursive representation of $\eta(\hat{w})$.

Proposition B.5. *Define*

$$\tau^{end} = \inf\{t \geq 0 : \Gamma_t \notin (\hat{w}^-, \hat{w}^+)\}$$

where $(\hat{w}^-, \hat{w}^+)$ is a Nash equilibrium. Then, the worker's share $\eta(\hat{w})$ satisfies the following Bellman equation

$$\eta(\hat{w}) = \mathbb{E}\left[\int_0^{\tau^{end}} e^{-(\hat{\rho}+\delta)t}(\hat{\rho}+\delta)\frac{e^{\Gamma_t} - \hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}}\,\mathrm{d}t + e^{-(\hat{\rho}+\delta)\tau^{end}}\mathbb{1}[\Delta z_{\tau^{end}} = \Delta^+] | \Gamma_0 = \hat{w}\right]$$

with

$$\mathrm{d}\Gamma_t = (\hat{\rho}+\delta)(-\hat{\gamma}\mathcal{T}(\Gamma_t,\hat{\rho}) + \sigma^2\mathcal{T}'_{\hat{w}}(\Gamma_t,\hat{\rho}))\,\mathrm{d}t + \sigma\sqrt{\mathcal{T}(\Gamma_t,\hat{\rho})(\hat{\rho}+\delta)}\,\mathrm{d}\mathcal{W}^z_t.$$

Proof. The HJB equations for the worker's value and the surplus of the match are

$$(\hat{\rho}+\delta)\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+)$$
$$(\hat{\rho}+\delta)\hat{S}(\hat{w}) = 1 - \hat{\rho}\hat{U} - \hat{\gamma}\hat{S}'(\hat{w}) + \frac{\sigma^2}{2}\hat{S}''(\hat{w}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+),$$

respectively. Replacing the definition of the worker's share $\eta(\hat{w}) = \hat{W}(\hat{w})/\hat{S}(\hat{w})$ into the worker's value function, we obtain

$$(\hat{\rho}+\delta)(\eta(\hat{w})\hat{S}(\hat{w})) = e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\left(\eta(\hat{w})\hat{S}'(\hat{w}) + \eta'(\hat{w})\hat{S}(\hat{w})\right) + \frac{\sigma^2}{2}\left(\eta(\hat{w})\hat{S}''(\hat{w}) + 2\eta'(\hat{w})\hat{S}'(\hat{w}) + \eta''(\hat{w})\hat{S}(\hat{w})\right) \forall \hat{w} \in (\hat{w}^-, \hat{w}^+).$$

Using the HJB equation of the surplus to replace $(\hat{\rho}+\delta)\hat{S}(\hat{w})$ on the left hand side,

$$(1-\hat{\rho}\hat{U})\eta(\hat{w}) = e^{\hat{w}} - \hat{\rho}\hat{U} + \eta'(\hat{w})(-\hat{\gamma}\hat{S}(\hat{w}) + \sigma^2\hat{S}'(\hat{w})) + \eta''(\hat{w})\frac{\sigma^2}{2}\hat{S}(\hat{w}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+).$$

Since $\hat{S}(\hat{w}) = (1-\hat{\rho}\hat{U})\mathcal{T}(\hat{w},\hat{\rho})$, multiplying by $(\hat{\rho}+\delta)$, we arrive at

$$(\hat{\rho}+\delta)\eta(\hat{w}) = (\hat{\rho}+\delta)\frac{e^{\hat{w}}-\hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}} + \eta'(\hat{w})(\hat{\rho}+\delta)(-\hat{\gamma}\mathcal{T}(\hat{w},\hat{\rho}) + \sigma^2\mathcal{T}'_{\hat{w}}(\hat{w},\hat{\rho})) + \eta''(\hat{w})\frac{\sigma^2}{2}(\hat{\rho}+\delta)\mathcal{T}(\hat{w},\hat{\rho}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+).$$

Finally, recall the value-matching conditions $\hat{W}(\hat{w}^-) = \hat{J}(\hat{w}^-) = \hat{W}(\hat{w}^+) = \hat{J}(\hat{w}^+) = 0$, and the smooth pasting conditions $\hat{W}'(-\Delta^-) = \hat{J}'(\Delta^+) = 0$. The L'Hôpital's rule implies

$$\begin{aligned}
\lim_{\hat{w}\downarrow\hat{w}^-} \eta(\hat{w}) &= \lim_{\hat{w}\downarrow\hat{w}^-} \frac{\hat{W}(\hat{w})}{\hat{S}(\hat{w})} = \lim_{\hat{w}\downarrow\hat{w}^-} \frac{\hat{W}'(\hat{w})}{\hat{J}'(\hat{w})} = 0 \\
\lim_{\hat{w}\uparrow\hat{w}^+} \eta(\hat{w}) &= \lim_{\hat{w}\uparrow\hat{w}^+} \frac{\hat{W}(\hat{w})}{\hat{S}(\hat{w})} = \lim_{\hat{w}\uparrow\hat{w}^+} \frac{\hat{W}'(\hat{w})}{\hat{W}'(\hat{w})} = 1,
\end{aligned}$$

which are the boundary values for the worker's share at the separation triggers.

Finally, the equivalence of the combined Dirichlet-Poisson problem (i.e., the mapping from the corresponding HJB equations and boundary conditions of $\eta(\hat{w})$ to the sequential formulation) gives us the following Bellman equation

$$\eta(\hat{w}) = \mathbb{E}\left[\int_0^{\tau^{end}} e^{-(\hat{\rho}+\delta)t}(\hat{\rho}+\delta)\frac{e^{\Gamma_t}-\hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}}\,\mathrm{d}t + e^{-(\hat{\rho}+\delta)\tau^{end}}\mathbb{1}[\Delta z_{\tau^{end}} = \Delta^+] | \Gamma_0 = \hat{w}\right],$$

where $\tau^{end} = \inf\{t \geq 0 : \Gamma_t \notin (\hat{w}^-, \hat{w}^+)\}$ and

$$\mathrm{d}\Gamma_t = (\hat{\rho}+\delta)(-\hat{\gamma}\mathcal{T}(\Gamma_t,\hat{\rho}) + \sigma^2\mathcal{T}'_{\hat{w}}(\Gamma_t,\hat{\rho}))\,\mathrm{d}t + \sigma\sqrt{\mathcal{T}(\Gamma_t,\hat{\rho})(\hat{\rho}+\delta)}\,\mathrm{d}\mathcal{W}^z_t.$$

□

Proof of Proposition 8. Below, we prove each property.

1. If $\Delta^+, \Delta^- \to \infty$, then $\mathcal{T}(\hat{w},\hat{\rho}) = \int_0^\infty e^{-(\hat{\rho}+\delta)t}\,\mathrm{d}t = \frac{1}{\hat{\rho}+\delta}$. The optimality condition for $\hat{w}^*$ implies

$$0 = -\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} = \eta'(\hat{w}^*)\left(\frac{\alpha}{\eta(\hat{w}^*)} - \frac{1-\alpha}{1-\eta(\hat{w}^*)}\right) \iff \alpha = \eta(\hat{w}^*).$$

Therefore, by the definition of $\eta(\hat{w})$,

$$\begin{aligned}
\alpha = \eta(\hat{w}^*) &= \frac{\mathbb{E}\left[\int_0^{\tau^m} e^{-\hat{\rho}t+\hat{w}_t}\,\mathrm{d}t | \hat{w}_0 = \hat{w}^*\right] - \hat{\rho}\hat{U}\mathcal{T}(\hat{w},\hat{\rho})}{(1-\hat{\rho}\hat{U})\mathcal{T}(\hat{w},\hat{\rho})} \\
\iff \left[\alpha + (1-\alpha)\hat{\rho}\hat{U}\right]\mathcal{T}(\hat{w},\hat{\rho}) &= \mathbb{E}\left[\int_0^{\tau^m} e^{-\hat{\rho}t+\hat{w}_t}\,\mathrm{d}t | \hat{w}_0 = \hat{w}^*\right].
\end{aligned}$$

Since $\mathcal{T}(\hat{w}, \hat{\rho})$ is constant, the HJB equation of the worker's share $\eta(\hat{w})$ is given by

$$(\hat{\rho}+\delta)\eta(\hat{w}) = (\hat{\rho}+\delta)\frac{e^{\hat{w}}-\hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}} - \hat{\gamma}\eta'(\hat{w}) + \eta''(\hat{w})\frac{\sigma^2}{2} \;\; \forall \hat{w} \in (-\infty, \infty). \tag{B.47}$$

Taking the derivative of (B.47) with respect to $\hat{w}$ yields

$$(\hat{\rho}+\delta)\eta'(\hat{w}) = (\hat{\rho}+\delta)\frac{e^{\hat{w}}}{1-\hat{\rho}\hat{U}} - \hat{\gamma}\eta''(\hat{w}) + \eta'''(\hat{w})\frac{\sigma^2}{2} \;\; \forall \hat{w} \in (-\infty, \infty).$$

This expression corresponds to the HJB of the function $\eta'(\hat{w})$, which can be expressed as

$$\eta'(\hat{w}^*) = (\hat{\rho}+\delta)\frac{\mathbb{E}\left[\int_0^{\tau^m} e^{-\hat{\rho}t+\hat{w}_t}\,\mathrm{d}t | \hat{w}_0 = \hat{w}^*\right]}{1-\hat{\rho}\hat{U}}$$

Combining all these results, we finally obtain

$$\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} = \frac{\eta'(\hat{w}^*)}{\alpha} = (\hat{\rho}+\delta)\frac{\mathbb{E}\left[\int_0^{\tau^m} e^{-\hat{\rho}t+\hat{w}_t}\,\mathrm{d}t | \hat{w}_0 = \hat{w}^*\right]}{\alpha(1-\hat{\rho}\hat{U})} = (\hat{\rho}+\delta)\frac{\left[\alpha+(1-\alpha)\hat{\rho}\hat{U}\right]\mathcal{T}(\hat{w},\hat{\rho})}{\alpha(1-\hat{\rho}\hat{U})} = \frac{\left[\alpha+(1-\alpha)\hat{\rho}\hat{U}\right]}{\alpha(1-\hat{\rho}\hat{U})}.$$

2. If $\gamma + \chi = 0$ and $\Delta^+ = \Delta^-$, then $\mathcal{T}'_{\hat{w}}(\hat{w}^*, \hat{\rho}) = 0$ and $\eta(\hat{w}^*) = \alpha$ (see the proof of Proposition B.4, item *a*). If $(\Delta^+ + \Delta^-)$ is small enough, then we can use a second-order approximation of $\eta(\hat{w})$ around $\hat{w} = \hat{w}^*$ to characterize $\eta'(\hat{w}^*)$ *only* using the border conditions. The approximation is given by

$$\eta(\hat{w}) = \eta(\hat{w}^*) + \eta'(\hat{w}^*)(\hat{w}-\hat{w}^*) + \frac{1}{2}\eta''(\hat{w}^*)(\hat{w}-\hat{w}^*)^2 + O((\hat{w}-\hat{w}^*)^3).$$

Evaluating this expression at $\hat{w}^-$ and $\hat{w}^+$, and omitting any terms of the order $O((\hat{w}-\hat{w}^*)^3)$, we obtain

$$\eta(\hat{w}^*) + \eta'(\hat{w}^*)(\hat{w}^- - \hat{w}^*) + \frac{1}{2}\eta''(\hat{w}^*)(\hat{w}^- - \hat{w}^*)^2 = 0,$$
$$\eta(\hat{w}^*) + \eta'(\hat{w}^*)(\hat{w}^+ - \hat{w}^*) + \frac{1}{2}\eta''(\hat{w}^*)(\hat{w}^+ - \hat{w}^*)^2 = 1,$$

respectively. The difference between both equations is given by $\eta'(\hat{w}^*) = \frac{1}{\Delta^+ + \Delta^-}$. From the proof of Proposition B.4 item *b*, we know that $\tilde{\mathcal{T}}(0,0) = 1/s = 1/(\delta + (\sigma/\Delta^+)^2) \Rightarrow s^{end} = (\sigma/\Delta^+)^2$. Replacing this result in the previous equation, we obtain, $\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} = \frac{1}{\alpha}\frac{1}{\Delta^+ + \Delta^-} = \frac{\sqrt{s^{end}}}{2\alpha\sigma}$.

□

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