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A Level-Agnostic Representation of Economic Agents*

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Abstract

The study of the interactions among economic agents, being rationality the main source of intentional behavior, requires mathematical tools able to capture systemic effects. Here we choose an alternative toolbox based on Category Theory. We examine potential *level-agnostic* formalisms, presenting three categories, $\mathcal{PR}$, $\mathcal{G}$ and an encompassing one, $\mathcal{I}$. The latter allows for representing dynamic rearrangements of the interactions among different agents.

Systems represented in $\mathcal{I}$, capture the dynamic interaction among the interfaces of their sub-agents, changing the connections among them based on their internal states. We illustrate the expressive power of this formalism in four different instances, providing practitioners with a toolbox for the representation of cases of interest, facilitating their modular analysis.

Keywords: Economic Agent, Interactions, Category Theory, Game Theory, Polynomial Functors.

MSC: 93A16; 18M99; 91A70.

1 Introduction

Economics can be understood as the study of the interactions among intentional agents, being rationality the main source of intentional behavior. The term *agent*, refers here also to firms, institutions, and other non-human economic agents, allowing the extension of economic analyses to all kinds of things able to exhibit agency, ranging from social groups to robots.

Agents can be seen as systems composed of other systems. While contemporary disciplines like Computer Science embraced this view ([10]), in this contribution we explore

*__Acknowledgements__: Thanks are due to Andrés Fioriti for the drawings and ideas, reflected in our previous joint publication ([27]).

possible formalisms that may lead to an extended conception of economic interactions among agents. We consider here two issues:

- How to deal with the decisions the sub-agents obtain inside a single agent.
- How to scale up the solutions of agents to larger systems, aggregating them.

As an example of the first issue we can think of a single agent having to solve two independent choice problems in parallel. It is natural to conceive the situation as if there were two agents exchanging information and resources to solve the two problems.

In the other direction, the problem of aggregation arises naturally in voting systems. Each voter has a preference and a government has to be chosen that can be seen as a single agent representing the society.

Each of these two issues is an instance of the same problem, one is the *bottom-up* and the other is its *top-down* version. Both reveal the need for a *level-agnostic* (or *continuous with respect to subagents*) representation of this "multi-level agency" phenomenon. This paper lays the ground for its formalization.

We start by noting that there exists a well-defined notion of agent defined in terms of a given preference relation over the space of alternatives. Then, the agent is said *rational* if she chooses the most preferred alternatives among those that are feasible for her.

In applications, it is customary to reduce the analysis to a subspace of the space of alternatives, simplifying the problem of making a decision. But this comes at the price of assuming the independence of the preferences over the subspace from the preferences over the rest of the larger space of alternatives.

In this initial version we first present a way of ensuring the consistency of the solutions found for the different subspaces. Then, another approach to the coordination of independent context is given, in this case involving games with shared players.

The final part of this paper presents a generalization, integrating both models, in which interactions are no longer fixed, but can evolve according to the inputs and outputs. In this, as well as in the previous two models, we apply the mathematical framework of Category Theory.
Category Theory provides a high-level abstract representation of formal structures, focusing on their interrelations. It has largely contributed to the advancement of the Math-

ematical Sciences by being "*math to scaffold accounts from many disciplines*" [24].

Our contribution can be understood in this sense, as a methodology to describe economic systems, using the same formalism for their components and for the larger systems that they may, in turn, integrate. In this sense, it provides a useful *theoretical* characterization that helps to understand in a modular form the interactions among those economic systems[1], independently of their position in structures in which they participate.

2 Mathematical Preliminaries

As it is well-known, Category Theory has provided a framework without which most of the contemporary results in both Algebraic Geometry and Topology would not have been found [12]. As repeatedly shown in actual mathematical practice, the language of Set Theory remains insufficient for capturing subtleties prevalent in those fields [16]. One reason is that, unlike Set Theory, the categorical approach allows for the maximization of the "external" scope of its formal results *and* the controlled "internal" sensitivity to particular differences in content within the representation of mathematical structures. Albeit Category Theory seems to provide a natural language for representing the decision-making problems outlined above, we have to note that some disciplines, like Economics, have been reluctant to adopt it.[2]

In this paper we draw heavily on the literature on Category Theory, although our results are clearly elementary. We will now present the basic concepts that will be used in subsequent sections. For further details and clarification, see the excellent general texts on Category Theory of Goldblatt ([11]), Barr & Wells ([3]), Adámek et al. ([2]), Lawvere and Shanuel ([14]), Spivak ([23]), Fong and Spivak ([6]), Southwell ([22]) or Cheng ([4]).

A category $\mathbf{C}$ consists of a set of *objects*, Obj and a class of *morphisms* between pairs of objects. Given two objects $a, b \in \text{Obj}$ a morphism f between them is notated by $f : a \to b$. Given another object c and a morphism $g : b \to c$, we have that f and g can be composed, yielding $g \circ f : a \to c$ (COMPOSITION). Additionally, for every $a \in \text{Obj}$, there exists an *identity* morphism, $\text{Id}_a : a \to a$. Morphisms are required to obey two rules: (i) if $f : a \to b$, $f \circ \text{Id}_a = f$ and $\text{Id}_b \circ f = f$ (IDENTITY); (ii) given $f : a \to b$, $g : b \to c$ and

[1]The formalism presented here has been extended to apply to all kinds of entities endowed with agency [27].

[2]Some notable exceptions are [9], [7], [1] and [20]. In turn, [5] presented arguments for the adoption of the categorical language in Economics.

$h : c \to d$, $(h \circ g) \circ f = h \circ (g \circ f) : a \to d$ (ASSOCIATIVITY).

Examples of categories are **SET** (the objects are sets, and the morphisms are functions between sets), **TOP** (the objects are topological spaces and the morphisms continuous functions), **POrd** (the objects are preorders and the morphisms are order-preserving functions), **Vec** (the objects are vector spaces and the morphisms linear maps), etc.

The terseness of categories facilitates diagrammatic reasoning. A diagram in which nodes represent objects and arrows represent morphisms allows to establish properties of a category. Diagrams that *commute*, i.e. such that all different direct paths of morphisms with the same start and end nodes are identified (that is, compose to a common morphism), indicate relations similar to those that can be established by means of equations.

Some of the most interesting constructions that can be defined in categories are *limits* and *colimits* (duals of limits). Any limit (or colimit) captures a *universal property* on a family of diagrams with the same basic shape. This basic shape is captured by a *cone*, that is, an object a and a family of arrows $\{f_a^{b_j} : a \to b_j\}_{j \in \mathcal{J}}$, such that for any pair $j, l \in \mathcal{J}$, if there exists a morphism $\gamma_{jl} : b_j \to b_l$ we have that $\gamma_{jl} \circ f_a^{b_j} = f_a^{b_l}$ (see Figure 1).

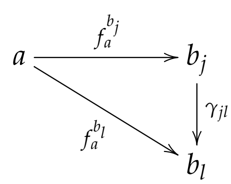


Figure 1: Commutative diagram

Then, given a class of cones of a given shape, a limit is an object L in this class such that for every other cone T in the class there exists a single morphism $T \to L$ such that the resulting combined diagram commutes. For instance, consider a family of cones of the shape depicted in Figure 2.

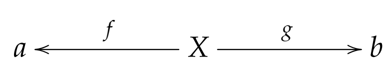


Figure 2: The limit of cones of this shape defines the product $a \times b$

then, the limit is the *product* $a \times b$ and with arrows p_1 and p_2, the projections on the first

(a) and second (b) components, respectively. For every other cone, with "apex" X there is a unique morphism $! : X \to a \times b$ such that $f = p_1 \circ !$ and $g = p_2 \circ !$.

Examples of colimits are *direct sums* (in **SET**, disjoint unions) and, somewhat confusingly called, *direct limits*, which in a self-contained description we will use to define *global solutions*.

Besides capturing interesting constructions common to many fields of Mathematics, Category Theory also provides tools for relating different categories to one another. This is achieved by means of mappings called *functors*. Given two categories **C** and **D** a functor F from **C** to **D** maps objects from **C** into objects of **D** as well as arrows from the former to the latter category such that, if

$$f : a \to b$$

in **C**, then:

$$F(f) : F(a) \to F(b)$$

in **D**. Furthermore $F(g \circ f) = F(g) \circ F(f)$ and $F(\mathrm{Id}_a) = \mathrm{Id}_{F(a)}$ for every object a in **C**.

These functors are called *covariant*. Another class, that of *contravariant* functors, is such that, if

$$f : a \to b$$

in **C**, then:

$$F(f) : F(a) \leftarrow F(b)$$

in **D**. Of particular interest are the contravariant functors $F : \mathbf{C} \to \mathbf{SET}$ (or a category of subsets of a given set), which are called *presheaves*. An intuitive interpretation is that given a morphism $a \to b$ in **C**, the morphism $F(b) \to F(a)$ in **SET** is the *restriction* of the "image" under F of b over the "image" of a. Given an object a in **C**, $F(a)$ is called a *section* of F over a. This can be extended to any family $B = \{b_j\}_{j \in \mathcal{J}}$ of objects in **C**: $F(B)$ is the section over B. In turn, given two families $B \subseteq B'$ and the section over B', namely $F(B')$ we can find its restriction over B, denoted $F(B')_{|B}$, yielding $F(B)$.

Given a presheaf $F : \mathbf{C} \to \mathbf{SET}$, consider a class of objects B in **C** and a cover $\{K_j\}_{j \in \mathcal{J}}$ (i.e. $B \subseteq \bigcup_{j \in \mathcal{J}} K_j$). Let $\{k_j\}_{j \in \mathcal{J}}$ be a sequence such that $k_j \in F(K_j)$ for each $j \in \mathcal{J}$. The presheaf F is said to be a *sheaf* if the following conditions are fulfilled:

- *Locality*: For every pair $i, j \in \mathcal{J}$, $k_{i|_{K_i \cap K_j}} = k_{j|_{K_i \cap K_j}}$ (i.e. the sections a_i, a_j coincide over $V_i \cap V_j$),
- *Gluing*: There exists a unique $\bar{b} \in F(B)$ such that $\bar{b}_{|K_j} = k_j$ for each $j \in \mathcal{J}$ (i.e. there exists a single object in the "image" of B that when restricted to each set in the covering yields the section corresponding to that set).

Another categorical notion that will be relevant in the next sections is that of a *symmetric monoidal category* (SMC). A category $\mathbf{C}$ is SMC if

- There exists an object $I \in \mathrm{Ob}(\mathbf{C})$ called the *monoidal unit*.
- There exists a functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, called the *monoidal product*, such that:
  - $I \otimes c \cong c \cong c \otimes I$ for every $c \in \mathrm{Ob}(\mathbf{C})$,
  - $(c \otimes d) \otimes e \cong c \otimes (d \otimes e)$ for every $c, d, e \in \mathrm{Ob}(\mathbf{C})$, and
  - $c \otimes d \cong d \otimes c$ for every $c, d \in \mathrm{Ob}(\mathbf{C})$.

Consider two monoidal categories $\mathbf{C}$ and $\mathbf{D}$ with monoidal products, $\otimes_{\mathbf{C}}$ and $\otimes_{\mathbf{D}}$, and monoidal units $I_{\mathbf{C}}$ and $I_{\mathbf{D}}$, respectively. A *lax monoidal functor* is a functor $F : \mathbf{C} \to \mathbf{D}$ together with a natural transformation

$$\phi_{XY} : F(X) \otimes_{\mathbf{D}} F(Y) \to F(X \otimes_{\mathbf{C}} Y)$$

and a morphism $\phi : I_{\mathbf{D}} \to F(I_{\mathbf{C})}$.

If $(\mathbf{C}, I, \otimes)$ is a symmetric monoidal category we can define an *operad* $\mathcal{O}_{\mathbf{C}}$ as follows:

- $\mathrm{Ob}\,(\mathcal{O}_{\mathbf{C}}) = \mathrm{Ob}(\mathbf{C})$.
- A morphism $(X_1, \ldots, X_n) \to Y$ in $\mathcal{O}_{\mathbf{C}}$ is defined as the morphism $X_1 \otimes \cdots \otimes X_n \to Y$ in $\mathbf{C}$.

Equipped with these notions we can consider a category $\mathcal{WD}$ such that:

- Each object is a *box* $X = \left(X^{\text{in}}, X^{\text{out}}\right)$, where $X^{\text{in}}, X^{\text{out}}$ are *typed* finite sets. Each element of $X^{\text{in}} \sqcup X^{\text{out}}$ is called a *port*.
- A morphism between two boxes X and Y is called a *wiring diagram* $\varphi : X \to Y$, such that $\varphi = \left(\varphi^{\text{in}}, \varphi^{\text{out}}\right)$ are defined as follows:

$$\varphi^{\text{in}} : X^{\text{in}} \longrightarrow Y^{\text{in}} \sqcup X^{\text{out}}$$
$$\varphi^{\text{out}} : Y^{\text{out}} \longrightarrow X^{\text{out}}$$

  where $\sqcup$ denotes the disjoint union of sets.

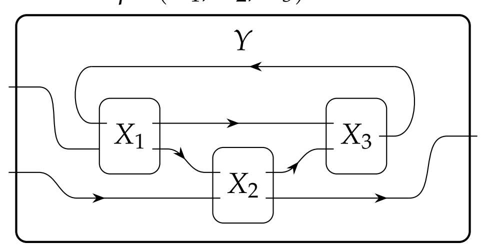
- Given two wiring diagrams $\varphi : X \to Y$ and $\psi : Y \to Z$ their composition makes the following diagrams commutative:

$$\begin{array}{ccc} X^{\text{in}} & \xrightarrow{(\psi\circ\phi)^{\text{in}}} & Z^{\text{in}} \sqcup X^{\text{out}} \\ \downarrow{\scriptstyle \phi^{\text{in}}} & & \uparrow{\scriptstyle Z^{\text{in}} \sqcup \phi^{\text{out}} \sqcup X^{\text{out}}} \\ Y^{\text{in}} \sqcup X^{\text{out}} & \xrightarrow[\psi^{\text{in}\sqcup X^{\text{out}}}]{} & Z^{\text{in}} \sqcup Y^{\text{out}} \sqcup X^{\text{out}} \end{array} \qquad \begin{array}{ccc} Z^{\text{out}} & \xrightarrow{(\psi\circ\phi)^{\text{out}}} & X^{\text{out}} \\ & {\scriptstyle \psi^{\text{out}}}\searrow & \uparrow{\scriptstyle \phi^{\text{out}}} \\ & & Y^{\text{out}} \end{array}$$

$\mathcal{WD}$ has a symmetric monoidal structure, where $\otimes$ is identified with $\sqcup : \mathcal{WD} \times \mathcal{WD} \to \mathcal{WD}$ while the unit I is $\emptyset$ (the box with an empty set of ports). Then, an operad $\mathcal{O}_{\mathcal{WD}}$ can be defined, to allow for the possibility of connecting different boxes into a single one.

For example, consider the morphism $\varphi : (X_1, X_2, X_3) \longrightarrow Y$ in $\mathcal{O}_{\mathcal{WD}}$. It can be depicted as follows:

$\varphi : (X_1, X_2, X_3) \to Y$



Another categorical formalism to be applied in this paper is that of *polynomial functors*. Since it is quite central for our argument we leave its presentation for section 6, in which we develop a unified level-agnostic model.

3 Sub-agents: Local vs. Global

The usual specification of decision-making under certainty by an agent starts with a space of possible **options**, $\mathcal{L}$ and a *utility* function, $U : \mathcal{L} \to \mathbb{R}$. Constraints on the set of options limit the available options to $\hat{L} \subseteq \mathcal{L}$. The agent seeks to find $\mathbf{x}^*$ maximizing U

over $\hat{L}$.

The space of options, $\mathcal{L}$, is a (real) Hilbert space, i.e. a complete metric space with an inner product. To ensure the existence of a $\mathbf{x}^*$, it is assumed that $\hat{L}$ is a compact subset of $\mathcal{L}$ and that U is a continuous function.

In a category-theoretical treatment of the global optimization of U over $\hat{L}$, $\mathbf{x}^*$ is represented as a *direct limit*. This approach allows, furthermore, to analyze the problem of obtaining a global result from local ones.

Consider first a family $\{L^k\}_{k=0}^{\kappa}$ of closed linear subspaces of $\mathcal{L}$ and, for any given k, the function

$$\mathrm{Proj}_k : \mathcal{L} \to \bigcup_{k=0}^{\kappa} L^k$$

such that $\mathrm{Proj}_k(x) = x^k \in L^k$, where x^k is the *projection* of x on L^k.[3]

Each L^k is the set of options of a *local* problem. The projection of a global solution $\mathbf{x}^*$ onto L^k will return the point in L^k which is the closest to $\mathbf{x}^*$. If the projection does not return a local solution, another operator can be defined, $\Gamma_k : \hat{L} \to \hat{L}^k$ yielding choices closest to the projection, if it does not belong to the subspace:

$$\Gamma_k(x) = \{x^k \in \hat{\mathbf{X}}^k : x^k \in \mathrm{argmin}_{y \in \hat{\mathbf{X}}^k} |y - \mathrm{Proj}_k(x)|\}.$$

If the global solution is not given, it must be sought by gluing together local ones. To formalize this we will introduce a category of local problems ([26]).

Definition 1 *Let $\mathcal{PR}$ be the category of local problems, where*

- *Obj($\mathcal{PR}$) is the class of objects. Each one, $s^k = \langle \hat{L}^k, u^k, \hat{\mathbf{X}}^k \rangle$ involves the maximization of the continuous utility function u^k over the compact set $\hat{L}^k \subset L^k$, a closed linear subspace of $\mathcal{L}$, yielding a family of solutions $\hat{\mathbf{X}}^k$.*
- *a morphism $\rho_{kj} : s^k \to s^j$ is defined as $\hat{L}^k \subseteq \hat{L}^j$, $u^k = u^j|_{L^k}$ and $dim(L^k) \leq dim(L^j)$.[4] It follows from this definition that an identity morphism $\rho_{kk} : s^k \to s^k$ trivially exists for every object s^k. Furthermore, given two morphisms $\rho_{kj} : s^k \to s^j$ and $\rho_{jl} : s^j \to s^l$ there exists*

[3] The existence of a projection is ensured by a straightforward application of the Linear Projection Theorem, according to which $|x - x^k| = \min_{y \in L^k} |x - y|$, where $|\cdot|$ is the norm of $\mathcal{L}$ [15].

[4] $\dim(\cdot)$ yields the dimension of a subspace of $\mathcal{L}$.

their composition $\rho_{jl} \circ \rho_{kl} = \rho_{kl}$, since $\hat{L}^k \subseteq \hat{L}^j \subseteq \hat{L}^l$, $dim(L^k) \leq dim(L^j) \leq dim(L^l)$ and by transitivity of the restrictions $u^k = u^j|_{L^k}$ and $u^j = u^l|_{L^j}$ we have that $u^k = u^l|_{L^k}$.

We also define $\mathcal{P}(\mathcal{L})$ as the category in which the objects are subsets of $\mathcal{L}$ and a morphism between two objects $f_{AB} : A \rightarrow B$ is defined as $A \subseteq B$.

Consider now a functor

$$\Sigma : \mathcal{PR} \longrightarrow \mathcal{P}(\mathcal{L})$$

which assigns to a problem $s^k = \langle \hat{L}^k, u^k, \hat{\mathbf{X}}^k \rangle$ the subset $\Sigma(s^k)$ of $\mathcal{L}$ defined by (see Figure 3)

$$\Sigma(s^k) = \{y \in \mathcal{L} \mid \Gamma_k(y) \in \hat{\mathbf{X}}^k\}.$$

A *section* σ_k over s^k is the assignment of the elements of $\Sigma(s^k)$ to s^k:

$$\sigma_k : s^k \mapsto \Sigma(s^k).$$

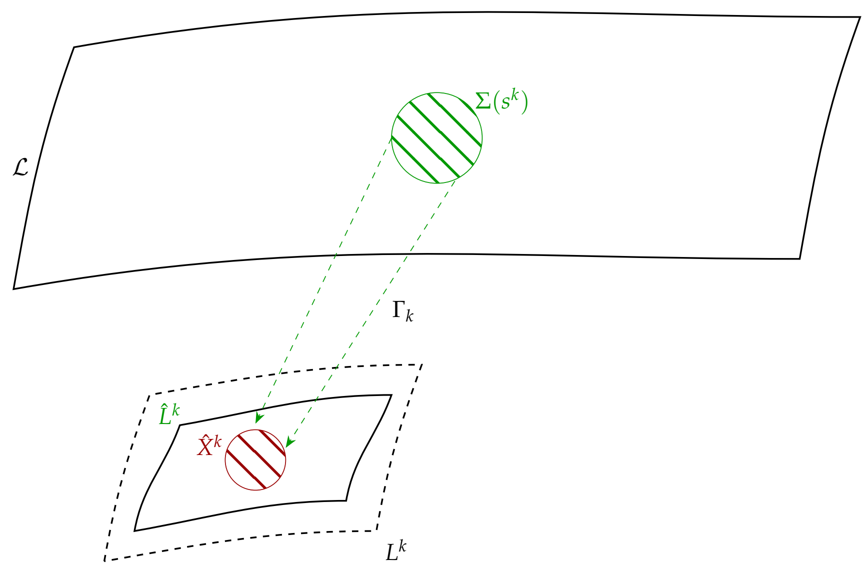


Figure 3: Representation of the relation between Γ_k and $\Sigma(s^k)$.

Given two problems, each one identified with a **sub-agent** in charge of solving it, $s^k = \langle \hat{L}^k, u^k, \hat{\mathbf{X}}^k \rangle$ and $s^j = \langle \hat{L}^j, u^j, \hat{\mathbf{X}}^j \rangle$, let us write $s^k \triangleleft s^j$ iff there exists a morphism ρ in $\mathcal{PR}$,

$\rho : s^k \to s^j$. That is, s^k is a restriction of s^j.

Let us define $r^j_k : \Sigma(s^j) \to \Sigma(s^k)$ such that to $\Sigma(s^j)$ it assigns $\Sigma(s^k)$. Given a section over s^j, r^j_k yields a section corresponding to its sub-problem s^k.

The following proposition shows that the functor Σ possesses an important property that is crucial for formalizing the possibility of patching up local problems and yielding a "larger" one:

Proposition 1 *Σ is a presheaf.*

Proof: *$\Sigma : \mathcal{PR} \to \mathcal{P}(\mathcal{L})$ is a functor. We can analyze its behavior by means of r^j_k:*

- *For any $s^k \in Obj(\mathcal{PR})$, since $s^k \lhd s^k$, $r^k_k = Id_{\Sigma(s^k)}$.*
- *If $s^k \lhd s^j \lhd s^l$ then $s^k \lhd s^l$. Thus, $r^j_k \circ r^l_j = r^l_k$.*

This means that $\Sigma : \mathcal{PR} \to \mathcal{P}(\mathcal{L})$ is a contravariant *functor. Or, in categorical terms, a presheaf.* $\square$

Consider now a family $\{s^k = \langle \hat{L}^k, u^k, \hat{\mathbf{X}}^k \rangle\}_{k \in K} \subseteq \text{Obj}(\mathcal{PR})$. It is said to be a *cover* of an object $s^j = \langle \hat{L}^j, u^j, \hat{\mathbf{X}}^j \rangle$ of $\text{Obj}(\mathcal{PR})$ if $s^k \lhd s^j$ for each $k \in K$ and $\hat{L}^j \subseteq \cup_{k \in K} \hat{L}^k$. That is, a problem s^j gets covered by the family $\{s^k\}_{k \in K}$ if the domain of problem s^j is included in the union of the domains of the problems of the family and furthermore, each s^k is a restriction of s^j.

The family of sections $\{\sigma_k\}_{k \in K}$ is said to be *compatible* if for any pair $k, l \in K$, given $\Sigma(s^k) = X^k$ and $\Sigma(s^l) = X^l$ (see Figure 4),

$$\Gamma_k(X^k) \cap \Gamma_l(X^k) = \Gamma_k(X^l) \cap \Gamma_l(X^l).$$

Given a cover $\{s^k\}_{k \in K}$ of a problem s^j with compatible sections, Σ satisfies the *sheaf property* if there exists a unique $\sigma_j = \Sigma(s^j)$ such that for each $k \in K$,

$$\sigma_k = \sigma_j \cap \Gamma_k^{-1}(\hat{L}^k).$$

That is, intuitively, the sheaf property is satisfied if σ_j in fact "glues" together all the assignments σ_k in $\mathcal{P}(\mathcal{L})$ (see Figure 5).

Summarizing the discussion up to this point, we can say that given a category of problems $\mathcal{PR}$ over a space $\mathcal{L}$, they can be seen as instances of a global one if there exists

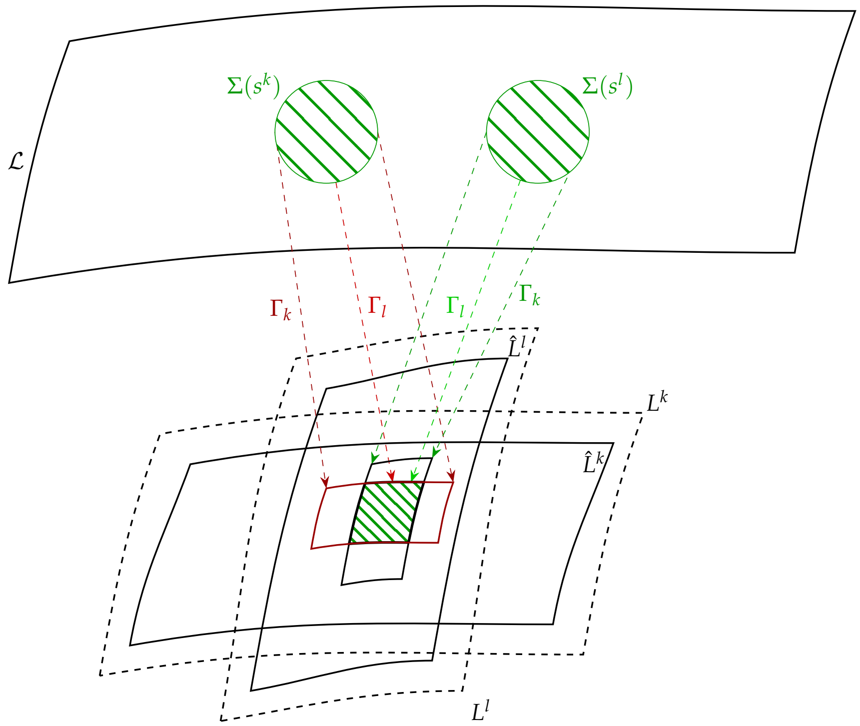


Figure 4: Compatibility of sections.

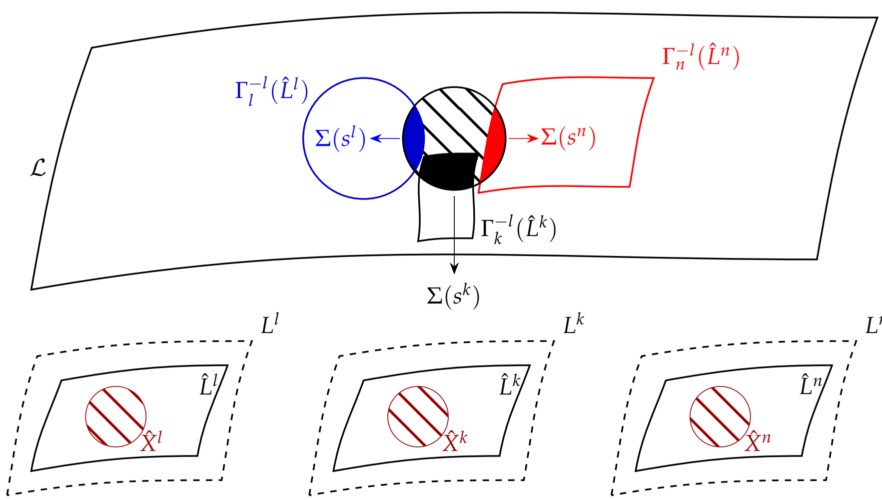


Figure 5: Sheaf property.

a presheaf $\Sigma : \mathcal{PR} \to \mathcal{P}(\mathcal{L})$, satisfying the sheaf property. Then, for any problem $\mathbf{s}^j$, covered by any compatible family of sub-problems, $\{s^k\}_{k\in K}$, $\Sigma(\mathbf{s}^j) \cap \Gamma_k^{-1}(\hat{L}^k) = \Sigma(s^k)$ for $k \in K$.

That is, the sheaf property ensures that the behavior of the **sub-agents** is consistent with that of the single **agent**.

4 A Categorical Representation of Games

Let us now consider, instead of the coordination of different local decision problems, the coordination of games. That is, decision problems involving several agents, instead of a single one. Thus, the approach discussed in this section generalizes the sheaf-theoretical framework presented above.[5]

Let us consider a category $\mathcal{G}$ of *games*. Each object G in the category corresponds to a game $G = \langle (I_G, S_G, \mathbf{O}_G, \rho_G), \pi_G \rangle$, where

- $(I_G, S_G, \mathbf{O}_G, \rho_G)$ is a game form:
  - I_G is the class of players.
  - $S_G = \prod_{i\in I_G} S_i^G$ is the *strategy set* of the game, where $S_i^G \subseteq S_i$ is the set of strategies that player i can deploy in game G, for each $i \in I_G$.[6]
  - $\mathbf{O}_G$ is the class of *outcomes* of the game and $\rho_G : S_G \to \mathbf{O}_G$ is a one-to-one function that associates each profile of strategies in the game with one of its outcomes.
- $\pi_G = \prod_{i\in I} \pi_i^G$ is a *profile of payoff functions*, where $\pi_i^G : \mathbf{O}_G \to \mathbb{R}^+$ is the payoff function of player i in game G, for each $i \in I_G$.

A game is defined in terms of the interactions of *players*. Each player can be seen as described in terms of the strategies she can play and the payoffs she can receive from the results of her action (jointly with those of the other players).

We can define a category $\mathcal{G}$, where the objects are games. Given two games

$$G = \langle (I_G, S_G, \mathbf{O}_G, \rho_G), \pi_G \rangle \quad \text{and} \quad G' = \langle (I_{G'}, S_{G'}, \mathbf{O}_{G'}, \rho_{G'}), \pi_{G'} \rangle,$$

[5] Alternative category-theoretic approaches to Game Theory were presented, for instance, in [7] and [28].

[6] S_i is the set of all the strategies that player i can play in the games in which she participates.

a morphism of games

$$G \to G'$$

is such that:

- $I_G \subseteq I_{G'}$.
- $S_i^G \subseteq S_i^{G'}$ for each $i \in I_G$.
- $\mathbf{O}_G \subseteq \mathbf{O}_{G'}$.

Thus, if a morphism $G \to G'$ exists, G can be conceived as a *subgame form* of G'.

To complete the characterization of $\mathcal{G}$ notice that it is immediate that we can define *pushouts* and an *initial object* in this category:

- **Pushouts**: Consider three objects G, G' and G'' and morphisms $G \xrightarrow{f} G'$ and $G \xrightarrow{g} G''$. Then, take the coproduct of G' and G'', denoted $G' + G''$, obtained as the direct sums of the strategies sets and the outcomes of both games. By identifying the subgame forms of G' and G'' corresponding to G we obtain the *pushout* of

$$G' \ \stackrel{f}{\leftarrow} \ G \ \stackrel{g}{\rightarrow} \ G''$$

- **Initial object**: Consider the *empty game* $G^{\emptyset}$, where $I_{G^{\emptyset}} = \emptyset$ and consequently $S_{G^{\emptyset}} = \emptyset$ and $\mathbf{O}_{G^{\emptyset}} = \emptyset$ (thus $\pi_{G^{\emptyset}}$ must be the empty function). It is immediate to see that $G^{\emptyset} \to G$ for every G in $\mathcal{G}$.

Then we have

Proposition 1 *$\mathcal{G}$ is a category with* colimits*.*

Since $\mathcal{G}$ is a category with colimits we can define *cospans* in it. Consider again three objects G, G' and G'' and two morphisms $G \xrightarrow{f} G'' \stackrel{g}{\leftarrow} G'$. This is called a cospan from G to G'. The interpretation of such a cospan is that G and G' are subgame forms of the same game (G'').

We can conceive each game G in $\mathcal{G}$ as a *box*, $G = (\mathrm{in}^G, \mathrm{out}^G)$, where in^G and out^G are, respectively *input* and *output* ports. in^G has type $\mathbf{O}_G$, i.e. the input is an outcome of G. In turn, the out^G port has type S_G, being each output a profile in G.

Notice that each player i can be conceived as a game $(\mathrm{in}^i, \mathrm{out}^i)$, where in^i has type $\cup_{G:i\in I_G}\mathbf{O}_G$ and out^i has type S_i.

Up to this point, our definition of morphisms in $\mathcal{G}$ does not involve the payoffs. They can be incorporated by redefining the games as *modal boxes*, in which an additional component is the set of *internal states* of the game. More precisely, given any G and the class of its internal states, Σ_G, we can identify G as a triple $\langle \mathrm{in}^G, \mathrm{out}^G, \Sigma_G\rangle$, associated to two correspondences:

- **payoff**: $\phi^1_G : \bar{\mathrm{in}}^G \times \Sigma_G \to \mathbb{R}^{+\mathbf{O}_G}$, such that for the vector $o \in \bar{\mathrm{in}}^G$ (the vector of all possible inputs of G, each entry being an outcome of the game) and state σ, $\phi^1_G(o,\sigma) = (\pi^i_G(o))_{o\in\mathbf{O}_G}$. That is, it yields the vector of payoffs corresponding to all the outcomes of G.
- **choice**: $\phi^2_G : \Sigma_G \to \bar{\mathrm{out}}^G$, such that for any state σ, $\phi^2_G(\sigma) = s \in \bar{\mathrm{out}}^G$ (the class of all possible strategy profiles in S_G) is a profile of strategies that may be chosen at that state.

Particularly relevant for our analysis is the definition of the internal states of each player i, Σ_i. Consider a game G such that $i \in I_G$, and a sequence of morphisms in $\mathcal{G}$

$$G^0_i \to G^1_i \to \quad \dots \quad \to G^{n-1}_i \to G^n_i$$

where G^0_i is a game in which i is the only player and $G = G^n_i$. We identify the state of player i when playing G as a sequence $\sigma^i_G = \langle \sigma^i_0, \dots, \sigma^i_{n-1}\rangle$, where $\sigma^i_k \in \Sigma_{G^k_i}$, for $k = 0 \dots, n-1$. Then, a distinguished object $\sigma^i_* \in \Sigma_i$ is defined, such that σ^i_G is one of its initial segments.[7]

Therefore, for each game G, σ^i_* can be instantiated yielding the corresponding state, and therefore the payoffs and the choices of player i in the game. The state σ_G of the entire game just obtains as the profile of states of its players.

A simple example is $\sigma^i_{G^n}$ yielding as payoff for i the product of the payoffs she gets in the subgames of G^n. This case will be elaborated a bit more in Example 1, below.

We can define the category of cospans in $\mathcal{G}$, denoted $\mathrm{cospan}_{\mathcal{G}}$ which has a symmetric monoidal structure. Its objects are the same as those of $\mathcal{G}$ and a morphism $G \overset{h}{\to} G'$ is a cospan from G to G', indicating that there exists a game of which G and G' are subgame

[7] Thus, σ^i_* has a *forest* structure.

forms. Thus, morphisms in $\text{cospan}_{\mathcal{G}}$ are actually isomorphisms.

Given two morphisms in $\text{cospan}_{\mathcal{G}}$, $G \xrightarrow{f} G'$ and $G' \xrightarrow{g} G''$ there exists a morphism $G \xrightarrow{g \circ f} G''$ that obtains as a composition of the corresponding cospans.

The monoidal structure of $\text{cospan}_{\mathcal{G}}$ is given by:

- The unit is $G^{\emptyset}$, the initial object in $\mathcal{G}$.
- The monoidal product of G and G', is the *coproduct* $G + G'$.

We now present a diagram language for open games. We start by considering the symmetric monoidal category $\mathbf{W}_{\mathcal{G}}$. By definition, we have that:

$$\mathbf{W}_{\mathcal{G}} \;=\; \text{cospan}_{\mathcal{G}}.$$

Each object, i.e. a game G, is seen as a $\langle \text{in}^G, \text{out}^G, \Sigma_G \rangle$-labeled *interface*, satisfying ϕ^1_G and ϕ^2_G. On the other hand, morphisms $G \rightarrow C \leftarrow G'$, are called $\langle \text{in}, \text{out}, \Sigma \rangle$-labeled *wiring diagrams*. The interpretation is that C is the overarching game that connects the subgames (not just the game forms) G and G'.

We write $\psi : G_1, G_2, \ldots, G_n \rightarrow \bar{G}$ to denote the wiring diagram $\phi : G_1 + G_2 + \ldots + G_n \rightarrow \bar{G}$. We can, in turn see this as

$$G_1 + G_2 + \ldots + G_n \xrightarrow{f} C \xleftarrow{\bar{f}} \bar{G}$$

which indicates that, being f and $\bar{f}$ isomorphisms,

Proposition 2 *$\bar{G}$ is the minimal game that includes the direct sum of $G_1, \ldots, G_n$ as a subgame.*

In $\mathbf{W}_{\mathcal{G}}$ the monoidal product of G and G', $G \otimes G'$ is defined as follows (where $\cup$ and $\sqcup$ represent set union and disjoint union of *sets*, respectively):

- $I_{G \otimes G'} = I_G \cup I_{G'}$.
- $\mathbf{O}_{G \otimes G'} = \mathbf{O}_G \sqcup \mathbf{O}_{G'}$.
- For each $i \in I_{G \otimes G'}$

$$S_i^{G \otimes G'} = \begin{cases} S_i^G & \text{if } i \in I_G \backslash I_{G'} \\ S_i^{G'} & \text{if } i \in I_{G'} \backslash I_G \\ S_i^G \times S_i^{G'} & \text{if } i \in I_G \cap I_{G'} \end{cases}$$

- $\pi_i^{G\otimes G'}(s) = \pi_i^G\left(s^G\right) + \pi_i^{G'}\left(s^{G'}\right)$, where $s^G, s^{G'}$ are the projections of $s \in \prod_{j\in I_{G\otimes G'}} S_j^{G\otimes G'}$.

5 Hypergraph Categories and Equilibria

We define a *hypergraph category* $\langle \mathcal{G}, \mathrm{Eq}\rangle$ with $\mathrm{Eq} : \mathbf{W}_{\mathcal{G}} \to \prod_i S_i$, such that, for every object G in $\mathbf{W}_{\mathcal{G}}$, $\mathrm{Eq}(G)$ is a class of vectors in $\prod_{i\in I} S_i^G$, the strategy set of game G. We assume that $\mathrm{Eq}(G)$ is a class of *equilibria* of G, for some notion of equilibrium (as for instance, dominant strategies equilibrium, admissible strategies, or Nash equilibrium).

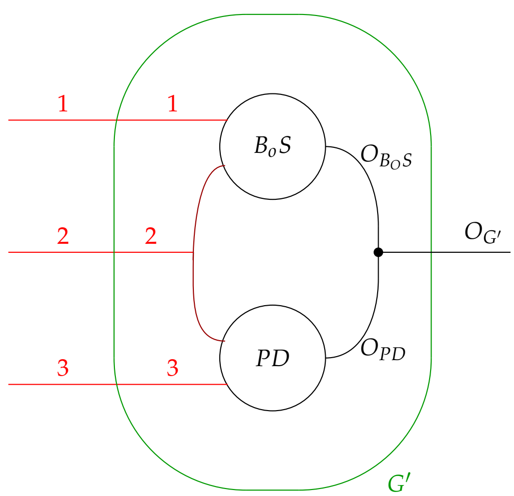
Example 1 *Consider two games, G between players* 1 *and* 2*:*[8]

	Bx	*Bll*
Bx	2,1	0,0
Bll	0,0	1,2

and G' between players 2 *and* 3*:*[9]

	C	*D*
C	2,2	0,3
D	3,0	1,1

The corresponding wiring diagram is:



[8]This a *Battle of the Sexes* game, where $S_1 = S_2 = \{\mathrm{Bx}, \mathrm{Bll}\}$.
[9]A *Prisoner's Dilemma*, where $S_2 = S_3 = \{C, D\}$.

In red we have highlighted $Eq(G) = \{(Bx,Bx),(Bll, Bll)\}$ and $Eq(G') = \{(D,D)\}$, where Eq corresponds to Nash equilibrium.[10]

Let us represent now $G \otimes G'$. We start by building its corresponding game form. We obtain two tables, where the first one corresponds to player 3 choosing C:

	$Bx \bowtie C$	$Bx \bowtie D$	$Bll \bowtie C$	$Bll \bowtie D$
Bx	$o_{1,1}$	$o_{1,2}$	$o_{1,3}$	$o_{1,4}$
Bll	$o_{2,1}$	$o_{2,2}$	$o_{2,3}$	$o_{2,4}$

and another corresponding to player 3 choosing D:

	$Bx \bowtie C$	$Bx \bowtie D$	$Bll \bowtie C$	$Bll \bowtie D$
Bx	$o'_{1,1}$	$o'_{1,2}$	$o'_{1,3}$	$o'_{1,4}$
Bll	$o'_{2,1}$	$o'_{2,2}$	$o'_{2,3}$	$o'_{2,4}$

For instance, o_{11} indicates that 1 and 2 go to Box and 2 and 3 Cooperate. On the other hand, $o'_{1,1}$ indicates that, again 1 and 2 go to Box, but while 2 keeps Cooperating, 3 Defects. The other entries can be interpreted likewise.

Suppose that the internal states of the players, σ^1_*, σ^2_* and σ^3_* are such that instantiated on $G \otimes G'$ yield the following payoffs and choices:

If 3 chooses C:

	$Bx \bowtie C$	$Bx \bowtie D$	$Bll \bowtie C$	$Bll \bowtie D$
Bx	$2,1 \times 2,2$	$2,1 \times 3,0$	$0,0 \times 2,2$	$0,0 \times 3,0$
Bll	$0,0 \times 2,2$	$0,0 \times 3,0$	$1,2 \times 2,2$	$1,2 \times 3,0$

while if 3 chooses D:

	$Bx \bowtie C$	$Bx \bowtie D$	$Bll \bowtie C$	$Bll \bowtie D$
Bx	$2,1 \times 0,3$	$2,1 \times 1,1$	$0,0 \times 0,3$	$0,0 \times 1,1$
Bll	$0,0 \times 0,3$	$0,0 \times 1,1$	$1,2 \times 0,3$	$1,2 \times 1,1$

In words, players 1 and 3 keep the payoffs they get in the subgames, while 2 takes the product of the payoffs in G and G'. In red, we have highlighted the equilibria of $G \otimes G'$, under this specification.

[10]Notice that here player 2, participates in two games.

Let us define an operation $\hat{\cup}$ such that given two equilibria $s \in \mathrm{Eq}(G)$ and $s' \in \mathrm{Eq}(G')$, yields a new profile $s \bowtie s' \in \mathrm{Eq}(G)\hat{\cup}\mathrm{Eq}(G')$ verifying that for each player $i \in I_G \cap I_{G'}$, a new strategy obtains combining s_i and s'_i, while in on all other cases the individual strategies are the same as in G and G'. Furthermore, $\pi_i^{G\hat{\cup}G'}(s \bowtie s') = \pi_i^G(s) \times \pi_i^{G'}(s')$ for $i \in I_G \cap I_{G'}$.[11]

In our example, since $\mathrm{Eq}(G \otimes G') = \{(\mathrm{Bx}, \mathrm{Bx} \bowtie \mathrm{D}, \mathrm{D}), (\mathrm{Bll}, \mathrm{Bll} \bowtie \mathrm{D}, \mathrm{D})\}$, we have that

$$\mathrm{Eq}(G)\hat{\cup}\mathrm{Eq}(G') = \mathrm{Eq}(G \otimes G').$$

This example illustrates the following claim:

Proposition 3 *For any pair of games G and G', $Eq(G)\hat{\cup}Eq(G') = Eq(G \otimes G')$.*

Proof: *Trivial. If $I_G \cap I_{G'} = \emptyset$, $G \otimes G' = G \cup G'$ with $G \cap G' = \emptyset$. Thus, each equilibrium of $G \otimes G'$ is just the disjoint combination of equilibria in G and G'.*

If, on the other hand, $I_G \cap I_{G'} \neq \emptyset$, given $i \in I_G \cap I_{G'}$, her strategy set in $G \otimes G'$ is $S_i^G \times S_i^{G'}$, where S_i^G and $S_i^{G'}$ are her strategy sets in G and G', respectively. Now suppose that s_i^G and $s_i^{G'}$ are equilibrium strategies of i in the individual games but that $(s_i^G, s_i^{G'})$ does not belong to an equilibrium in $G \otimes G'$.

Then, there exist an alternative combined strategy $(\hat{s}_i^G, \hat{s}_i^{G'})$ such that on the new profile π_i yields a higher payoff, but since this equilibrium can be decomposed in two profiles, one in G and the other in G', the payoff of i is the product of the payoffs over those two profiles. But then either $\hat{s}_i^G$ yields a higher payoff than s_i^G or $\hat{s}_i^{G'}$ yields a higher payoff than $s_i^{G'}$ (recall that they are all positive real numbers).

Thus, either s_i^G or $s_i^{G'}$ is not an equilibrium in the corresponding game. Absurd. $\square$

Proposition 3 indicates that there exist a trivial *natural isomorphism*

$$\mathrm{Eq}(G)\hat{\cup}\mathrm{Eq}(G') \ \rightarrow \ \mathrm{Eq}(G \otimes G').$$

[11] An alternative yielding also Proposition 3 obtains if, instead, we take $\pi_i^{G\hat{\cup}G'}(s \bowtie s') = \pi_i^G(s) + \pi_i^{G'}(s')$ for $i \in I_G \cap I_{G'}$.

Furthermore, taking the unit in $\prod_i S_i$ to be the empty set, we have also that $\emptyset = \text{Eq}(G^{\emptyset})$, where $G^{\emptyset}$ is the initial object in $\mathcal{G}$ and thus in $\mathbf{W}_{\mathcal{G}}$.

Recalling the definition of a *lax monoidal functor* as a functor $F : \mathbf{C} \to \mathbf{D}$ together with a natural transformation

$$F(X) \otimes_{\mathbf{D}} F(Y) \to F(X \otimes_{\mathbf{C}} Y)$$

we have, trivially, that

Proposition 4 *Eq is a lax monoidal functor.*

Thus, the corresponding algebra associates the composition of games with the equilibria of the components.

Proposition 4 depends critically on the possibility of defining $\hat{\cup}$ in terms of a function $\mathbf{f}$, defined as follows. Given a player $i \in I_G \cap I_{G'}$, a combined strategy $s_i \bowtie s'_i$ is such that for $s = (s_i, s_{-i}) \in \text{Eq}(G)$ and $s' = (s'_i, s'_{-i}) \in \text{Eq}(G')$, satisfying $\pi_i(s \bowtie s') = \mathbf{f}(\pi_i^G(s), \pi_i^{G'}(s'))$ and with $s \bowtie s' \in \text{Eq}(G \otimes G')$. As we saw above if $\mathbf{f}$ is the arithmetic product or sum, Eq will be indeed a lax monoidal functor.

But this restricts the compositionality of games to just trivial cases. We are interested in more general and non-obvious cases. To do that consider an alternative characterization of the hypergraph category $\langle \mathcal{G}, \text{Eq} \rangle$:

$$\text{Eq} : \text{W}_{\mathcal{G}} \to \prod_i S_i \times \cup_{G \in \mathbf{Obj}(\mathcal{G})} \Sigma_G$$

Furthermore, we need another definition of $\hat{\cup}$:

$$\otimes : (\prod_i S_i \times \cup_{G \in \mathbf{Obj}(\mathcal{G})} \Sigma_G) \ \times \ (\prod_i S_i \times \cup_{G \in \mathbf{Obj}(\mathcal{G})} \Sigma_G) \ \ \to \ \ \prod_i S_i \times \bigcup_{G \in \mathbf{Obj}(\mathcal{G})} \Sigma_G$$

such that given two games G and G' with $s \in \prod_{i \in I_G} S_i$ and σ_G, and $s' \in \prod_{i \in I_{G'}} S_i$ and $\sigma_{G'}$ we have:

$$(s, \sigma_G) \hat{\cup} (s', \sigma_{G'}) = (\bar{s}, \sigma_{G+G'}) \in \prod_{i \in I_{G+G'}} S_i \times \Sigma_{G+G'}$$

where $\bar{s} \in S_{G+G'}$ is a Nash equilibrium if and only if s and s' are Nash equilibria of G and G' respectively.

$\hat{\cup}$ is well-defined. To see this, just recall that, by definition $G + G'$ obtains in terms of the game forms of G and G' (the strategy sets and the outcomes), allowing different possible internal states and thus payoffs. The view of games as boxes presented in Section 4 indicates that there exist sequences of internal states of games, in parallel to sequences of morphisms between games, allowing to define $\sigma_{G+G'}$, and thus payoffs that make $\bar{s}$ a Nash equilibrium if s and s' are also equilibria.

We can see that $\prod_i S_i \times \bigcup_{G \in \mathrm{Obj}(\mathcal{G})} \Sigma_G$ with $\hat{\cup}$, defined as above can be seen as a monoidal category, with morphisms defined in terms of those of $\mathcal{G}$, with $(\emptyset, \emptyset)$ as its initial object. It allows to define Eq in such a way that by definition:

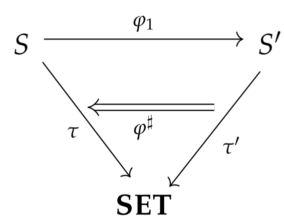
Proposition 5 *Eq is a lax monoidal functor satisfying* $Eq(G + G') = Eq(G) \hat{\cup} Eq(G')$.

6 A more general model

$\langle \mathcal{G}, \mathrm{Eq} \rangle$, in any of the two versions of Eq seems too rigid to capture the dynamics of economic interactions. A more flexible structure is needed.

Let us start with the following category:

- Objects: pairs (S, τ), where $S \in \mathrm{Ob}(\mathbf{Set})$ and $\tau : I \to \mathrm{Set}$.
- Morphisms: $(S, \tau) \xrightarrow{\varphi} (S', \tau')$: pairs $(\varphi_1, \varphi^\sharp)$, such that



  That is, $\varphi_1 : S \to S'$ while $\varphi^\sharp : \tau\prime(s') \mapsto \tau(s)$ for $s' \in S'$ and $s \in S$.

These "two-sided" morphisms generalize the "one-sided" ones we have been considering up to this point. The $\varphi^\sharp$ component facilitates the composition of objects that are somehow incompatible. To show precisely what this means, we present a much more evocative and functional presentation of this category, called **Poly** [17]:

- Each object $p \in \text{Ob}(\mathbf{Poly})$ is written as:

$$p = \sum_{i \in I} y^{p[i]}$$

where each term $y^{p[i]}$ is a functor with domain $p[i]$ into **Set**. Each i can be conceived as a *problem* while $p[i]$ is a set of its *solutions*.

- Given $p = \sum_{i \in I} y^{p[i]}$ and $q = \sum_{j \in J} y^{q[j]}$ a morphism $\phi : p \to q$ is $\phi = (\phi_{\to}, \phi^{\leftarrow})$ such that
  - $\phi^{\to} : I \to J$ and,
  - $\phi^{\leftarrow} : q[\phi^{\to}(i)] \mapsto p[i]$.

We can see how this specification captures the previously given definition of **Poly**. Each $y^{p[i]}$ is identified with $\tau : S \to \mathbf{Set}$, where $S \equiv p[i]$. Then, p represents

$$\sqcup_i \{\tau_i : p[i] \to \mathbf{Set}\}$$

Furthermore, $\phi^{\to}$, which sends problems indexed by I into problems indexed by J, represents φ_1, while $\phi^{\leftarrow}$, which sends the solutions in $q[\phi^{\to}(i)]$ back to the solutions in $p[i]$, corresponds to $\varphi^{\sharp}$.

Interestingly, the usefulness of considering this specification of **Poly** is that we can use it to represent a relation between a class of problems, indexed by I and their solutions $\{p[i]\}_{i \in I}$. Thus, it disregards the codomain of the τ_i's, to just focus on the S_i's and their indexes.

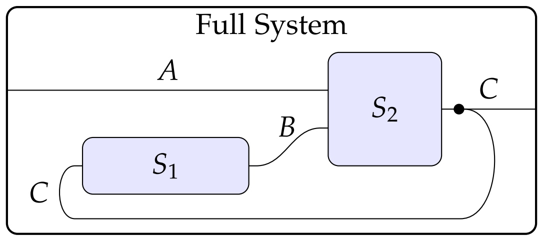
We can conceive any $p \in Ob(\mathbf{Poly})$ as an *interface* between inputs and outputs, being the inputs problems and the outputs their solutions. There are different ways of creating new interfaces up from other interfaces. We focus on the following construction:

- $[p, q] = \sum_{\phi : p \to q} y^{\sum_{i \in I} q[\phi^{\to}(i)]}$, an *internal hom* in **Poly**. It can be seen as a process that takes as inputs (*problems*) the morphisms from p to q and as outputs (*solutions*) all the possible solutions to the images of p in q.

- Given $[p, q]$, a $[p, q] - \mathbf{Coalg}$ is a category in which each object is triple $\langle s, \rho, \mu \rangle$:
  - $s \in S$, where S is a space of *states*, capturing the dynamics of the interface,
  - $\rho : s \mapsto (\phi, i, q[\phi^{\to}(i)])$, where $\phi : p \to q$ is a morphism. That is, it assigns to the current state one of the solutions in $[p, q]$,
  - μ updates the state in response to that pattern, i.e. $\mu(\phi, i, q[\phi^{\to}(i)]) = s' \in S$.

Example 2 *Consider a system in which two subsystems, S_1 and S_2 acting in parallel, are described by $p \simeq By^C \otimes Cy^{AB}$, yielding the full system, represented by $q \simeq Cy^A$.*

For any state $s \in S$ of a $[p, q]$-coalgebra (S, ρ, μ), we have that $\rho(s)$ gives a morphism $p \to q$ in Poly, that can be depicted as:



Given $(a, b, c) \in A \times B \times C, \mu(a, b, c)$ is the updated state in S, which in turn may yield a new connection among S_1 and S_2.

This example shows that $[\cdot, \cdot]$-coalgebras provide flexible and dynamic connections among subsystems. This inspires the following extension of **Poly**, a category $\mathbb{O}$**rg** such that [21]:

- $Ob(\mathbb{O}\mathbf{rg}) = Ob(\mathbf{Poly})$ and,
- $Morph(\mathbb{O}\mathbf{rg}) = [p, q] - \mathbf{Coalg}$.

This means that two interfaces (connecting problems with their solutions) p and q are related by dynamic procedures of reconnection between them.

Our generalized model, covering *both* $\mathcal{PR}$ and $\langle \mathcal{G}, \mathrm{Eq} \rangle$ is a category $\mathcal{I}$ based on $\mathbb{O}$**rg** such that, briefly:

- Each object $a = \left\langle a^{\text{in}}, a^{\text{out}} \right\rangle \in \mathrm{Ob}(\mathcal{I})$ is identified with

$$p_a \simeq a^{\text{out}} y^{a^{\text{in}}} \in \mathrm{Ob}(\mathbb{O}\mathbf{rg})$$

- for objects $a_1, \ldots, a_n, b$ there corresponds a $[p_{a_1} \otimes \ldots \otimes p_{a_n}, p_b] - \mathbf{Coalg}$ of states $S_{a_1, \ldots, a_n, b}$.[12]
- Each object a has an *identity* morphism.

[12]The operation $p_a \otimes p_b$, where $p_a = \sum_{i \in I} y^{p_a[i]}$ and $p_b = \sum_{j \in J} y^{p_b[j]}$, is such that to each problem $(i, j) \in I \times J$ it correspond the pair of solutions to i and j, $(p_a[i], p_b[j])$.

- Pairs of morphisms compose.

The last two requirements indicate, roughly, that morphisms inherit the identity and compositionality properties of $\mathbb{O}$**rg**.

Then, we can prove that $\mathcal{I}$ is a category of level-agnostic dynamic arrangements:

Theorem 1 *There exist two categories $\overline{\mathcal{PR}}$ and $\overline{\mathcal{G}}$, isomorphic to $\mathcal{PR}$ and $\mathcal{G}$, respectively, such that $Ob(\overline{\mathcal{G}}), Ob(\overline{\mathcal{PR}}) \subseteq Ob(\mathcal{I})$ while $Morph(\overline{\mathcal{PR}}), Morph(\overline{\mathcal{G}}) \subseteq Morph(\mathcal{I})$, consist of trivial internal hom coalgebras with single states.*

Proof: *Each problem in $\mathcal{PR}$ can be interpreted as an interface between the problem itself and its optimal solutions. The same applies to any interactive decision-making setting in $\mathcal{G}$.*

More precisely, a local problem $s^k \in Ob(\mathcal{PR})$ and a game $G \in Ob(\langle \mathcal{G}, Eq \rangle)$ can be represented by polynomial functor p_{s^k} or p_G, respectively. In the former case, p_{s^k} is an interface between the specification of the local problem $(\hat{L}^k, u^k)$ and its solutions $\hat{\mathbf{X}}^k$. In the case of a game, p_G is an interface between the game G and its equilibria $Eq(G)$.

Each state in the morphism between two interfaces p_{s^k} and p_{s^j} represents a particular $r^k_j : \Sigma(s^k) \rightarrow \Sigma(s^j)$ that sends a section of solutions over s^k to a corresponding section over s^j, yielding a sheaf.

Analogously, each state in the morphism between two interfaces p_G and $p_{G'}$ represents a particular wiring, *connecting the games G and G', such that the equilibrium obtains by tensoring those of the two games.*

Since in $\mathcal{PR}$ and $\mathcal{G}$, morphisms cannot be rearranged they can be seen as hom coalgebras with a single state. □

Thus, $\mathcal{I}$ incorporates all the representational advantages of $\mathcal{PR}$ and $\mathcal{G}$, adding the possibility of capturing the dynamics of actual systems.

The following two examples exhibit the advantages of formalizing problems in $\mathcal{I}$:

Example 3 *([18]): Consider a Principal-Agent problem defined by two functions:*

$$\Phi_{\rightarrow} : X \times Y \times \mathbb{R} \rightarrow \mathbb{R} \text{ and } \Pi : X \times Y \times \mathbb{R} \rightarrow \mathbb{R}$$

where:

- X *is the compact set of types of the Agent.*
- Y *is the compact set of possible decisions made by the Agent.*
- $\Phi_{\rightarrow}$ *is continuous, strictly decreasing in the third argument.*
- $\Phi_{\rightarrow}$ *is full range in the third argument:* $\Phi_{\rightarrow}(x, y, \cdot)[\mathbb{R}] = \mathbb{R}$ *for every* $(x, y) \in X \times Y$.
- Π *is continuous and increasing in the third argument.*
- Π *is full range in the third argument:* $\Pi(x, y, \cdot)[\mathbb{R}] = \mathbb{R}$ *for every* $(x, y) \in X \times Y$.

Given a type x *of the Agent, her decision* y *and* v*, the money transfer to the Principal,* $\Phi_{\rightarrow}(x, y, v) = u_A$ *is the utility of the Agent, while* $\Pi(x, y, v) = u_P$ *is the utility of the Principal.*

An inverse generating function *is*

$$\Phi^{\leftarrow} : Y \times X \times \mathbb{R} \rightarrow \mathbb{R}$$

such that given $u_A = \Phi_{\rightarrow}(x, y, \Phi^{\leftarrow}(y, x, u_A))$ *there exists* $v = \Phi^{\leftarrow}(y, x, \Phi_{\rightarrow}(x, y, v))$.

Given $\lambda \in \mathbb{M}$*, the class of* Borel measures *over* $X \times Y$ *and* $\underline{u}$*, a reservation utility of the Agent, the Principal's problem amounts to choosing* $\langle \lambda, \bar{u}_A, \bar{v} \rangle$ *as to maximize*

$$\int_X \int_Y \Pi(x, y, \Phi^{\leftarrow}(y, x, \bar{u}_A)) d\lambda(x, y)$$

s.t. $\bar{v} = \Phi^{\leftarrow}(y, x, \bar{u}_A)$ *and* $\bar{u}_A \geq \underline{u}$.
This setting can be naturally represented by defining two objects in $\mathcal{I}$*,* A*, and* P *(the Agent and the Principal, respectively). The corresponding polynomial functors are:*

- p_P *takes as input* $\underline{u}$ *and returns the optimal values* λ^**,* u_A^* *and* $\bar{v}^*$*. That is,* $p_P = \sum_{\underline{u} \in \mathbb{R}} y^{p_P[\underline{u}]}$*, such that* $p_P[\underline{u}] = \langle \lambda^*, u_A^*, \bar{v}^* \rangle$.
- p_A *takes as input* $\bar{v}$ *and returns her decision* y *and the Principal's utility* u_P*. That is,* $p_A = \sum_{\bar{v} \in \mathbb{R}} y^{p_A[\bar{v}]}$*, such that* $p_A[\bar{v}] = \langle y, u_P \rangle$.

Then, the entire problem can be understood in terms of the identity *morphism of* $p_A \otimes p_P$*, yielding the adjunction between* $\Phi^{\rightarrow}$ *and* $\Phi^{\leftarrow}$.

A promising area of research in which $\mathcal{I}$ could be relevant for the design of mechanisms:

Example 4 *([13] [7]): Mechanisms (and* institutions *in general) can be conceived as game forms. That is, each mechanism* M *can be represented as* $M = (I_M, S_M, \mathbf{O}_M, \rho_M)$ *(see Section 4).*

Each $i \in I_M$ can be given different incentives according the environment $\mathbf{e} \in E$ in which she interacts with the others. Each $\mathbf{e} \in E$ will have an associated profile of payoff functions that correspond to the outcomes in M, $\pi^{\mathbf{e}}_M$.

The task of a mechanism designer D is to assign to a given environment a mechanism $M \in \mathbb{M}$, in order to ensure a target $\mathbf{o}^*$. Thus, in $\mathcal{I}$, D has an associated $p_D = \sum_{\mathbf{e} \in E} y^{p_D[\mathbf{e}]}$ where

$$p_D[\mathbf{e}] = \{\langle M, \pi^{\mathbf{e}}_M\rangle : M \in \mathbb{M} \text{ such that } s^*_M \in Eq(\langle M, \pi^{\mathbf{e}}_M\rangle) \text{ and } \rho(s^*_M) = \mathbf{o}^* \in \mathbf{O}_M\}$$

Each game form $M \in \mathbb{M}$ constitutes a local problem. *The polynomial corresponding to these problems is $p_{\mathbb{M}}$. In turn, given the choice of* Nature *(represented by a constant polynomial $p_E = E$), the whole problem can be described by a $[p_D \times p_E, p_{\mathbb{M}}]$-coalgebra, where:*

$$[p_D \times p_E, p_{\mathbb{M}}] = \sum_{\phi : p_D \times p_E \to p_{\mathbb{M}}} y^{\sum_{\mathbf{e} \in E} p_{\mathbb{M}}[\phi^{\to}(\mathbf{e})]}$$

and $p_{\mathbb{M}}[\phi^{\to}(\mathbf{e})] = \langle M, \pi^{\mathbf{e}}_M\rangle$.

7 Conclusions

This paper discussed the question of representing interactions among economic agents. We resorted to the language of Category Theory and, in particular, constructions like *sheaves*, *hypergraph categories*, and *polynomial functors*.

The category defined in terms of the latter, $\mathcal{I}$, has as objects the interfaces between problems and their solutions, while the interaction among them is captured by coalgebras based on the internal homs of the interfaces. That is, sets of states that determine the arrangement of connections among the problems and their solutions. Furthermore, the connections are rearranged in response to the outputs obtained previously.

We intend to explore further this formalism and use it to represent specific problems. While a first step involves showing that $\mathcal{I}$ can reformulate known models, the real gist of this development is to capture new phenomena, establishing their relations to the former.

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