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Lattice operations for the stable set in substitutable matching markets via re-equilibration dynamics*

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Abstract

We compute the lattice operations for the (pairwise) stable set in two-sided matching markets where only substitutability on agents' choice functions is imposed. To do this, we use Tarski operators defined on the lattices of worker-quasi-stable and firm-quasi-stable matchings. These operators resemble lay-off and vacancy chain dynamics, respectively. First, we compute the lattice operations in the many-to-one model. Then, we extend these operations to a many-to-many model with substitutable choice functions on one side and responsive preferences on the other, via a morphism that relates many-to-one with many-to-many matchings in a natural way. Finally, we present the lattice operations in the many-to-many model with substitutable choice functions on both sides.

JEL classification: C78, D47.

Keywords: two-sided matching, worker-quasi-stability, firm-quasi-stability, Tarski operator, lattice operations, re-equilibration.

1 Introduction

The lattice structure of the set of stable allocations is a fundamental tool in two-sided matching theory. It is the basis of the results on the conflict of interests between sides, the

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coincidence of interests within sides, and the solution of several algorithmic issues. In this paper, we study the lattice structure of two-sided matchings when only substitutability on agents' choice functions is imposed. This condition, introduced in the matching literature by Kelso and Crawford (1982), states that agents are still chosen when the set of alternatives shrinks and they are still available, so no complementarities among agents prevail. Substitutability has also been identified as a maximal domain condition for the existence of stable matchings (Hatfield and Kominers, 2017).

Since the seminal paper of Blair (1988) it is known that, when partial orders are properly defined, the set of stable matchings under substitutability alone has a dual lattice structure. Nevertheless, even though Blair (1988) obtains the lattice operations, his method is not transparent and lacks an economic interpretation. To provide a more complete answer either more stringent conditions on the model are necessary or a much more cumbersome theoretical structure is to be imposed in order to obtain them. Following the first approach, the lattice operations have been obtained invoking, in addition to substitutability, the property of "separability with quota" in many-to-one models (Martínez et al., 2001) and the "law of aggregate demand" in many-to-many models (Alkan, 2002).[1] Following the second approach, Echenique and Oviedo (2004) compute, in many-to-one models, the join and meet between two stable matchings at the expense of performing their analysis in the realm of pre-matchings, entities whose economic interpretation –again– is difficult to grasp. They define a Tarski operator within the set of pre-matchings and show that the formulas for the join and meet obtained by Martínez et al. (2001) define only pre-matchings when substitutability alone is required, but the fixed points obtained by starting their operator in such pre-matchings deliver the desired join and meet stable matchings.

Our proposal refines the second approach by both simplifying the analysis and adding economic content. We still only impose substitutability, but do not use pre-matchings. Instead, we consider quasi-stable matchings. A quasi-stable matching may have blocking pairs but only those which do not affect pre-existing relations for agents on one side of the market. Therefore, these matchings come in two flavors. In many-to-one models, worker-quasi-stable matchings demand that workers involved in blocking pairs are single; whereas firm-quasi-stable matchings demand that firms involved in blocking pairs cannot fire their current workers.

Worker-quasi-stable matchings in many-to-one models are introduced in Bonifacio et al. (2022), where it is shown that they form a lattice with respect to the (Blair's partial) order for the firms under substitutability. Bonifacio et al. (2022) also presents a Tarski operator defined in the worker-quasi-stable lattice that describes how, starting from any worker-quasi-stable matching, a decentralized sequence of offers in which unemployed

[1] Alkan (2002) calls this property "cardinal monotonicity".

workers are hired (causing new unemployments), produces a sequence of worker-quasi-stable matchings that converges to a stable matching. This lay-off chain dynamics is fundamental to compute the join with respect to firms' order. Given two stable matchings, we can define a new matching selecting, for each firm, the best subset of workers among those that this firm is matched to in either matching. Roth (1985) claims that such new matching is stable. However, as Li (2013) points out, this is not always the case. Only the worker-quasi-stability of such matching can be ensured (Bonifacio et al., 2022). Here, we show that the fixed point obtained by applying (iteratively) the Tarski operator to this worker-quasi-stable matching turns out to be the join (in the stable set) between the two original stable matchings. This provides a definite answer to the question left open in Li (2013).

Firm-quasi-stable matchings in many-to-one models generalize the one-to-one notion of "simple" matchings (Sotomayor, 1996) and have been studied, among others, by Cantala (2004, 2011). Under responsive preferences, Wu and Roth (2018) show that firm-quasi-stable matchings[2] form a lattice with respect to the (unanimous) order for the workers, and present a Tarski operator defined on this lattice, that can be interpreted as modeling vacancy chain dynamics, whose iterative application produces a sequence of firm-quasi-stable matchings that converges to a stable matching. In this paper, we extend such results to a model with substitutable choice functions. In our broader model, we show that firm-quasi-stable matchings still form a lattice and generalize Wu and Roth (2018)'s operator to compute the join between two stable matchings with respect to the order for the workers. To do this, given two stable matchings, we first define a new matching by allowing each worker to select the best firm matched with them through either matching. This new matching, in general, will not be stable (see Martínez et al., 2001). However, we show that it is firm-quasi-stable. Moreover, we also show that applying our Tarski operator for firm-quasi-stable matchings iteratively to this new matching produces a stable matching that turns out to be the join (in the stable set) between the two original stable matchings.

Notice that once we have found the join of two stable matchings with respect to the order for one side of the market, the duality between the order for the firms and the order for the workers under substitutability (Blair, 1988) allows us to conclude that such join is exactly the meet with respect to the order for the other side of the market. Hence, both lattice operations are computed for both sides of the market. Also notice that, at first glance, our way of producing the join between stable matchings mimics the workings of Echenique and Oviedo (2004). Given two stable matchings, Echenique and Oviedo (2004) first compute the natural candidate to be the join following the insights provided by the one-to-one case. This natural candidate, in general, is not even a matching (it is a pre-

[2] Actually, Wu and Roth (2018) study the slightly stronger notion of "worker-envy-free" matching.

matching) in their setting. Nevertheless, our approach is simpler and more intuitive: our natural candidates to be the join for each side of the market are quasi-stable matchings, and the Tarski operators used to "re-equilibrate" them have deep economic meaning: they can be regarded as vacancy chain dynamics and lay-off chain dynamics, as described in Wu and Roth (2018) and Bonifacio et al. (2022), in firm-quasi-stable matchings and worker-quasi-stable matchings, respectively.

To complement our results, we also compute the lattice operations in two different many-to-many models.

Firstly, and building on previous work of Manasero (2018), for a many-to-many market in which agents on one side have substitutable choice functions and agents on the other side have responsive preferences, we construct a related many-to-one market that possesses a stable set that is order-isomorphic to the (pairwise) stable set of the original many-to-many market.[3] Therefore, the join and meet operations can be carried out by means of this order-isomorphism straightforwardly. This "bridge" between these two-sided matchings is analogous to the one presented by Gale and Sotomayor (1985) between the many-to-one problem with responsive preferences and the classical one-to-one problem. We believe that this order-isomorphic construction is valuable on its own, and could be use to prove other results as well.

Secondly, for a many-to-many market in which both sides have substitutable choice functions, we (i) generalize the notions of quasi-stability, (ii) show that they form lattices with respect to Blair's partial orders, and (iii) present Tarski operators to compute the lattice operations. Remember that Blair (1988) provides a way to compute the lattice operations, without explicit mention of quasi-stability or any Tarski operator. Blair's insight is actually equivalent to our fixed point method applied to the lattices of many-to-many quasi-stable matchings. Thus, our approach provides a sound economic foundation for Blair's original construct.

The remainder of the paper is organized as follows. In Section 2, we present some preliminaries and the notion of substitutability. Section 3 is devoted to computing the lattice operations for the substitutable many-to-one model, by means of Tarski operators defined in the sets of worker-quasi-stable and firm-quasi-stable matchings. The extensions to many-to-many models are presented in Section 4. In Subsection 4.1, the isomorphism between stable matchings in the (substitutable-responsive) many-to-many model and the (substitutable) many-to-one model is analyzed. In Subsection 4.2, the extension to the substitutable many-to-many market is treated. Finally, some conclusions are gathered in Section 5.

[3]There are several notions of stability in many-to-many matching. The straightforward generalization of many-to-one stability is the so-called *pairwise stability*.

2 Preliminaries

A **two-sided matching market** consists of two disjoint sets, the set of firms F and the set of workers W. Depending on how many workers a firm is allowed to hire, matching markets are classified as **many-to-one** or **many-to-many**. Throughout the paper, we assume that each firm $f \in F$ has a choice function $C_f : 2^W \longrightarrow 2^W$ that satisfies **substitutability**: for $S' \subseteq S \subseteq W$, we have $C_f(S) \cap S' \subseteq C_f(S')$.[4] Under a regularity condition called **consistency**,[5] substitutability is equivalent to **path-independence**, which says that

$$C_f\left(S \cup S'\right) = C_f\left(C_f\left(S\right) \cup S'\right) \tag{1}$$

for each pair of subsets S and S' of W.[6] A profile of choice functions for all firms is denoted by $\boldsymbol{C_F}$. A matching associates firms and workers. Formally,

Definition 1 *A **matching** μ is a function from set $F \cup W$ into $2^{F \cup W}$ such that, for each $w \in W$ and each $f \in F$:*

(i) $\mu(w) \subseteq F$,

(ii) $\mu(f) \subseteq W$, *and*

(iii) $w \in \mu(f)$ *if and only if* $f \in \mu(w)$.

Agent $a \in F \cup W$ is **matched** if $\mu(a) \neq \emptyset$, otherwise a is **unmatched**. A matching μ is **blocked by firm f** if $\mu(f) \neq C_f\left(\mu(f)\right)$; that is, firm f wants to fire some workers in $\mu(f)$.

Within the set of matchings, we can define a partial order from the firms' standpoint as follows.[7] Let $\mu, \mu' \in \mathcal{M}$. We say that **$\mu$ is (Blair) preferred to μ' by the firms**, and write $\boldsymbol{\mu \succeq_F \mu'}$, if $C_f(\mu(f) \cup \mu'(f)) = \mu(f)$ for each $f \in F$.

In the following sections, we will specify three different matching markets, each one more general than its predecessor: (i) a many-to-one market where workers have strict preferences over firms, (ii) a many-to-many market where workers have responsive preferences over sets of firms, and (iii) a many-to-many market where workers have substitutable choice functions over sets of firms.

[4] Substitutability is equivalent to the following: for each $w \in W$ and each $S \subseteq W$ such that $w \in S$, $w \in C_f\left(S\right)$ implies that $w \in C_f\left(S' \cup \{w\}\right)$ for each $S' \subseteq S$.

[5] **Consistency:** $C_f(S') = C_f(S)$ whenever $C_f(S) \subseteq S' \subseteq S \subseteq W$.

[6] See Alkan (2002).

[7] Given a set $\mathcal{X}$, a *partial order* $\geq$ over $\mathcal{X}$ is a reflexive, antisymmetric, and transitive binary relation. If this is the case, sometimes we refer to the *partially ordered set* $(\mathcal{X}, \geq)$.

3 The substitutable many-to-one model

Besides each $f \in F$ having a choice function C_f over subsets of W, a **(substitutable) many-to-one market** is specified by endowing each worker $w \in W$ with a strict preference relation P_w over the individual firms and the prospect of being unmatched, denoted by $\emptyset$. The weak preference associated with P_w is denoted by R_w. A many-to-one market is denoted by $(\boldsymbol{C_F}, \boldsymbol{P_W})$, where C_F is the profile of choice functions for all firms and P_W is the profile of preferences for all workers. A matching μ in this model satisfies Definition 1 with the additional requirement that $|\mu(w)| \leq 1$ for each $w \in W$.[8] Let $\boldsymbol{\mathcal{M}}$ denote the set of all matchings for market (C_F, P_W).

A matching μ is **blocked by worker w** if $\emptyset P_w \mu(w)$; that is, worker w prefers being unemployed rather than working for firm $\mu(w)$. A matching is **individually rational** if it is not blocked by any individual agent. A matching μ is **blocked by a firm-worker pair (f, w)** if $w \in C_f(\mu(f) \cup \{w\})$ and $f P_w \mu(w)$; that is, if f and w are not matched through μ, firm f wants to hire w, and worker w prefers firm f rather than $\mu(w)$. A matching μ is **stable** if it is individually rational and it is not blocked by any firm-worker pair. Let $\boldsymbol{\mathcal{S}}$ denote the set of all stable matchings for market (C_F, P_W).

Within $\mathcal{M}$, we can also define a partial order from the workers' standpoint. Let $\mu, \mu' \in \mathcal{M}$. We say that **$\mu$ is (unanimously) preferred to μ' by the workers**, and write $\boldsymbol{\mu \geqslant_W \mu'}$, if $\mu(w) R_w \mu'(w)$ for each $w \in W$.

An important fact about the set of stable matchings is that it is a lattice with respect to both partial orders $\succeq_F$ and $\geqslant_W$.[9] Moreover, $\succeq_F$ and $\geqslant_W$ are dual orders within $\mathcal{S}$ (see Blair, 1988).[10] In order to compute the lattice operations for such lattices, we will consider two enlargements of the set of stable matchings: the set of worker-quasi-stable matchings and the set of firm-quasi-stable matchings. In each one, a Tarski operator is to be used to compute the join for the partial order that endowed it with a lattice structure.

3.1 A Tarski operator for worker-quasi-stable matchings

In many-to-one matching, worker-quasi-stability allows for blocking pairs as long as the worker involved in the blocking is unemployed. Formally,

[8]In this many-to-one model, instead of condition (iii) we will write: "$w \in \mu(f)$ if and only if $\mu(w) = f$".

[9]Given a partially ordered set $(\mathcal{X}, \geq)$, and two elements $x, y \in \mathcal{X}$, an element $z \in \mathcal{X}$ is an *upper bound of x and y* if $z \geq x$ and $z \geq y$. An element $x \veebar y \in \mathcal{X}$ is the *join* (or *supremum*) *of x and y* if and only if (i) $x \veebar y$ is an upper bound of x and y, and (ii) $t \geq x \veebar y$ for each upper bound t of x and y. The definitions of *lower bound* and *meet* (or *infimum*) *of x and y*, denoted $x \barwedge y$, are dual and we omit them. Furthermore, $(\mathcal{X}, \geq)$ is a *lattice* if $x \veebar y$ and $x \barwedge y$ exist for each pair $x, y \in \mathcal{X}$.

[10]Partial orders $\geq$ and $\geq'$ defined over $\mathcal{X}$ are *dual* whenever for each pair $x, y \in \mathcal{X}$ we have $x \geq y$ if and only if $y \geq' x$.

Definition 2 *Matching μ is **worker-quasi-stable** if it is individually rational and, whenever (f,w) blocks μ, we have $\mu(w)=\emptyset$.*

Let $\boldsymbol{\mathcal{Q}^W}$ denote the set of all worker-quasi-stable matchings for market (C_F,P_W). Notice that $\mathcal{Q}^W$ is non-empty, since the empty matching, in which every agent is unmatched, belongs to $\mathcal{Q}^W$.

Given $\mu,\mu'\in\mathcal{Q}^W$, define matching $\boldsymbol{\lambda_{\mu,\mu'}}$ as follows:

(i) for each $f\in F$, $\lambda_{\mu,\mu'}(f)=C_f\left(\mu(f)\cup\mu'(f)\right)$,

(ii) for each $w\in W$, $\lambda_{\mu,\mu'}(w)=\{f\in F: w\in\lambda_{\mu,\mu'}(f)\}$.

In Bonifacio et al. (2022) it is shown that matching $\lambda_{\mu,\mu'}$ is well-defined, i.e., it is a worker-quasi-stable matching. Moreover, it is the join between μ and μ' with respect to $\succeq_F$ within the worker-quasi-stable set. Then, given that the empty matching is the minimum of the worker-quasi-stable set with respect to $\succeq_F$, this set is a join-semilattice[11] with a minimum, and thus, a lattice. We summarized all these results in the following proposition.

Proposition 1 *(Facts about worker-quasi-stable matchings)*

(i) Let $\mu,\mu'\in\mathcal{Q}^W$. Then, $\lambda_{\mu,\mu'}\in\mathcal{Q}^W$. Furthermore, it is the join of μ and μ' with respect to $\succeq_F$ in $\mathcal{Q}^W$.

(ii) $(\mathcal{Q}^W,\succeq_F)$ is a lattice.

Proof. (i) follows from Lemmata 1 and 2 in Bonifacio et al. (2022), and (ii) from Theorem 1 in Bonifacio et al. (2022). □

Next, we present the Tarski operator within worker-quasi-stable matchings first studied in Bonifacio et al. (2022). Given $\mu\in\mathcal{M}$ and $w\in W$, let

$$F_w^\mu=\{f\in F: w\in C_f(\mu(f)\cup\{w\})\}.$$

The set F_w^μ comprises w's partner at μ and all firms that want to block μ with worker w. Similarly, given $\mu\in\mathcal{M}$ and $w\in W$, let

$$W_f^\mu=\{w\in W: fR_w\mu(w)\}.$$

The set W_f^μ comprises all workers that are matched with f in μ and all workers that want to block μ with firm f. Given $\mu\in\mathcal{Q}^W$ and $f\in F$, let

$$B_f^\mu=\left\{w\in W_f^\mu: f=\max_{P_w} F_w^\mu\right\}\cup\{\mu(f)\}.$$

[11] A partially order set $(\mathcal{X},\geq)$ is a *join-semilattice* if $x\vee y$ exists for each pair $x,y\in\mathcal{X}$.

The set B_f^μ gathers all workers that are matched with f in μ and all workers that consider f as their best blocking partner available. The Tarski operator for worker-quasi-stable matchings is defined, for each matching μ, by making each firm f to choose among the elements of B_f^μ.

Definition 3 *(Many-to-one Tarski operator for worker-quasi-stable matchings) For each $\mu \in \mathcal{Q}^W$, operator $\mathcal{T}^F : \mathcal{Q}^W \longrightarrow \mathcal{Q}^W$ assigns*

(i) for each $f \in F$, $\mathcal{T}^F[\mu](f) = C_f(B_f^\mu)$

(ii) for each $w \in W$, $\mathcal{T}^F[\mu](w) = \begin{cases} f & \text{if } w \in \mathcal{T}^F[\mu](f) \\ \emptyset & \text{otherwise} \end{cases}$

This operator is (i) well-defined and Pareto-improving for the firms, (ii) isotone, and (iii) has as its fixed points the set of stable matchings. We formalize these results in the following proposition.

Proposition 2 *For operator $\mathcal{T}^F : \mathcal{Q}^W \longrightarrow \mathcal{Q}^W$ we have:*

(i) For each $\mu \in \mathcal{Q}^W$, $\mathcal{T}^F[\mu] \in \mathcal{Q}^W$ and $\mathcal{T}^F[\mu] \succeq_F \mu$.

(ii) If $\mu, \mu' \in \mathcal{Q}^W$ and $\mu \succeq_F \mu'$, then $\mathcal{T}^F[\mu] \succeq_F \mathcal{T}^F[\mu']$.

(iii) $\mathcal{T}^F[\mu] = \mu$ if and only if $\mu \in \mathcal{S}$.

Proof. (i) and (iii) come from Theorem 2 (ii) and (iii) in Bonifacio et al. (2022), respectively; whereas (ii) from Lemma 4 in Bonifacio et al. (2022). □

3.2 A Tarski operator for firm-quasi-stable matchings

In this subsection we define an analogous Tarski operator for firm-quasi-stable matchings. Such matchings allow for blocking pairs as long as they do not compromise previous relations from the firms' perspective. Formally,

Definition 4 *Matching μ is **firm-quasi-stable** if it is individually rational and, for each $f \in F$ and each $S \subseteq W_f^\mu$,[12] we have*

$$\mu(f) \subseteq C_f(\mu(f) \cup S).$$

[12] Remember that $W_f^\mu = \{w \in W : f R_w \mu(w)\}$.

Denote by $\boldsymbol{\mathcal{Q}^F}$ the set of all firm-quasi-stable matchings for market (C_F, P_W). Notice that $\mathcal{Q}^F$ is non-empty, since the empty matching, in which every agent is unmatched, belongs to $\mathcal{Q}^F$.

First, we show that $\mathcal{Q}^F$ is a lattice. Given $\mu, \mu' \in \mathcal{Q}^F$, define matching $\boldsymbol{\gamma_{\mu,\mu'}}$ as follows:

(i) for each $w \in W$, $\gamma_{\mu,\mu'}(w) = \begin{cases} \mu(w) & \text{if } \mu(w) P_w \mu'(w) \\ \mu'(w) & \text{otherwise} \end{cases}$

(ii) for each $f \in F$, $\gamma_{\mu,\mu'}(f) = \{w \in W : \gamma_{\mu,\mu'}(w) = f\}$

Next, we prove results for matching $\gamma_{\mu,\mu'}$ similar to those presented in Subsection 3.1 for matching $\lambda_{\mu,\mu'}$. This allows us to show the lattice structure of $\mathcal{Q}^F$, extending the result of Wu and Roth (2018) to a model with substitutable choice functions.

Theorem 1 *(Facts about firm-quasi-stable matchings)*

(i) Let $\mu, \mu' \in \mathcal{Q}^F$. Then, $\gamma_{\mu,\mu'} \in \mathcal{Q}^F$. Furthermore, it is the join of μ and μ' with respect to $\geqslant_W$ in $\mathcal{Q}^F$.

(ii) $(\mathcal{Q}^F, \geqslant_W)$ is a lattice.

Proof. See Subsection A.5 in the Appendix. □

Following the idea used to construct the Tarski operator for worker-quasi-stable matchings, we now present a Tarski operator for firm-quasi-stable matchings.

Given $\mu \in \mathcal{Q}^F$, for each $w \in W$, let

$$B_w^\mu = \{f \in F : w \in C_f(W_f^\mu)\} \cup \{\emptyset\}.$$

Set B_w^μ consists of the firm matched with w at μ (if any),[13] all firms that w considers to block with and are willing to do so, and the prospect of being unmatched. Then, the Tarski operator for firm-quasi-stable matchings is defined, for each matching μ, by making each worker w to select the best element of B_w^μ.

Definition 5 *(Many-to-one Tarski operator for firm-quasi-stable matchings) For each $\mu \in \mathcal{Q}^F$, operator $\mathcal{T}^W : \mathcal{Q}^F \longrightarrow \mathcal{Q}^F$ assigns*

(i) for each $w \in W$, $\mathcal{T}^W[\mu](w) = \max_{P_w} B_w^\mu$, and

(ii) for each $f \in F$, $\mathcal{T}^W[\mu](f) = \{w \in W : \mathcal{T}^W[\mu](w) = f\}$.

[13]Let $w \in W$. If $\mu(w) = \emptyset$, clearly $\mu(w) \in B_w^\mu$. Otherwise, if there is $f \in F$ such that $f = \mu(w)$ and $w \notin C_f(W_f^\mu)$, substitutability implies $w \notin C_f\left(\mu(f) \cup \left(W_f^\mu \setminus \mu(f)\right)\right)$, contradicting that $\mu \in \mathcal{Q}^F$ since $(W_f^\mu \setminus \mu(f)) \subseteq W_f^\mu$. Therefore, $\mu(w) = f \in B_w^\mu$.

Remark 1 *It is easy to see that when firms have responsive preferences instead of substitutable choice functions, operator $\mathcal{T}^W$ specializes in operator T defined by Wu and Roth (2018).*

We obtain analogous results for operator $\mathcal{T}^W$ to those presented in Proposition 2 for operator $\mathcal{T}^F$.

Theorem 2 *For operator $\mathcal{T}^W : \mathcal{Q}^F \longrightarrow \mathcal{Q}^F$ we have:*

(i) For each $\mu \in \mathcal{Q}^F$, $\mathcal{T}^W[\mu] \in \mathcal{Q}^F$ and $\mathcal{T}^W[\mu] \geqslant_W \mu$.

(ii) If $\mu, \mu' \in \mathcal{Q}^F$ and $\mu \geqslant_W \mu'$, then $\mathcal{T}^W[\mu] \geqslant_W \mathcal{T}^W[\mu']$.

(iii) $\mathcal{T}^W[\mu] = \mu$ if and only if $\mu \in \mathcal{S}$.

Proof. See Subsection A.6 in the Appendix. □

Remark 2 *Since $\mathcal{T}^W$ is isotone by Theorem 2 (ii), the set of its fixed points is a non-empty lattice with respect to $\geqslant_W$ according to Tarski's Fixed Point Theorem. Since this set is exactly $\mathcal{S}$ by Theorem 2 (iii), as a byproduct, we obtain an alternative proof that $\mathcal{S}$ is non-empty and has a lattice structure.*

3.3 Lattice operations

Given two stable matchings μ and μ', the natural candidates to be the join between them with respect to partial orders $\succeq_F$ and $\geqslant_W$ in the stable set are $\lambda_{\mu,\mu'}$ and $\gamma_{\mu,\mu'}$, respectively. However, in general, $\lambda_{\mu,\mu'}$ and $\gamma_{\mu,\mu'}$ are only *quasi-stable* matchings (see Proposition 1 (i), Theorem 1 (i), and Example 1). The following result shows that $\lambda_{\mu,\mu'}$ can be re-equilibrated by applying iteratively operator $\mathcal{T}^F$ to obtain the join between μ and μ' with respect to $\succeq_F$ within $\mathcal{S}$. Similarly, $\gamma_{\mu,\mu'}$ can be re-equilibrated by applying iteratively operator $\mathcal{T}^W$ to obtain the join between μ and μ' with respect to $\geqslant_W$ within $\mathcal{S}$. Given $\mu \in \mathcal{Q}^F$, denote by $\boldsymbol{\mathcal{F}^W(\mu)}$ to the fixed point of $\mathcal{T}^W$ starting from μ. Similarly, given $\mu \in \mathcal{Q}^W$, denote by $\boldsymbol{\mathcal{F}^F(\mu)}$ to the fixed point of $\mathcal{T}^F$ starting from μ.

Theorem 3 *Let $\mu, \mu' \in \mathcal{S}$. Then,*

(i) $\mu \curlyvee_F \mu' = \mathcal{F}^F(\lambda_{\mu,\mu'})$, and

(ii) $\mu \curlywedge_F \mu' = \mathcal{F}^W(\gamma_{\mu,\mu'})$.

Proof. See Subsection A.7 in the Appendix. □

The duality between $\succeq_F$ and $\geqslant_W$ in the stable set (Blair, 1988) allows us to present the following corollary.

Corollary 1 *Let $\mu, \mu' \in \mathcal{S}$. Then,*

(i) $\mu \veebar_W \mu' = \mathcal{F}^W(\gamma_{\mu,\mu'})$, *and*

(ii) $\mu \triangle_W \mu' = \mathcal{F}^F(\lambda_{\mu,\mu'})$.

The following example, taken from Li (2013) illustrates our results. It was originally presented by Li (2013) to show, among other things, that $\lambda_{\mu,\mu'}$ is not always stable, as mistakenly claimed by Roth (1985).

Example 1 *Let (C_F, P_W) be a market with $F = \{f_1, f_2, f_3, f_4, f_5\}$ and $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$. Preferences of the agents are given in Table 1. Firms' choice functions are derived from these preferences in the standard way.*[14]

P_{f_1}	$\boxed{\overline{w_4}^{\star}}$	$\underline{w_1}$	$w_5^{\dagger}$	$\cdots$	$\enclose{roundedbox}{\emptyset}$	$\cdots$	P_{w_1}	$\enclose{circle}{\overline{f_2}^{\dagger}}$	$\underline{f_1}$	$f_3^{\star}$	$\cdots$	$\boxed{\emptyset}$	$\cdots$
P_{f_2}	$\boxed{\underline{w_2}^{\star}}$	$\enclose{roundedbox}{\overline{w_1, w_3}^{\dagger}}$	$\cdots$				P_{w_2}	$\enclose{circle}{\overline{f_3}^{\dagger}}$	$\boxed{\underline{f_2}^{\star}}$	$\cdots$			
P_{f_3}	$w_1^{\star}$	$\boxed{\underline{w_3}}$	$\enclose{roundedbox}{\overline{w_2}^{\dagger}}$	$\cdots$			P_{w_3}	$\enclose{circle}{\overline{f_2}^{\dagger}}$	$\boxed{\underline{f_3}}$	$\cdots$	$\emptyset^{\star}$	$\cdots$	
P_{f_4}	$\boxed{\overline{w_5}^{\star}}$	$\enclose{roundedbox}{\underline{w_4, w_6}^{\dagger}}$	$\cdots$				P_{w_4}	$\enclose{circle}{\underline{f_4}^{\dagger}}$	$\boxed{\overline{f_1}^{\star}}$	$\cdots$			
P_{f_5}	$\boxed{\overline{w_6}^{\star}}$	$\enclose{roundedbox}{\underline{w_5}}$	$\cdots$	$\emptyset^{\dagger}$	$\cdots$		P_{w_5}	$f_1^{\dagger}$	$\enclose{circle}{\underline{f_5}}$	$\boxed{\overline{f_4}^{\star}}$	$\cdots$		
							P_{w_6}	$\enclose{circle}{\underline{f_4}^{\dagger}}$	$\boxed{\overline{f_5}^{\star}}$	$\cdots$			

Table 1: Preference profile for Example 1.

Let

$$\underline{\mu} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 \\ \{w_1\} & \{w_2\} & \{w_3\} & \{w_4, w_6\} & \{w_5\} \end{pmatrix}$$

and

$$\overline{\mu} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 \\ \{w_4\} & \{w_1, w_3\} & \{w_2\} & \{w_5\} & \{w_6\} \end{pmatrix}.$$

Then,

$$\boxed{\mu} = \lambda_{\underline{\mu},\overline{\mu}} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & \emptyset \\ \{w_4\} & \{w_2\} & \{w_3\} & \{w_5\} & \{w_6\} & \{w_1\} \end{pmatrix}$$

[14]For example, $C_{f_1}(W) = \{w_4\}$ and $C_{f_4}(\{w_1, w_2, w_3, w_4, w_6\}) = \{w_4, w_6\}$.

and

$$\textcircled{\mu} = \gamma_{\underline{\mu},\overline{\mu}} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 \\ \emptyset & \{w_1, w_3\} & \{w_2\} & \{w_4, w_6\} & \{w_5\} \end{pmatrix}.$$

Notice that both $\underline{\mu}$ and $\overline{\mu}$ are stable matchings. We know, by Bonifacio et al. (2022) , that $\boxed{\mu}$ is the join of $\underline{\mu}$ and $\overline{\mu}$ with respect to $\succeq_F$ in the worker-quasi-stable set. However, as Li (2013) points out, $\boxed{\mu}$ is not stable, since (f_3, w_1) blocks it. Similarly, $\textcircled{\mu}$ is the join of $\underline{\mu}$ and $\overline{\mu}$ with respect to $\geqslant_W$ in the firm-quasi-stable set by Theorem 1. However, $\textcircled{\mu}$ is not stable either, since (f_1, w_5) blocks it. Applying the respective Tarski operator once, we get

$$\mu^\star = \mathcal{T}^F\left[\boxed{\mu}\right] = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & \emptyset \\ \{w_4\} & \{w_2\} & \{w_1\} & \{w_5\} & \{w_6\} & \{w_3\} \end{pmatrix}$$

and

$$\mu^\dagger = \mathcal{T}^W\left[\textcircled{\mu}\right] = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 \\ \{w_5\} & \{w_1, w_3\} & \{w_2\} & \{w_4, w_6\} & \emptyset \end{pmatrix}.$$

It is readily seen that $\mu^\star$ is the firm-optimal matching, so it is stable. By Theorem 3 (i), it is $\mu \curlyvee_F \mu'$. Similarly, $\mu^\dagger$ is the worker-optimal matching, so it is stable and by Theorem 3 (ii), it is $\mu \curlywedge_F \mu'$. $\diamondsuit$

4 Extensions to many-to-many models

In this section, we extend our results to many-to-many matching models. The first extension, in Subsection 4.1, deals with a substitutable-responsive many-to-many model. We do not explicitly compute the lattice operations, but instead construct an order-isomorphism between a many-to-many market and a related many-to-one market by which the lattice operations are "lifted". The second extension, in Subsection 4.2, generalizes all our results of Section 3 to the substitutable (on both sides) many-to-many setting, providing an economic rationale to the previous work of Blair (1988).

4.1 The substitutable-responsive many-to-many model

Besides each $f \in F$ having a choice function C_f over subsets of W, a **sustitutable-responsive many-to-many market** market is specified by endowing each worker $w \in W$ with a quota q_w that bounds the number of firms this worker can be matched to and defines a **responsive** preference $P_w^{q_w}$ over 2^F that satisfies:

(i) for each $T \subseteq F$ such that $|T| > q_w$, $\emptyset \; P_w^{q_w} \; T$.

(ii) for each $T \subseteq F$ such that $|T| \leq q_w$, each $f \in F \setminus T$, and each $f' \in T \cup \{\emptyset\}$,

$$(T \setminus \{f'\}) \cup \{f\} \; P_w^{q_w} \; T \text{ if and only if } \{f\} \; P_w^{q_w} \; \{f'\}.$$

This implies that adding "good" firms to a set leads to a better set, whereas adding "bad" firms to a set leads to a worse set. In addition, for any two subsets that differ in only one firm, the firm prefers the subset containing the most preferred worker.

A substitutable-responsive many-to-many market is denoted by $(\boldsymbol{C_F}, \boldsymbol{P_W^q})$, where C_F is the profile of choice functions for all firms and P_W^q is the profile of preferences for all workers. A matching μ in this model satisfies Definition 1 with the aditional requirement that $|\mu(w)| \leq q_w$ for each $w \in W$. Let $\boldsymbol{\mathcal{M}^q}$ denote the set of all matchings of market (C_F, P_W^q).

A matching $\mu \in \mathcal{M}^q$ is **blocked by a firm-worker pair** $(\boldsymbol{f}, \boldsymbol{w})$ if $f \notin \mu(w)$, $w \in C_f(\mu(f) \cup \{w\})$, and

(i) $|\mu(w)| = q_w$ and there is $f' \in \mu(w)$ such that $\{f\}\ P_w^{q_w}\ \{f'\}$; or

(ii) $|\mu(w)| < q_w$ and $\{f\}\ P_w^{q_w}\ \emptyset$.

A matching is **(pairwise) stable** if it is individually rational[15] and it is not blocked by any firm-worker pair. Let $\boldsymbol{\mathcal{S}^q}$ denote the set of all stable matchings of market (C_F, P_W^q).

Analogously to the many-to-one model, we can define the two partial orders $\succeq_F^q$ and $\geqslant_W^q$ within $\mathcal{M}^q$. We already know that partially-order sets $(\mathcal{S}^q, \succeq_F^q)$ and $(\mathcal{S}^q, \geqslant_W^q)$ are endow with a (dual) lattice structure (Blair, 1988). Remember that our goal is to obtain their lattice operations. To do this, we are going to define a related many-to-one market. The idea is simple. In the new market, each worker is replicated as many times as her quota, and workers' preferences are the linear preferences over the set of firms that underlie their responsive preferences in the many-to-many market. In turn, firms' choice functions are adapted so that sets with more than one replica of a worker are unacceptable, and sets with at most one replica of each worker are chosen as in the original choice function.

Before formally presenting the new market, we need some definitions. Given set of workers $W = \{w_1, \ldots, w_m\}$ and quotas $q = (q_{w_1}, \ldots, q_{w_m})$, the ***q*-replica of *W*** is the set

$$W^q = \{w_1^1, \ldots, w_1^{q_{w_1}}, \ldots, w_m^1, \ldots, w_m^{q_{w_m}}\}.$$

Generic elements of W^q are denoted w^t. The **natural projection** is the function $\pi : W^q \longrightarrow W$ such that $\pi(w_i^t) = w_i$ for each $i \in \{1, \ldots, m\}$ and each $t \in \{1, \ldots, q_{w_i}\}$.

Given $f \in F$, choice function $C_f : 2^W \longrightarrow 2^W$, and workers' quotas $q = (q_{w_1}, \ldots, q_{w_m})$, the ***q*-extension of $\boldsymbol{C_f}$** is the choice function $C_f^q : 2^{W^q} \longrightarrow 2^{W^q}$ such that:

(i) for each $S \subseteq W^q$,

$$C_f^q(S) = \left\{w^t \in S\ :\ t \leq t' \text{ for each } w^{t'} \in S \text{ with } \pi(w^{t'}) = \pi(w^t)\right\}, \text{ and}$$

[15]This definition is equal to the many-to-one version.

(ii) for each $S \subseteq W^q$, $\pi(C_f^q(S)) = C_f(\pi(S))$.

Within any subset of the q-replica of W, the q-extension of C_f: (i) selects the replicas with the lowest indices, and (ii) chooses among replicas of different workers in the same way as C_f does. An important fact is that substitutability is preserved under q-extensions.[16]

Given a many-to-many market, its related many-to-one market, then, consists of the q-replica of the set of workers, each endowed with the linear order of the firms that underlies her responsive preference, and the set of firms endowed with the q-extensions of their original choice functions. Formally,

Definition 6 *Given a substitutable-responsive many-to-many market (C_F, P_W^q), its* ***(many-to-one) related market*** *is the market where:*

(i) each firm $f \in F$ is endowed with C_f^q, the q-extension of C_f, and

(ii) the set of workers is the q-replica of W, and each $w^t \in W^q$ is endowed with P_{w^t}, the restriction of $P_{\pi(w^t)}^{q_{\pi(w^t)}}$ to the set $\{T \subseteq F : |T| \leq 1\}$.

Denote this market by $\boldsymbol{(C_F^q, P_{W^q})}$. Since this is a substitutable many-to-one market, we will denote the set of all its matchings by $\mathcal{M}$, and the set of all its stable matchings by $\mathcal{S}$. Furthermore, the partial orders for the firms and the workers defined on $\mathcal{M}$ are denoted by $\succeq_F$ and $\geqslant_W$, respectively.

There is a natural way to reconstruct a many-to-many matching from a matching in its related many-to-one market: for each worker in the many-to-many market, just bundle together all firms that are matched with her many-to-one replicas. Formally,

Definition 7 *Given many-to-many market (C_F, P_W^q) and its related many-to-one market (C_F^q, P_{W^q}), the* ***natural morphism***

$$\Phi : \mathcal{M} \longrightarrow \mathcal{M}^q$$

assigns, for each $\mu \in \mathcal{M}$, matching $\Phi[\mu] \in \mathcal{M}^q$ such that:

(i) for each $w \in W$, $\Phi[\mu](w) = \bigcup_{w^t \in \pi^{-1}(\{w\})} \mu(w^t)$.

(ii) for each $f \in F$, $\Phi[\mu](f) = \{w \in W : f \in \Phi[\mu](w)\}$.

[16] To see this, assume C_f is substitutable and let $S' \subseteq S \subseteq W^q$. Then,

$$\pi(C_f^q(S) \cap S') \subseteq \pi(C_f^q(S)) \cap \pi(S') = C_f(\pi(S)) \cap \pi(S') \subseteq C_f(\pi(S')) = \pi(C_f^q(S')),$$

which implies $\pi(C_f^q(S) \cap S') \subseteq \pi(C_f^q(S'))$ and thus $C_f^q(S) \cap S' \subseteq C_f^q(S')$. Hence, C_f^q is substitutable.

The natural morphism behaves as expected: (i) it is surjective, meaning that we can always describe a matching in the many-to-many market as a matching in the many-to-one market;[17] (ii) preserves stability; and, additionally, (iii) establishes a one-to-one correspondence between stable matchings in both markets. Moreover, (iv) it is an order-isomorphism between stable matchings.

Theorem 4 *For natural morphism $\Phi : \mathcal{M} \longrightarrow \mathcal{M}^q$ we have:*

(i) Φ *is onto.*

(ii) $\Phi(\mathcal{S}) = \mathcal{S}^q$.

(iii) $\Phi|_{\mathcal{S}}$ *is bijective.*

(iv) Let $\mu, \mu' \in \mathcal{S}$*. Then,* $\mu \succeq_F \mu'$ *if and only if* $\Phi[\mu] \succeq_F^q \Phi[\mu']$.

Proof. See Subsection A.1 in the Appendix. □

The order-preserving nature of this morphism allows us to translate to the substitutable-responsive many-to-many market the computation of the lattice operations from the related many-to-one market.

Theorem 5 *Let* (C_F, P_W^q) *be a substitutable-responsive many-to-many market and let* (C_F^q, P_{W^q}) *be its related many-to-one market. Let* $\mathcal{S}^q$ *and* $\mathcal{S}$ *be their stable sets, respectively. Then, for each pair* $\mu, \mu' \in \mathcal{S}^q$,

(i) $\mu \curlyvee_F^q \mu' = \Phi\left[\Phi^{-1}[\mu] \curlyvee_F \Phi^{-1}[\mu']\right]$, *and*

(ii) $\mu \curlywedge_F^q \mu' = \Phi\left[\Phi^{-1}[\mu] \curlywedge_F \Phi^{-1}[\mu']\right]$,

where Φ *is the natural morphism, and* $\curlyvee_F$ *and* $\curlywedge_F$ *are the operations defined on the many-to-one lattice* $(\mathcal{S}, \succeq_F)$.

Proof. It follows from Theorem 4. □

Remark 3 *By duality, for each pair* $\mu, \mu' \in \mathcal{S}^q$*, lattice operations* $\mu \veebar_W^q \mu'$ *and* $\mu \triangle_W^q \mu'$ *with respect to* $\geqslant_W^q$ *are obtained in a similar way.*

[17] Note, however, that Φ need not be injective. To see this, consider the many-to-many market $F = \{f_1, f_2\}$, $W = \{w\}$, $C_{f_i}(\{w\}) = \{w\}$ for $i \in \{1,2\}$, $q_w = 2$, and $P_w^{q_w} : \{f_1, f_2\}, \{f_1\}, \{f_2\}, \emptyset$. Then, in the related many-to-one market we have $W^q = \{w^1, w^2\}$ and $f_1 P_{w^i} f_2 P_{w^i} \emptyset$ for each $i \in \{1,2\}$. Therefore,

$$\Phi \begin{bmatrix} f_1 & f_2 \\ w^1 & w^2 \end{bmatrix} = \Phi \begin{bmatrix} f_1 & f_2 \\ w^2 & w^1 \end{bmatrix} = \begin{bmatrix} \{f_1, f_2\} \\ w \end{bmatrix}.$$

4.2 The substitutable many-to-many model

Besides each $f \in F$ having a choice function C_f over subsets of W, a **sustitutable many-to-many market** market is specified by endowing each worker $w \in W$ with a choice function C_w over subsets of F, making this model completely symmetric in F and W. A substitutable many-to-many market is denoted by $(\boldsymbol{C_F, C_W})$, where C_F is the profile of choice functions for all firms and C_W is the profile of choice functions for all workers. Let $\widetilde{\boldsymbol{\mathcal{M}}}$ denote the set of all matchings for market (C_F, C_W).

We can define a partial order from the workers' standpoint as follows. Let $\mu, \mu' \in \widetilde{\mathcal{M}}$. We say that $\boldsymbol{\mu}$ **is (Blair) preferred to** $\boldsymbol{\mu'}$ by the workers, and write $\boldsymbol{\mu \succeq_W \mu'}$ if $C_w(\mu(w) \cup \mu'(w)) = \mu(w)$ for each $w \in W$. Remember that $\succeq_F$ was already defined, in an analoguos way, in Section 2.

A matching $\mu \in \widetilde{\mathcal{M}}$ is **blocked by worker** w if $\mu(w) \neq C_w(\mu(w))$. A matching is **individually rational** if it is not blocked by any firm or worker. A matching $\mu \in \widetilde{\mathcal{M}}$ is **blocked by a firm-worker pair** $(\boldsymbol{f, w})$ if $f \notin \mu(w)$, $w \in C_f(\mu(f) \cup \{w\})$ and $f \in C_w(\mu(w) \cup \{f\})$. A matching is **stable** if it is individually rational and it is not blocked by any firm-worker pair. Let $\widetilde{\boldsymbol{\mathcal{S}}}$ denote the set of all stable matchings of market (C_F, C_w). It is a well-known fact that lattices $(\widetilde{\mathcal{S}}, \succeq_F)$ and $(\widetilde{\mathcal{S}}, \succeq_W)$ are dual (Blair, 1988).

Since market (C_F, C_W) is symmetric in F and W, in what follows we construct a many-to-many Tarski operator for worker-quasi-stable matchings. Of course, the dual construction for firm-quasi-stable matchings is straightforward.

The notion of worker-quasi-stability is easily adapted to the many-to-many setting. Blocking pairs are allowed as long as they do not compromise the already existing relations for the workers in the matching. For $\mu \in \widetilde{\mathcal{M}}$ and $w \in W$, let $\widetilde{F}^{\mu}_w = \{f \in F : w \in C_f(\mu(f) \cup \{w\})\}$.

Definition 8 *Matching μ is **worker-quasi-stable** if it is individually rational and, for each $w \in W$ and each $T \subseteq \widetilde{F}^{\mu}_w$, we have*

$$\mu(w) \subseteq C_w(\mu(w) \cup T).$$

Denote by $\widetilde{\boldsymbol{\mathcal{Q}}}^{\boldsymbol{W}}$ the set of all worker-quasi-stable matchings for market (C_F, C_W). Notice that, since the empty matching belongs to this set, $\widetilde{\mathcal{Q}}^W \neq \emptyset$. In Appendix A.2 (see Lemma 2 (i)) we show that if $\mu \in \widetilde{\mathcal{Q}}^W$ then $\mu(w) \subseteq \widetilde{F}^{\mu}_w$ for each $w \in W$.

Remark 4 *The dual definition of* firm-quasi-stability *can be obtained by interchanging the roles of sets W and F in Definition 8. Notice also that Definition 8 specializes to Definition 2 in a many-to-one model where workers have linear preferences, since in this case, for each $w \in W$ and each $T \subseteq F$, $C_w(T) = \max_{P_w} T$. So, in order to have $\mu(w) \subseteq C_w(\mu(w) \cup \{f\})$ when (f, w) blocks μ, necessarily $\mu(w) = \emptyset$.*

Next, we generalize matching $\lambda_{\mu,\mu'}$. Given $\mu, \mu' \in \widetilde{\mathcal{Q}}^W$, define matching $\widetilde{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\boldsymbol{\mu'}}$ as follows:

(i) for each $f \in F$, $\widetilde{\lambda}_{\mu,\mu'}(f) = C_f\left(\mu(f) \cup \mu'(f)\right)$,

(ii) for each $w \in W$, $\widetilde{\lambda}_{\mu,\mu'}(w) = \{f \in F : w \in \widetilde{\lambda}_{\mu,\mu'}(f)\}$.

The straightforward generalization of Proposition 1 is presented next.

Theorem 6 *(Facts about worker-quasi-stable matchings)*

(i) Let $\mu, \mu' \in \widetilde{\mathcal{Q}}^W$. Then, $\widetilde{\lambda}_{\mu,\mu'} \in \widetilde{\mathcal{Q}}^W$. Furthermore, $\widetilde{\lambda}_{\mu,\mu'}$ is the join of μ and μ' with respect to $\succeq_F$ in $\widetilde{\mathcal{Q}}^W$.

(ii) $(\widetilde{\mathcal{Q}}^W, \succeq_F)$ is a lattice.

Proof. See Subsection A.2 in the Appendix. □

To define the Tarski operator for worker-quasi-stable matchings in the many-to-many setting, first we need to generalize the definition of set B_f^μ. Given $\mu \in \widetilde{\mathcal{Q}}^W$ and $f \in F$, let

$$\widetilde{B}_f^\mu = \left\{ w \in W : f \in C_w(\widetilde{F}_w^\mu) \right\} \cup \{\mu(f)\}. \tag{2}$$

We are now in a position to define the many-to-many version of the operator for matchings in $\widetilde{\mathcal{Q}}^W$.

Definition 9 *(Many-to-many Tarski operator for worker-quasi-stable matchings) For each $\mu \in \widetilde{\mathcal{Q}}^W$, operator $\widetilde{\mathcal{T}}^F : \widetilde{\mathcal{Q}}^W \longrightarrow \widetilde{\mathcal{Q}}^W$ assigns*

(i) for each $f \in F$, $\widetilde{\mathcal{T}}^F[\mu](f) = C_f(\widetilde{B}_f^\mu)$

(ii) for each $w \in W$, $\widetilde{\mathcal{T}}^F[\mu](w) = \{f \in F : w \in \widetilde{\mathcal{T}}^F[\mu](f)\}$.

Remark 5 *The dual definition of* many-to-many Tarski operator for firm-quasi-stable matchings *can be obtained by interchanging the roles of sets W and F in Definition 9. Note also that operator $\widetilde{\mathcal{T}}^F$ specializes in operator $\mathcal{T}^F$ (Definition 3) in the many-to-one model.*

Operator $\widetilde{\mathcal{T}}^F$ is (i) well-defined and Pareto-improving for the firms, (ii) isotone, and (iii) has as its set of fixed points to the stable set.

Theorem 7 *For operator $\widetilde{\mathcal{T}}^F : \widetilde{\mathcal{Q}}^W \longrightarrow \widetilde{\mathcal{Q}}^W$ we have:*

(i) For each $\mu \in \widetilde{\mathcal{Q}}^W$, $\widetilde{\mathcal{T}}^F[\mu] \in \widetilde{\mathcal{Q}}^W$ and $\widetilde{\mathcal{T}}^F[\mu] \succeq_F \mu$.

(ii) If $\mu, \mu' \in \widetilde{\mathcal{Q}}^W$ and $\mu \succeq_F \mu'$, then $\widetilde{\mathcal{T}}^F[\mu] \succeq_F \widetilde{\mathcal{T}}^F[\mu']$.

(iii) $\widetilde{\mathcal{T}}^F[\mu] = \mu$ *if and only if* $\mu \in \widetilde{\mathcal{S}}$.

Proof. See Subsection A.3 in the Appendix. □

Given $\mu \in \widetilde{\mathcal{Q}}^W$, denote by $\widetilde{\boldsymbol{\mathcal{F}}}^{\boldsymbol{F}}(\boldsymbol{\mu})$ to the fixed point of $\widetilde{\mathcal{T}}^F$ starting from μ. Similarly, given $\mu \in \widetilde{\mathcal{Q}}^F$, denote by $\widetilde{\boldsymbol{\mathcal{F}}}^{\boldsymbol{W}}(\boldsymbol{\mu})$ to the fixed point of $\widetilde{\mathcal{T}}^W$ starting from μ. Our main result is the following.

Theorem 8 *Let* $\mu, \mu' \in \widetilde{\mathcal{S}}$. *Then,*

(i) $\mu \curlyvee_F \mu' = \widetilde{\mathcal{F}}^F(\widetilde{\lambda}_{\mu,\mu'})$, *and*

(ii) $\mu \curlywedge_F \mu' = \widetilde{\mathcal{F}}^W(\widetilde{\gamma}_{\mu,\mu'})$.

Proof. See Subsection A.4 in the Appendix. □

By the duality between $\succeq_F$ and $\succeq_W$ in $\widetilde{\mathcal{S}}$, the following corollary also holds.

Corollary 2 *Let* $\mu, \mu' \in \widetilde{\mathcal{S}}$. *Then,*

(i) $\mu \curlyvee_W \mu' = \widetilde{\mathcal{F}}^W(\widetilde{\gamma}_{\mu,\mu'})$, *and*

(ii) $\mu \curlywedge_W \mu' = \widetilde{\mathcal{F}}^F(\widetilde{\lambda}_{\mu,\mu'})$.

The following example (Example 2 in Blair, 1988) illustrates the workings of our operator.

Example 2 *Let* $F = \{f_1, \ldots, f_7\}$ *and* $W = \{w_1, \ldots, w_{10}\}$. *Consider the substitutable many-to-many market induced by the preferences in Table 2. It is readily seen that matchings*

$$\underline{\mu} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 \\ \{w_2, w_3, w_4\} & \{w_1, w_5\} & \{w_1, w_6\} & \{w_{10}\} & \{w_8\} & \{w_9\} & \{w_7\} \end{pmatrix}$$

and

$$\overline{\mu} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 \\ \{w_2, w_3, w_4\} & \{w_1, w_5\} & \{w_9\} & \{w_1, w_7\} & \{w_8\} & \{w_6\} & \{w_{10}\} \end{pmatrix}$$

are stable. If we compute matching $\lambda_{\underline{\mu},\overline{\mu}}$ *we obtain*

$$\boxed{\mu} = \lambda_{\underline{\mu},\overline{\mu}} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 \\ \{w_2, w_3, w_4\} & \{w_1, w_5\} & \{w_9\} & \{w_{10}\} & \{w_8\} & \{w_6\} & \{w_7\} \end{pmatrix}.$$

Matching $\boxed{\mu}$ *is worker-quasi-stable but not stable since, for instance,* (f_1, w_1) *blocks* $\boxed{\mu}$. *Let us now use the Tarski operator to compute* $\widetilde{\mathcal{T}}^F\left[\boxed{\mu}\right]$. *We get*

P_{f_1}	$\enclose{circle}{w_1{}^\star}$	$\boxed{\overline{\underline{w_2, w_3, w_4}}}$	$\cdots$							
P_{f_2}	$w_2{}^\star$	w_8	$\enclose{circle}{\boxed{\overline{\underline{w_1, w_5}}}}$	$\cdots$						
P_{f_3}	$w_3{}^\star$	$\enclose{circle}{\boxed{\overline{w_9}}}$	$\underline{w_1, w_6}$	$\cdots$						
P_{f_4}	$w_4{}^\star$	$\enclose{circle}{\boxed{\overline{\underline{w_{10}}}}}$	$\overline{w_1, w_7}$	$\cdots$						
P_{f_5}	$w_5{}^\star$	$\enclose{circle}{\boxed{\overline{\underline{w_8}}}}$	$\cdots$							
P_{f_6}	$\enclose{circle}{\boxed{\overline{w_6{}^\star}}}$	$\underline{w_9}$	$\cdots$							
P_{f_7}	$\enclose{circle}{\boxed{\underline{w_7{}^\star}}}$	$\overline{w_{10}}$	$\cdots$							
P_{w_1}	f_2, f_3, f_4	$\underline{f_2, f_3}$	f_3, f_4	$\overline{f_2, f_4}$	$\enclose{circle}{f_1, f_2}$	f_1, f_3	f_1, f_4	$f_1{}^\star$	$\boxed{f_2}$	$\cdots$
P_{w_2}	$\boxed{\overline{\underline{f_1}}}$	$f_2{}^\star$	$\enclose{circle}{\emptyset}$							
P_{w_3}	$\boxed{\overline{\underline{f_1}}}$	$f_3{}^\star$	$\enclose{circle}{\emptyset}$							
P_{w_4}	$\boxed{\overline{\underline{f_1}}}$	$f_4{}^\star$	$\enclose{circle}{\emptyset}$							
P_{w_5}	$\enclose{circle}{\boxed{\overline{\underline{f_2}}}}$	$f_5{}^\star$	$\cdots$							
P_{w_6}	$\underline{f_3}$	$\enclose{circle}{\boxed{\overline{\underline{f_6{}^\star}}}}$	$\cdots$							
P_{w_7}	$\overline{f_4}$	$\enclose{circle}{\boxed{\underline{f_7{}^\star}}}$	$\cdots$							
P_{w_8}	$\enclose{circle}{\boxed{\overline{\underline{f_5}}}}$	f_2	$\emptyset^\star$							
P_{w_9}	$\underline{f_6}$	$\enclose{circle}{\boxed{\overline{f_3}}}$	$\emptyset^\star$							
$P_{w_{10}}$	$\overline{f_7}$	$\enclose{circle}{\boxed{\underline{f_4}}}$	$\emptyset^\star$							

Table 2: Preference profile for Example 2.

$$\textcircled{\mu} = \widetilde{\mathcal{T}}^F \left[\!\left[\boxed{\mu}\right]\!\right] = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & \emptyset \\ \{w_1\} & \{w_1, w_5\} & \{w_9\} & \{w_{10}\} & \{w_8\} & \{w_6\} & \{w_7\} & \{w_2, w_3, w_4\} \end{pmatrix}.$$

Again, matching $\textcircled{\mu}$ *is worker-quasi-stable but not stable since, for example,* (f_2, w_2) *blocks* $\textcircled{\mu}$*. A second application of the operator generates matching* $\widetilde{\mathcal{T}}^{F^{(2)}} \left[\!\left[\boxed{\mu}\right]\!\right]$*:*

$$\mu^\star = \widetilde{\mathcal{T}}^{F^{(2)}} \left[\!\left[\boxed{\mu}\right]\!\right] = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & \emptyset \\ \{w_1\} & \{w_2\} & \{w_3\} & \{w_4\} & \{w_5\} & \{w_6\} & \{w_7\} & \{w_8, w_9, w_{10}\} \end{pmatrix}.$$

Matching $\mu^\star$ *is stable. In fact, it is the firm-optimal matching. Therefore,* $\mu^\star = \underline{\mu} \curlyvee_F \overline{\mu}$*. It is clear that Blair had in mind the previously presented re-equilibration process to achieved the join between* $\underline{\mu}$ *and* $\overline{\mu}$*. In his explanation of this example, he says: "the join between* $\underline{\mu}$ *and* $\overline{\mu}$ *cannot have worker* w_1 *hired by firms* f_3 *or* f_4*. Thus, worker* w_1 *will want to work for firm* f_1*. This means that firm* f_1 *will not want workers* w_2, w_3*, and* w_4*, who will want to work for firms* f_2, f_3*, and* f_4*, respectively. Since workers* w_5, w_6*, and* w_7 *have no alternative employment, they will work for* f_5, f_6*, and* f_7*, respectively. Thus, the join has every firm get its first choice".* $\diamondsuit$

5 Conclusions

In this paper, we compute the lattice operations for the stable set when only substitutability on agents' choice functions is imposed. A few last remarks are in order.

If besides being substitutable, choice functions are assumed to satisfy the "law of aggregate demand", that says that when a firm chooses from an expanded set it hires at least as many workers as before,[18] then $\widetilde{\lambda}_{\mu,\mu'}$ and $\widetilde{\gamma}_{\mu,\mu'}$ are actually stable matchings. Under this additional requirement, the lattice operations are obtained by Alkan (2002). Moreover, Alkan (2002) also shows that the lattice of stable matchings is distributive, something that does not hold with substitutability alone (these results are also obtained by Li, 2014, through different methods). Again under the law of aggregate demand, Manasero (2021) also obtains the lattice operations from a many-to-one model, using a morphism similar to the one used in Subsection 4.1.

In a setting of many-to-many matchings with contracts with substitutable choice functions on one side and responsive preferences on the other, Bonifacio et al. (2024) show the lattice structure of the set of envy-free matchings. Bonifacio et al. (2024) also study re-equilibration by means of a Tarski operator. Firm-quasi-stability generalizes their notion

[18] **Law of aggregate demand:** $S' \subseteq S \subseteq W$ implies $|C_f(S')| \leq |C_f(S)|$ (see Alkan, 2002; Hatfield and Milgrom, 2005).

of "envy-freeness". Disregarding contracts, our operator in Definition 9 generalizes the one presented in Bonifacio et al. (2024). In fact, our results of Subsection 4.2 can easily be extended to a matching with contracts environment.

In order to prove that the set of many-to-one firm-quasi-stable matchings and both many-to-many (worker and firm) quasi-stable matchings are lattices, we follow Wu and Roth (2018) and Bonifacio et al. (2022): we show that such sets are join-semilattices with a minimum. However, how to compute the meet between any pair of quasi-stable matching remains an open question, even invoking additional restrictions such as the law of aggregate demand.

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A Proofs

A.1 Proof of Theorem 4

Before presenting the proof of Theorem 4, we prove an auxiliary result that says that for firms matched in μ, the natural morphism acts as the natural projection. Formally,

Lemma 1 *Let $\mu \in \mathcal{M}$ and $f \in F$ such that $\Phi[\mu](f) \neq \emptyset$. Then, $\Phi[\mu](f) = \pi(\mu(f))$.*

Proof. Let $\mu \in \mathcal{M}$ and $f \in F$ such that $\Phi[\mu](f) \neq \emptyset$. Thus , there is $w \in \Phi[\mu](f)$. This is equivalent to the existence of $w^t \in \pi^{-1}(\{w\})$ such that $\mu(w^t) = f$. This, in turn, is equivalent to $w = \pi(w^t)$ and $w^t \in \mu(f)$. This last assertion is equivalent to $w \in \pi(\mu(f))$. □

Remark 6 *Let $\mu \in \mathcal{M}$ and $f \in F$. Then, $\mu(f) = \emptyset$ if and only if $\Phi[\mu](f) = \emptyset$.*

Proof of Theorem 4. (i) To see that Φ is onto, let $\nu \in \mathcal{M}^q$. We need to show that there is $\mu \in \mathcal{M}$ such that $\Phi(\mu) = \nu$. Let $w \in W$. Clearly, $|\nu(w)| \leq q_w$. Label firms in $\nu(w)$ according to order $P_w^{q_w}$. Then, $\nu(w) = \{f_1, f_2, \ldots, f_r\}$ with $r \leq q_w$ and $\{f_i\} P_w^{q_w} \{f_j\}$ when $i < j$. For $\pi^{-1}(\{w\}) = \{w^1, w^2, \ldots, w^{q_w}\}$, define

$$\mu(w^t) = \begin{cases} f_t & \text{if } t \leq r \\ \emptyset & \text{otherwise} \end{cases}$$

Finally, for each $f \in F$, let $\mu(f) = \{w^t \in W^q : \mu(w^t) = f\}$. It is clear that μ thus defined belongs to $\mathcal{M}$. Moreover, by the definition of natural morphism, $\Phi(\mu) = \nu$, so Φ is surjective.

(ii) Let $\mu \in \mathcal{S}$. We divide the proof into two steps.
Step 1: $\Phi[\mu]$ is individually rational. For each $w \in W$, we need to see that

$$\Phi[\mu](w) \; R_w^{q_w} \; \emptyset. \tag{3}$$

Let $w \in W$. If $\Phi[\mu](w) = \emptyset$, then (3) clearly holds, so let $f \in \Phi[\mu](w)$. Thus, there is $w^t \in \pi^{-1}(\{w\})$ such that $f = \mu(w^t)$. Since μ is individually rational, $\mu(w^t) = f R_{w^t} \emptyset$. By definition of P_{w^t}, we have that $\{f\} R_w^{q_w} \emptyset$. By responsiveness, (3) holds. For each $f \in F$, we need to see that

$$C_f(\Phi[\mu](f)) = \Phi[\mu](f). \tag{4}$$

Let $f \in F$. If $\Phi[\mu](f) = \emptyset$, then (4) clearly holds, so assume $\Phi[\mu](f) \neq \emptyset$.

Therefore, by Lemma 1, the definition of C_f^q, the individual rationality of $\mu(f)$, and the by Lemma 1 again, we have

$$C_f(\Phi[\mu](f)) = C_f(\pi(\mu(f))) = \pi(C_f^q(\mu(f))) = \pi(\mu(f)) = \Phi[\mu](f),$$

so (4) holds.
Step 2: $\Phi[\mu]$ has no blocking pair. Assume, on contradiction, that (f, w) is a blocking pair of $\Phi[\mu]$. Then,

$$w \notin \Phi[\mu](f) \text{ and } w \in C_f(\Phi[\mu](f) \cup \{w\}). \tag{5}$$

First, we show that for each $w^t \in \pi^{-1}(\{w\})$ we have

$$w^t \in C^q_f(\mu(f) \cup \{w^t\}). \tag{6}$$

To prove (6), there are two cases to consider:

1. $\mathbf{\Phi[\mu](f) = \emptyset}$. By Remark 6, $\mu(f) = \emptyset$. Aditionally, by (5), $w \in C_f(\{w\})$. Let $w^t \in \pi^{-1}(\{w\})$. Since $C_f(\{w\}) = C_f(\{\pi(w^t)\}) = \pi(C^q_f(\{w^t\}))$, we have $w \in \pi(C^q_f(\{w^t\}))$. This implies that $w^t \in C^q_f(\{w^t\})$, and together with $\mu(f) = \emptyset$, that (6) holds.

2. $\mathbf{\Phi[\mu](f) \neq \emptyset}$. By (5) and Lemma 1, $w \in C_f(\pi(\mu(f) \cup \{w\}))$. Let $w^t \in \pi^{-1}(\{w\})$. Since

$$C_f(\pi(\mu(f) \cup \{w\})) = C_f(\pi(\mu(f) \cup \pi(w^t))) = \\ = C_f(\pi(\mu(f) \cup \{w^t\})) = \pi(C^q_f(\mu(f) \cup \{w^t\})),$$

it follows that $w \in \pi(C^q_f(\mu(f) \cup \{w^t\}))$. Therefore, there is $w^s \in \pi^{-1}(\{w\})$ such that $w^s \in C^q_f(\mu(f) \cup \{w^t\})$. If $w^s \neq w^t$, then $w^s \in \mu(f)$. This, in turn, implies that $f \in \cup_{w^r \in \pi^{-1}(\{w\})}\mu(w^r) = \Phi[\mu](w)$, contradicting the first part of (5). Hence, $w^s = w^t$ and (6) holds.

Next, to fully describe the block of $\Phi[\mu]$ by pair (f, w), there are two cases to consider:

1. $\mathbf{|\Phi[\mu](w)| < q_w}$. Then, $\{f\}P^{q_w}_w\emptyset$ and there is $w^t \in \pi^{-1}(\{w\})$ such that $\mu(w^t) = \emptyset$. Moreover, by the definition of P_{w^t}, $fP_{w^t}\emptyset = \mu(w^t)$. This last fact, together with (5), implies that (w^t, f) blocks μ. This contradicts that $\mu \in \mathcal{S}$.

2. $\mathbf{|\Phi[\mu](w)| = q_w}$. Then, there is $f' \in \Phi[\mu](w)$ such that $\{f\}P^{q_w}_w\{f'\}$. Since $f' \in \Phi[\mu](w)$, there is $w^t \in \pi^{-1}(w)$ such that $\mu(w^t) = f'$. By the definition of P_{w^t}, we have $fP_{w^t}f' = \mu(w^t)$. This last fact, together with (5), implies that (w^t, f) blocks μ. This contradicts that $\mu \in \mathcal{S}$.

We conclude, then, that $\Phi[\mu]$ has no blocking pair. By Step 1 and Step 2, we have shown that $\Phi(\mathcal{S}) = \mathcal{S}^q$.

(iii) By (i) and (ii), it only remains to show that $\Phi|_{\mathcal{S}}$ is injective. Let $\mu, \mu' \in \mathcal{S}$ be such that $\mu \neq \mu'$. Then, there is $w^t \in W^q$ such that $\mu(w^t) \neq \mu'(w^t)$. Let $w = \pi(w^t)$. Define

$$\mathscr{T} = \left\{ t' \in \{1, \ldots, q_w\} \ : \ \pi(w^{t'}) = w \text{ and } \mu(w^{t'}) \neq \mu'(w^{t'}) \right\}.$$

Therefore, $\mathscr{T} \neq \emptyset$ $\left(\text{because } t \in \mathscr{T}\right)$. Let $t^\star$ the minimum element of $\mathscr{T}$. Next, assume on contradiction, that $\Phi[\mu](w) = \Phi[\mu'](w)$. Without loss of generality, consider the case

$$\{\mu(w^{t^\star})\}\ P_w^{q_w}\ \{\mu'(w^{t^\star})\}. \tag{7}$$

Notice that, since $\mu \in \mathcal{S}$, (7) implies $\mu(w^{t^\star}) \neq \emptyset$. Let $f = \mu(w^{t^\star})$. There are two cases to consider:

1. $\boldsymbol{\mu'(w^{t^\star}) = \emptyset}$. As $w^{t^\star} \in C_f^q(\mu'(f) \cup \{w^{t^\star}\})$ (by (i) in the definition of q-extension of C_f) and considering that (7) implies $f\ P_{w^{t^\star}}\ \mu'(w^{t^\star})$, it follows that $(f, w^{t^\star})$ blocks μ', contradicting that $\mu' \in \mathcal{S}$.

2. $\boldsymbol{\mu'(w^{t^\star}) = f'}$ **for some** $\boldsymbol{f' \in F \setminus \{f\}}$. As $f \in \Phi[\mu](w) = \Phi[\mu'](w)$, there is $w^{t'} \in \pi^{-1}(\{w\})$ such that $\mu'(w^{t'}) = f$. Since $t^\star < t'$, by (i) in the definition of q-extension of C_f, we have that $w^{t^\star} \in C_f^q(\mu'(f) \cup \{w^{t^\star}\})$. Furthermore, (7) implies that $f\ P_{w^{t^\star}}\ \mu'(w^{t^\star})$. Therefore, $(f, w^{t^\star})$ blocks μ', contradicting that $\mu' \in \mathcal{S}$.

Since in both cases we reach a contradiction, we conclude that $\Phi[\mu](w) \neq \Phi[\mu'](w)$. Hence, $\Phi[\mu] \neq \Phi[\mu']$ and $\Phi|_{\mathcal{S}}$ is injective.

(iv) Let $\mu, \mu' \in \mathcal{S}$ and let $f \in F$.
($\Longrightarrow$) Assume $\mu \succeq_F \mu'$. Then, $C_f^q(\mu(f) \cup \mu'(f)) = \mu(f)$. If $\mu'(f) = \emptyset$. Remark 6 implies $\Phi[\mu'](f) = \emptyset$. Therefore,

$$C_f\left(\Phi[\mu](f) \cup \Phi[\mu'](f)\right) = C_f\left(\Phi[\mu](f)\right) = \Phi[\mu](f)$$

and the result follows. Assume next that $\mu'(f) \neq \emptyset$. Since $\mu, \mu' \in \mathcal{S}$ and $\mu \succeq_F \mu'$, we have $\mu(f) \neq \emptyset$. Using Remark 6, $\Phi[\mu](f) \neq \emptyset$ and $\Phi[\mu'](f) \neq \emptyset$. Thus,

$$C_f\left(\Phi[\mu](f) \cup \Phi[\mu'](f)\right) = C_f\left(\pi(\mu(f)) \cup \pi(\mu'(f))\right) = C_f\left(\pi(\mu(f) \cup \mu'(f))\right) =$$

$$= \pi\left(C_f^q(\mu(f) \cup \mu'(f))\right) = \pi\left(\mu(f)\right) = \Phi[\mu](f),$$

so $C_f\left(\Phi[\mu](f) \cup \Phi[\mu'](f)\right) = \Phi[\mu](f)$ and, since f is arbitrary, $\Phi[\mu] \succeq_F^q \Phi[\mu']$.
($\Longleftarrow$) Assume $\Phi[\mu] \succeq_F^q \Phi[\mu']$. Then, $C_f\left(\Phi[\mu](f) \cup \Phi[\mu'](f)\right) = \Phi[\mu](f)$. If $\Phi[\mu'](f) = \emptyset$, Remark 6 implies $\mu'(f) = \emptyset$. Therefore,

$$C_f^q(\mu(f) \cup \mu'(f)) = C_f^q(\mu(f)) = \mu(f)$$

and the result follows. Assume next that $\Phi[\mu'](f) \neq \emptyset$. Since $\Phi[\mu] \succeq_F^q \Phi[\mu']$, we have $\Phi[\mu](f) \neq \emptyset$. If $\Phi[\mu](f) = \Phi[\mu'](f)$ injectivity of Φ implies that $\mu(f) = \mu'(f)$ and the result follows. Assume, then, that $\Phi[\mu](f) \neq \Phi[\mu'](f)$. Thus, there is $w \in \Phi[\mu](f)$ such that $w \notin \Phi[\mu'](f)$. Hence, $\mu'(f) \neq w^t$ for each $w^t \in \pi^{-1}(\{w\})$. Let $\widetilde{w}^t \in \pi^{-1}(\{w\})$ be such that $\widetilde{w}^t = \mu(f)$. Using that $\Phi[\mu] \succeq_F^q \Phi[\mu']$, Lemma 1, and the definition of q-extension of C_f, we obtain

$$\pi(\mu(f)) = C_f(\pi(\mu(f)) \cup \pi(\mu'(f))) = C_f(\pi(\mu(f) \cup \mu'(f))) = \pi\left(C_f^q(\mu(f) \cup \mu'(f))\right).$$

Thus, $\pi(\mu(f)) = \pi\left(C_f^q(\mu(f) \cup \mu'(f))\right)$. This last equality, together with $\widetilde{w}^t = \mu(f)$ and $\mu'(f) \neq w^t$ for each $w^t \in \pi^{-1}(\{w\})$ imply that $C_f^q(\mu(f) \cup \mu'(f)) = \mu(f)$. Since f is arbitrary, $\mu \succeq_F \mu'$. □

A.2 Proof of Theorem 6

First, we present a useful result about the behavior of set $\widetilde{F}_w^\mu$.

Lemma 2 *(Facts about $\widetilde{F}_w^\mu$)*

(i) Let $\mu \in \widetilde{\mathcal{M}}$ be such that $C_f(\mu(f)) = \mu(f)$ for each $f \in F$. Then, $\mu(w) \subseteq \widetilde{F}_w^\mu$ for each $w \in W$.

(ii) Let $\mu, \mu' \in \widetilde{\mathcal{M}}$ and $S_f, S'_f \subseteq W$ be such that $S_f \subseteq S'_f$, $\mu(f) = C_f(S_f)$, and $\mu'(f) = C_f(S'_f)$ for each $f \in F$. Then, $\widetilde{F}_w^{\mu'} \subseteq \widetilde{F}_w^\mu$ for each $w \in W$.

Proof. (i) Let $\mu \in \widetilde{\mathcal{M}}$ be such that $C_f(\mu(f)) = \mu(f)$ for each $f \in F$, and let $w \in W$. If $f \in \mu(w)$, then $w \in \mu(f)$. Thus, $\mu(f) = \mu(f) \cup \{w\}$ and $w \in \mu(f) = C_f(\mu(f) \cup \{w\})$, so $f \in \widetilde{F}_w^\mu$.

(ii) Let $\mu, \mu' \in \widetilde{\mathcal{M}}$ and $S_f, S'_f \subseteq W$ fo each $f \in F$ be as stated in the Lemma, and let $w \in W$. If $f \in \widetilde{F}_w^{\mu'}$, by (1) we have

$$w \in C_f(\mu'(f) \cup \{w\}) = C_f\left(C_f(S'_f) \cup \{w\}\right) = C_f(S'_f \cup \{w\}).$$

Since $S_f \subseteq S'_f$, by substitutability and (1) we obtain

$$w \in C_f(S_f \cup \{w\}) = C_f\left(C_f(S_f) \cup \{w\}\right) = C_f(\mu(f) \cup \{w\}).$$

Hence, $f \in \widetilde{F}_w^\mu$. □

Remark 7 *Part (i) of Lemma 2 is a restatement of Proposition 4.7 in Blair (1988), whereas part (ii) is a restatement of Proposition 4.9 therein. We include the easy proofs for completeness.*

Proof of Theorem 6. Let $\mu, \mu' \in \widetilde{\mathcal{Q}}^W$. For short, let $\widetilde{\lambda} \equiv \widetilde{\lambda}_{\mu,\mu'}$.

(i) First, we show that $\widetilde{\lambda}$ is individually rational. Let $f \in F$. Then, by (1),

$$C_f\left(\widetilde{\lambda}(f)\right) = C_f\left(C_f\left(\mu(f) \cup \mu'(f)\right)\right) = C_f\left(\mu(f) \cup \mu'(f)\right) = \widetilde{\lambda}(f),$$

and $C_f\left(\widetilde{\lambda}(f)\right) = \widetilde{\lambda}(f)$.

Claim: $\widetilde{\lambda}(w) \subseteq C_w(\widetilde{F}_w^{\widetilde{\lambda}})$ for each $w \in W$. Let $w \in W$. Define, for each $f \in F$, $S_f = \mu(f)$ and $S'_f = \mu(f) \cup \mu'(f)$. Then, $S_f \subseteq S'_f$, $\mu(f) = C_f(S_f)$ and $\widetilde{\lambda}(f) = C_f(S'_f)$. Therefore, by Lemma 2 (ii), it follows that $\widetilde{F}_w^{\widetilde{\lambda}} \subseteq \widetilde{F}_w^{\mu}$. This last fact, together with $\mu \in \widetilde{\mathcal{Q}}^W$, $\mu(w) \subseteq \widetilde{F}_w^{\mu}$ (by Lemma 2 (i)), and substitutability, imply

$$\mu(w) \cap \widetilde{F}_w^{\widetilde{\lambda}} \subseteq C_w(\widetilde{F}_w^{\mu}) \cap \widetilde{F}_w^{\widetilde{\lambda}} \subseteq C_w(\widetilde{F}_w^{\widetilde{\lambda}}),$$

so $\mu(w) \cap \widetilde{F}_w^{\widetilde{\lambda}} \subseteq C_w(\widetilde{F}_w^{\widetilde{\lambda}})$ and, since by Lemma 2 (i) we have $\widetilde{\lambda}(w) \subseteq \widetilde{F}_w^{\widetilde{\lambda}}$, it follows that

$$\mu(w) \cap \widetilde{\lambda}(w) \subseteq C_w(\widetilde{F}_w^{\widetilde{\lambda}}).$$

In an analogous way we can prove that $\mu'(w) \cap \widetilde{\lambda}(w) \subseteq C_w(\widetilde{F}_w^{\widetilde{\lambda}})$. Finally, since $\widetilde{\lambda}(w) \subseteq \mu(w) \cup \mu'(w)$, we get $\widetilde{\lambda}(w) \subseteq C_w(\widetilde{F}_w^{\widetilde{\lambda}})$. This proves the claim.

Now, let $w \in W$. By the Claim, $\widetilde{\lambda}(w) \subseteq C_w(\widetilde{F}_w^{\widetilde{\lambda}})$. Then, since $\widetilde{\lambda}(w) \subseteq \widetilde{F}_w^{\widetilde{\lambda}}$ by Lemma 2 (i), substitutability implies $\widetilde{\lambda}(w) \subseteq C_w(\widetilde{\lambda})$. Moreover, as $C_w(\widetilde{\lambda}) \subseteq \widetilde{\lambda}(w)$, we get $C_w(\widetilde{\lambda}(w)) = \widetilde{\lambda}(w)$. Hence, $\widetilde{\lambda}$ is individually rational.

To finish the proof that $\widetilde{\lambda}$ is worker-quasi-stable, let $w \in W$ and $T \subseteq \widetilde{F}_w^{\mu}$. By the Claim, $\widetilde{\lambda}(w) \subseteq C_w(\widetilde{F}_w^{\widetilde{\lambda}})$. Then, since $\widetilde{\lambda}(w) \subseteq \widetilde{F}_w^{\widetilde{\lambda}}$ by Lemma 2 (i), substitutability implies $\widetilde{\lambda}(w) \subseteq C_w(\widetilde{\lambda}(w) \cup T)$.

To see that $\widetilde{\lambda}$ is the join of μ and μ' with respect to $\succeq_F$ in $\widetilde{\mathcal{Q}}^W$, first notice that, by (1),

$$C_f(\widetilde{\lambda}(f) \cup \mu(f)) = C_f(C_f(\mu(f) \cup \mu'(f)) \cup \mu(f)) = C_f(\mu(f) \cup \mu'(f)) = \widetilde{\lambda}(f)$$

for each $f \in F$, so
$widetilde\lambda \succeq_F \mu$. Similarly, $\widetilde{\lambda} \succeq_F \mu'$, and thus $\widetilde{\lambda}$ is an upper bound of μ and μ'. Let $\nu \in \widetilde{\mathcal{Q}}^W$ be another upper bound of μ and μ'. Then, by (1),

$$\nu(f) = C_f(\nu(f) \cup \mu(f)) = C_f(C_f(\nu(f) \cup \mu'(f)) \cup \mu(f)) = C_f(\nu(f) \cup \mu(f) \cup \mu'(f)) =$$

$$= C_f(\nu(f) \cup C_f(\mu(f) \cup \mu'(f))) = C_f(\nu(f) \cup \widetilde{\lambda}(f))$$

for each $f \in F$, so $\nu \succeq_F \widetilde{\lambda}$.

(ii) Consider the empty matching $\mu_\emptyset$ where every agent is unmatched. Clearly, $\mu_\emptyset \in \widetilde{\mathcal{Q}}^W$. Moreover, $\mu \succeq_F \mu_\emptyset$ for each $\mu \in \widetilde{\mathcal{Q}}^W$, so $(\widetilde{\mathcal{Q}}^W, \succeq_F)$ has a minimum. By Theorem 6 (i), $(\widetilde{\mathcal{Q}}^W, \succeq_F)$ is a join-semilattice. Any join-semilattice with a minimum is a lattice (see, for example, Stanley, 2011). $\square$

A.3 Proof of Theorem 7

(i) Let $\mu \in \widetilde{\mathcal{Q}}^W$ and let $f \in F$. Then,

$$C_f(\widetilde{\mathcal{T}}^F[\mu](f)) = C_f(C_f(\widetilde{B}_f^{\mu})) = C_f(\widetilde{B}_f^{\mu}) = \widetilde{\mathcal{T}}^F[\mu](f).$$

Let $w \in W$. Using a similar reasoning as the one applied to matching $\widetilde{\lambda}$ to obtain the Claim of part (i), but this time to matching $\widetilde{\mathcal{T}}^F[\mu]$, it follows that

$$\widetilde{\mathcal{T}}^F[\mu](w) \subseteq C_w\left(\widetilde{F_w}^{\widetilde{\mathcal{T}}^F[\mu]}\right). \tag{8}$$

Since, by Lemma 2 (i), $\widetilde{\mathcal{T}}^F[\mu](w) \subseteq \widetilde{F_w}^{\widetilde{\mathcal{T}}^F[\mu]}$, substitutability and (8) imply $\widetilde{\mathcal{T}}^F[\mu](w) \subseteq C_w\left(\widetilde{\mathcal{T}}^F[\mu](w)\right)$. As $C_w\left(\widetilde{\mathcal{T}}^F[\mu](w)\right) \subseteq \widetilde{\mathcal{T}}^F[\mu](w)$ as well, we have $C_w\left(\widetilde{\mathcal{T}}^F[\mu](w)\right) = \widetilde{\mathcal{T}}^F[\mu](w)$. This proves that $\widetilde{\mathcal{T}}^F[\mu]$ is individually rational. Next, Let $T \subseteq \widetilde{F_w}^{\widetilde{\mathcal{T}}^F[\mu]}$. The fact that $\widetilde{\mathcal{T}}^F[\mu](w) \subseteq \widetilde{F_w}^{\widetilde{\mathcal{T}}^F[\mu]}$, (8), and substitutability, imply $\widetilde{\mathcal{T}}^F[\mu](w) \subseteq C_w\left(\widetilde{\mathcal{T}}^F[\mu](w) \cup T\right)$. Thus, $\widetilde{\mathcal{T}}^F[\mu] \in \widetilde{\mathcal{Q}}^W$.

Moreover, $\mu(f) \subseteq \widetilde{B}_f^{\mu}$ and (1) imply that

$$C_f(\widetilde{\mathcal{T}}^F[\mu](f) \cup \mu(f)) = C_f\left(C_f(\widetilde{B}_f^{\mu}) \cup \mu(f)\right) = C_f(\mu(f) \cup \widetilde{B}_f^{\mu}) = C_f(\widetilde{B}_f^{\mu}) = \widetilde{\mathcal{T}}^F[\mu](f).$$

As f is arbitrary, $\widetilde{\mathcal{T}}^F[\mu] \succeq_F \mu$.

(ii) Let $\mu, \mu' \in \widetilde{\mathcal{Q}}^W$ be such that $\mu \succeq_F \mu'$. Assume $\widetilde{\mathcal{T}}^F[\mu] \succeq_F \widetilde{\mathcal{T}}^F[\mu']$ does not hold. Then, there is $f \in F$ such that

$$\widetilde{\mathcal{T}}^F[\mu](f) \neq C_f\left(\widetilde{\mathcal{T}}^F[\mu](f) \cup \widetilde{\mathcal{T}}^F[\mu'](f)\right). \tag{9}$$

By (1), $C_f\left(\widetilde{\mathcal{T}}^F[\mu](f) \cup \widetilde{\mathcal{T}}^F[\mu'](f)\right) = C_f\left(C_f(\widetilde{B}_f^{\mu}) \cup C_f(\widetilde{B}_f^{\mu'})\right) = C_f\left(\widetilde{B}_f^{\mu} \cup \widetilde{B}_f^{\mu'}\right)$. Thus, (9) becomes

$$C_f(\widetilde{B}_f^{\mu}) \neq C_f\left(\widetilde{B}_f^{\mu} \cup \widetilde{B}_f^{\mu'}\right). \tag{10}$$

As substitutability implies $C_f\left(\widetilde{B}_f^{\mu} \cup \widetilde{B}_f^{\mu'}\right) \cap \widetilde{B}_f^{\mu} \subseteq C_f(\widetilde{B}_f^{\mu})$, by (10) it follows that there is $w \in \widetilde{B}_f^{\mu'} \setminus \widetilde{B}_f^{\mu}$ such that $w \in C_f\left(\widetilde{B}_f^{\mu} \cup \widetilde{B}_f^{\mu'}\right)$. Since $\mu(f) \subseteq \widetilde{B}_f^{\mu}$, substitutability implies then that $w \in C_f(\mu(f) \cup \{w\})$. Therefore, $f \in \widetilde{F}_w^{\mu}$.

Claim: $\widetilde{F_w}^{\mu} \subseteq \widetilde{F_w}^{\mu'}$**.** For each $f' \in F$, let $S'_{f'} = \mu(f') \cup \mu'(f')$ and $S_{f'} = \mu'(f')$. Then, $S_{f'} \subseteq S'_{f'}$. Moreover, $\mu'(f') = C_{f'}(S_{f'})$ for each $f' \in F$ by the individual rationality of μ', and $\mu(f') = C_{f'}(S'_{f'})$ for each $f' \in F$ because $\mu \succeq_F \mu'$. Therefore, by Lemma 2 (ii), $\widetilde{F}_w^{\mu} \subseteq \widetilde{F}_w^{\mu'}$. This proves the Claim.

Since $w \notin \widetilde{B}_f^{\mu}$, $f \notin C_w(\widetilde{F}_w^{\mu})$. As $f \in \widetilde{F}_w^{\mu}$, substitutability and the Claim imply that $f \notin C_w(\widetilde{F}_w^{\mu'})$. This contradicts that $w \in \widetilde{B}_f^{\mu'}$. Hence, $\widetilde{\mathcal{T}}^F[\mu] \succeq_F \widetilde{\mathcal{T}}^F[\mu']$, as desired.

(iii) ($\Longrightarrow$) Assume that $\widetilde{\mathcal{T}}^F[\mu] = \mu$. Let $f \in F$. To check that $\mu \in \widetilde{\mathcal{S}}$ it is sufficient to see that $\widetilde{B}_f^{\mu} = \mu(f)$. By (2), $\mu(f) \subseteq \widetilde{B}_f^{\mu}$. Let $w \in \widetilde{B}_f^{\mu}$. Then, $f \in C_w(\widetilde{F}_w^{\mu}) \subseteq \widetilde{F}_w^{\mu}$ and thus $w \in C_f(\mu(f) \cup \{w\})$. Moreover, as $C_f(\widetilde{B}_f^{\mu}) = \mu(f)$, by (1) we have

$$C_f(\mu(f) \cup \{w\}) = C_f\left(C_f(\widetilde{B}_f^{\mu}) \cup \{w\}\right) = C_f\left(\widetilde{B}_f^{\mu} \cup \{w\}\right) = C_f\left(\widetilde{B}_f^{\mu}\right) = \mu(f).$$

Hence, $w \in \mu(f)$ and $\widetilde{B}_f^{\mu} \subseteq \mu(f)$. Therefore, $\widetilde{B}_f^{\mu} = \mu(f)$ and $\mu \in \widetilde{\mathcal{S}}$.

($\Longleftarrow$) Assume $\widetilde{\mathcal{T}}^F[\mu] \neq \mu$. Then, there is $f \in F$ such that $C_f(\widetilde{B}_f^{\mu}) \neq \mu(f)$. Therefore, there is $w \in C_f(\widetilde{B}_f^{\mu})$ with $w \notin \mu(f)$. Since $\mu(f) \subseteq \widetilde{B}_f^{\mu}$, by substitutability,

$$w \in C_f(\mu(f) \cup \{w\}). \tag{11}$$

Moreover, since $w \in \widetilde{B}_f^{\mu}$, $f \in C_w(\widetilde{F}_w^{\mu})$. Since $\mu(w) \subseteq F_w^{\mu}$ by Lema 2 (i), substitutability implies

$$f \in C_w(\mu(w) \cup \{f\}). \tag{12}$$

Therefore, as $w \notin \mu(f)$, by (11) and (12) it follows that (f, w) blocks μ. Hence, $\mu \notin \widetilde{\mathcal{S}}$. □

A.4 Proof of Theorem 8

Let $\mu, \mu' \in \widetilde{\mathcal{S}}$. First, let us see (i). For short, let $\widetilde{\lambda} \equiv \widetilde{\lambda}_{\mu,\mu'}$. By Theorem 6 (i), $\widetilde{\lambda} \in \widetilde{\mathcal{Q}}^W$ and, furthermore, $\widetilde{\lambda}$ is the join of μ and μ' with respect to $\succeq_F$ in $\widetilde{\mathcal{Q}}^W$. Define

$$\Lambda = \{\nu \in \widetilde{\mathcal{Q}}^W : \nu \succeq_F \widetilde{\lambda}\}.$$

Claim: $\widetilde{\mathcal{T}}^F[\Lambda] \subseteq \Lambda$. Let $\nu \in \Lambda$. Then, $\nu \succeq_F \widetilde{\lambda} \succeq_F \mu$ and $\nu \succeq_F \widetilde{\lambda} \succeq_F \mu'$. By Proposition 2 (ii) and (iii), $\mathcal{T}^F(\nu) \succeq_F \mathcal{T}^F(\mu) = \mu$ and $\mathcal{T}^F(\nu) \succeq_F \mathcal{T}^F(\mu') = \mu'$. This implies that $\mathcal{T}^F(\nu)$ is an upper bound of μ and μ' in $\widetilde{\mathcal{Q}}^W$. By definition of join, $\mathcal{T}^F(\nu) \succeq_F \widetilde{\lambda}$ and thus $\mathcal{T}^F(\nu) \in \Lambda$. This proves the claim.

Next, consider the restriction of $\widetilde{\mathcal{T}}^F$ to Λ. By the previous claim, $\widetilde{\mathcal{T}}^F|_{\Lambda} : \Lambda \longrightarrow \Lambda$. Since $\widetilde{\mathcal{T}}^F$ is isotone by Theorem 7 (ii) and $\widetilde{\lambda}$ is the $\succeq_F$-smallest element in Λ, $\widetilde{\mathcal{F}}^F(\widetilde{\lambda})$ is the $\succeq_F$-smallest fixed point larger than μ and μ'. Otherwise, if δ is an upper bound of μ and μ', $\widetilde{\mathcal{F}}^F(\widetilde{\lambda}) \succ_F \delta$, and δ is a fixed point of $\widetilde{\mathcal{T}}^F$; as $\delta \succeq_F \widetilde{\lambda}$ isotonicity implies $\delta \succeq_F \widetilde{\mathcal{F}}^F(\widetilde{\lambda})$, a contradiction. Finally, since $\widetilde{\mathcal{F}}^F(\widetilde{\lambda}) \in \widetilde{\mathcal{S}}$ by Theorem 7 (iii), we get $\widetilde{\mathcal{F}}^F(\widetilde{\lambda}) = \mu \curlyvee_F \mu'$, as desired.

To see (ii), first apply *mutatis mutandis* the same reasoning as before, but this time to operator $\widetilde{\mathcal{T}}^W$ using results dual to those of in Theorem 7, to obtain $\mu \curlyvee_W \mu' = \widetilde{\mathcal{F}}^W(\widetilde{\gamma}_{\mu,\mu'})$. Next, by Theorem 4.5 in Blair (1988), $\succeq_F$ and $\succeq_W$ are dual orders in $\widetilde{\mathcal{S}}$. Therefore, $\mu \curlywedge_F \mu' = \mu \curlyvee_W \mu'$ and the result follows. □

A.5 Proof of Theorem 1

When a substitutable many-to-many market (C_F, C_W) specializes to a many-to-one market (C_F, P_W), the choice function takes the following particular form. Let $w \in W$. Then, for each $T \subseteq F$, $C_w(T) = \max_{P_w} T$. In particular, Blair's order for the workers, $\succeq_W$, becomes equivalent to the unanimous order for the workers, $\geqslant_W$.[19] Furthermore, the notion of many-to-many firm-quasi-stability dual to Definition 8 specializes to Definition 4. To see this, notice that many-to-many firm-quasi-stability for a matching μ implies that it is individually rational and that, for each $f \in F$ and each $S \subseteq \widetilde{W}_f^\mu \equiv \{w \in W : f \in C_w(\mu(w) \cup \{f\})\}$, we have $\mu(f) \subseteq C_f(\mu(f) \cup S)$. But for a many-to-one market, $\widetilde{W}_f^\mu = \{w \in W : f \in C_w(\mu(w) \cup \{f\})\} = \{w \in W : f R_w \mu(w)\} = W_f^\mu$, so both definitions coincide. Hence, Theorem 1 is implied by Theorem 6. □

A.6 Proof of Theorem 2

Symmetrically to Definition 9, we can define operator $\widetilde{\mathcal{T}}^W : \widetilde{\mathcal{Q}}^F \longrightarrow \widetilde{\mathcal{Q}}^F$ that assigns, for any given $\mu \in \widetilde{\mathcal{Q}}^F$,

(i) for each $w \in W$, $\widetilde{\mathcal{T}}^W[\mu](w) = C_w(\widetilde{B}_w^\mu)$, and

(ii) for each $f \in F$, $\widetilde{\mathcal{T}}^W[\mu](f) = \{w \in W : f \in \widetilde{\mathcal{T}}^W[\mu](w)\}$,

where $\widetilde{B}_w^\mu = \left\{f \in F : w \in C_f(\widetilde{W}_f^\mu)\right\} \cup \{\mu(w)\}$.

Let $\mu \in \mathcal{Q}^F$. As, for each $T \subseteq F$, $C_w(T) = \max_{P_w} T$ for each $w \in W$ and $\succeq_W$ is equivalent to $\geqslant_W$, we have that $\widetilde{W}_f^\mu \equiv \{w \in W : f \in C_w(\mu(w) \cup \{f\})\} = W_f^\mu$ for each $f \in F$ and $\widetilde{B}_w^\mu = B_w^\mu$ for each $w \in W$. Therefore, the dual of Definition 9 (for firm-quasi-stable matchings) specializes to Definition 5, since

$$\widetilde{\mathcal{T}}^W[\mu](w) = C_w(\widetilde{B}_w^\mu) = C_w(B_w^\mu) = \max_{P_w} B_w^\mu = \mathcal{T}^W[\mu](w).$$

Consequently, Theorem 2 follows from the dual of Theorem 7. □

A.7 Proof of Theorem 3

In the many-to-one model, operators $\widetilde{\mathcal{T}}^F$ and $\widetilde{\mathcal{T}}^W$ are $\mathcal{T}^F$ and $\mathcal{T}^W$, respectively; and matchings $\widetilde{\gamma}_{\mu,\mu'}$ and $\widetilde{\lambda}_{\mu,\mu'}$ are $\gamma_{\mu,\mu'}$ and $\lambda_{\mu,\mu'}$, respectively. Therefore, Theorem 3 is a special case of Theorem 8. □

[19] $\mu \geqslant_W \mu' \iff \mu(w) R_w \mu'(w)$ for each $w \in W \iff \mu(w) = \max_{P_w}\{\mu(w), \mu'(w)\} = C_w(\{\mu(w), \mu'(w)\})$ for each $w \in W \iff \mu \succeq_W \mu'$.