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Obvious Manipulations, Stability, and Efficiency in Matching Markets with No, Unitary, and Multiple Contracts: Three Different Results *

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Abstract

In two-sided many-to-many matching markets under substitutable preferences —both with and without contracts— all stable-dominating mechanisms are manipulable. In light of this, we examine whether some of these mechanisms are at least not obviously manipulable (NOM). To this end, we discuss three established models that are encompassed by our general framework: the no-contract case, the unitary contract case, and the multiple-contract case. Our results reveal fundamental differences among the three models. We transition from a no-contracts model, where all stable-dominating mechanisms are NOM, to a multiple-contracts model, where all stable mechanisms and all efficient stable-dominating mechanisms are obviously manipulable (OM). In the intermediate case of unitary contracts the doctor-proposing DA mechanism remains NOM, but the hospital-proposing DA mechanism and all efficient stable-dominating mechanisms are OM. These findings reveal fundamental trade-offs between stability, efficiency, and NOM in these markets.

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1 Introduction

Two-side many-to-many matching markets, both with and without contracts, have been extensively studied in economics. These markets where agents on both sides of a market have preferences over potential matches on the other side, play a crucial role in various applications including labor markets, school choice, and medical residency, among others. (see,among many others, Gale and Shapley, 1962; Roth and Sotomayor, 1990; Sotomayor, 1996; Echenique and Oviedo, 2006; Hatfield and Milgrom, 2005; Klaus and Walzl, 2009; Kominers, 2012; Pepa Risma, 2015; Hatfield and Kominers, 2017; Kamada and Kojima, 2024)

A central concern in these models is the design of stable-dominating mechanisms that prevent strategic manipulation by one side of the market (in this case, doctors).[1] However, in many-to-many settings with substitutable preferences,[2] all stable-dominating mechanisms are manipulable.[3] This limitation raises a crucial question: Is it possible to identify mechanisms that, although manipulable, are not obviously manipulable (NOM)? The notion of NOM, introduced by Troyan and Morrill (2020), has gained increasing relevance in the literature as a viable alternative to strategy-proofness when the latter is unattainable

[1]A mechanism is stable-dominating if it is either stable or Pareto-dominates a stable mechanism from the perspective of doctors. The focus on stable-dominating mechanisms rather than solely stable ones arises because stability is incompatible with Pareto efficiency. Indeed, even the doctor-proposing DA mechanism, which Pareto-dominates all other stable mechanisms, may be Pareto inefficient. Seeking a relaxation of stability that is both normatively justified and compatible with efficiency, stable-dominating mechanisms have been explored in several works (see, among others, Kesten, 2010; Hirata and Kasuya, 2017; Troyan and Morrill, 2020; Doğan and Ehlers, 2021).

[2]Substitutability is a fundamental condition widely used in the matching literature, ensuring that agents do not consider contracts as complements. This assumption is typically required to guarantee the existence of stable allocations.

[3]See Martínez et al. (2004); Hatfield and Milgrom (2005) for stable mechanisms without and with contracts, respectively, and Abdulkadiroğlu et al. (2009); Kesten (2010); Alva and Manjunath (2019) for stable-dominating mechanisms.

(see, e.g., Troyan et al., 2020; Aziz and Lam, 2021; Ortega and Segal-Halevi, 2022; Psomas and Verma, 2022; Arribillaga and Bonifacio, 2024, 2025; Arribillaga and Risma, 2025). This concept captures whether an agent can easily recognize a profitable deviation, even with limited strategic reasoning. A manipulation is considered obvious if either (i) the best possible outcome under the manipulation is strictly better than the best possible outcome under truth-telling, or (ii) the worst possible outcome under the manipulation is strictly better than the worst possible outcome under truth-telling.

To explore this question, we examine three different domains of doctors' preferences, each giving rise to a known and distinct many-to-many matching model embedded in a general framework: (i) A model with no-contracts, where each doctor-hospital pair has at most one feasible (individually rational) contract; (ii) A model with unitary contracts, where each doctor-hospital pair can sign at most one contract; and (iii) A model with multiple contracts, where each doctor-hospital pair can sign several contracts. Each of these models has been extensively studied and applied in a broad literature. The classical no-contracts model has been analyzed in Roth and Sotomayor (1990); Sotomayor (2004); Echenique and Oviedo (2006); Bonifacio et al. (2024); Kamada and Kojima (2024), among others. The unitary contracts model is central to many contributions (see, e.g., Kominers, 2012; Klaus and Walzl, 2009; Millán and Pepa Risma, 2018). In fact, Kominers (2012) demonstrates that unitarity is essentially necessary for the embedding result of Echenique (2012), which maps the model with contracts into a wage-based model as in Kelso and Crawford (1982). Finally, several studies have focused on the general model in which multiple contracts can be signed between each doctor-hospital pair. For instance, Hatfield and Kominers (2017) shows that non-unitarity arises in various important applications, such as the United Kingdom Medical Intern Match. Additionally, Pepa Risma (2022) proposes an algorithm to compute the full set of stable allocations in the multiple contracts setting.

Our findings reveal fundamental differences between the three models. We move from a model with no-contracts, where all stable-dominating mechanisms are NOM, to a model with multiple contracts, where all stable mechanisms and all efficient stable-dominating mechanisms become obviously manipulable.[4] In the intermediate case of

[4]The result for multiple contracts holds even when preferences are responsive. A preference is said to be responsive when it follows a structured ordering in which the ranking of sets of options depends solely on the ranking of the individual elements within those sets.

unitary contracts, the doctor-proposing DA mechanism remains NOM, while the hospital-proposing DA mechanism and any efficient stable-dominating mechanisms become obviously manipulable. Beyond the theoretical implications, our results contribute to the broader discussion on strategic behavior in matching markets. They highlight a fundamental difference between the three models and underscore the limitations of the stability- or efficiency-based mechanisms in mitigating manipulation incentives.

Our result for the no-contracts setting extends the one stated by Troyan and Morrill (2020), which was established for a many-to-one model with responsive preferences, to a many-to-many model with substitutable preferences. It also generalizes a recent results by two of the authors of this paper, Arribillaga and Risma (2025), which was previously stated for a many-to-one model —both with no and (unitary) contracts— to the many-to-many model.

The remainder of this paper is organized as follows. Section 2 presents the formal model and definitions. Section 3 discusses our main results for each matching model. Finally, Section 4 concludes with a summary of key findings and final remarks.

2 Preliminaries

2.1 Matching with contracts. Stability and Efficiency

Let D and H be finite sets of doctors and hospitals, respectively, and define the set of all agents as $F = D \cup H$. There is also a finite set X of contracts, where each contract $x \in X$ represents an agreement between a doctor-hospital pair. Specifically, each contract is bilateral, involving exactly one doctor $x_D \in D$ and one hospital $x_H \in H$. The set X may contain two or more contracts involving the same doctor-hospital pair of $(d, h) \in D \times H$, each specifying different conditions.

In the general many-to-many matching model studied here, each agent can sign multiple contracts with the same or different hospitals. An **allocation** is a subset of contracts, $Y \subseteq X$. Note that the empty set, representing a situation where no contracts are signed, is also an allocation. We denote by $\mathcal{X}$ the set of all allocations, i.e., the set of all subsets of X. Given a set of contracts $Y \subseteq X$, a subset of agents $S \subseteq F$, and an agent $i \in F$, let Y_S denote the set of agents in S that appear in at least one contract in Y, and let Y_i denote the set of all contracts in Y that involve i.

Given a set of contracts X, a particular market is determined by a preference relation P_i for each agent $i \in F$ over the subsets of X_i.[5] The weak preference relation associated with P_i will be denoted by R_i.[6] A subset of contracts $Y \subseteq X$ is an **acceptable allocation** for i if $YP_i\varnothing$, and a contract $y \in X_i$ is an **acceptable contract** for i if $\{y\}$ is an acceptable allocation for i.

The **choice set** of $i \in F$ given $Y \subseteq X$ is the subset of Y_i that i likes the most according to P_i,

$$C_i(P_i, Y) = \max_{P_i}\{Z \subseteq X : Z \subseteq Y_i\}.$$

To simplify the notation, we will omit preferences in the choice set notation when they are clear from the context: we will write $C_i(Y)$ instead of $C_i(P_i, Y)$. Given $Y \subseteq X$ we will denote $C_H(Y) := \cup_{h\in H}C_h(Y)$ and $C_D(Y) := \cup_{d\in D}C_d(Y)$.

As is common in matching literature, we will assume throughout the paper that agents' preferences are substitutable. This means that doctors and hospitals do not consider contracts to be complementary with one another. Given $i \in F$, we will say that P_i satisfies **substitutability** if $x \in C_i(Z)$ implies $x \in C_i(Y)$ whenever $x \in Y \subseteq Z \subseteq X$.

Given $P \in \mathcal{P}$, we say that $Y \subseteq X$ is *individually rational* if $C_D(Y) = C_H(Y) = Y$, i.e., if Y does not contain unwanted contracts. Moreover, Y is **unblocked** if it does not exist a non-empty **blocking set** $Z \subseteq X$ such that $Z \cap Y = \varnothing$ and $Z_i \subseteq C_i(Z \cup Y)$ for all $i \in Z_F$.

A **stable allocation** is defined as one that is individually rational and is not blocked by any other allocation. If we add the condition that the blocking set has exactly one element, i.e., $|Z| = 1$, we arrive at the concept of pairwise stability. Hatfield and Kominers (2017) shows that when agents' preferences are substitutable, stability and pairwise stability are equivalent and the set of stable allocations is non-empty. Furthermore, the unanimously most preferred stable allocations for each side of the market exist and can be obtained through adapted versions of both the doctor-proposing and hospital-proposing DA algorithms.

An allocation Y **Pareto-dominates** another allocation Y' if $Y_dR_dY'_d$ for all $d \in D$ and $Y_dP_dY'_d$ for some $d \in D$. Y is said to be **stable-dominating** if it is stable or if it is individually rational and Pareto-dominates some stable allocation. Finally, Y is said to be **efficient** if no other allocation Pareto-dominates it.

[5] Preferences are antisymmetric, transitive, and complete binary relations on $\{Y : Y \subseteq X_i\}$.

[6] That is, for all $Z, Y \subseteq X_i$, ZR_iY if and only if either $Z = Y$ or ZP_iY.

2.2 Mechanisms and obvious manipulations

In our analysis, we assume that only one side of the market is strategic: the doctors, while hospitals' preferences are fixed and common knowledge. Throughout (most of) the paper P_h will be substitutable and fixed (but arbitrary) for each hospital h, and we will often omit the reference to hospital preferences.

An arbitrary preference for a doctor d is denoted by P_d. For each doctor d, let $\mathcal{P}_d$ denote the set of all substitutable preferences for d. A preference profile $P = (P_d)_{d\in D}$ specifies a preference for each doctor. The set of all substitutable preference profiles that could arise in the market is denoted by $\mathcal{P} = \prod_{d\in D} \mathcal{P}_d$. For each profile P and doctor $d \in D$, we denote by P_{-d} the sub-profile in $\mathcal{P}_{-d} = \prod_{i\in D\setminus\{d\}} \mathcal{P}_i$ obtained by deleting P_d from P. Finally, we will need to consider a generic subdomains of preferences, $\mathcal{T}_d \subseteq \mathcal{P}_d$ for each doctor d, and its corresponding set of preference profiles, $\mathcal{T} = \prod_{d\in D} \mathcal{T}_d$.

Definition 1 *A **(matching) mechanism** on $\mathcal{T}$ is a function $\varphi : \mathcal{T} \to \mathcal{X}$ that assigns an allocation to each preference profile $P \in \mathcal{T}$.*

Given $d \in D$, we will denote by $\varphi_d(P)$ the set of contracts involving d in $\varphi(P)$. Let $\varphi : \mathcal{T} \to \mathcal{X}$ be a mechanism. It is **stable** if $\varphi(P)$ is a stable allocation at P, for each $P \in \mathcal{T}$. It is **stable-dominating** if $\varphi(P)$ is a stable-dominating allocation at P, for each $P \in \mathcal{T}$. It is **efficient** if $\varphi(P)$ is an efficient allocation at P, for each $P \in \mathcal{T}$.

Non-manipulability (or strategy-proofness) has played a central role in mechanism design. A doctor manipulates a matching mechanism if there exists a situation in which they achieve a better outcome by declaring a preference different from their true one. In our context, any stable-dominating mechanism is manipulable, so we aim to identify mechanisms that at least avoid the obvious manipulations introduced by Troyan and Morrill (2020). Before presenting formal definitions, we first introduce some notation.

Given a mechanism $\varphi : \mathcal{T} \to \mathcal{X}$, a doctor $d \in D$ and a preference $P_d \in \mathcal{T}_d$, we define the **option set** left open by P_d at φ as the set of all allocations that can be selected by φ once that doctor d has declared P_d,

$$O^{\varphi}(P_d) = \{\varphi_d(P_d, P_{-d}) : P_{-d} \in \mathcal{P}_{-d}\}$$

Given a doctor d, a preference $P_d \in \mathcal{T}_d$, and a set of subsets of contracts involving d, $\mathcal{Y} \subseteq \{Y : Y \subseteq X_d\}$, we denote by $W_d(P_d, \mathcal{Y})$ the worst set of contracts in $\mathcal{Y}$ according to P_d. Given a mechanism φ on $\mathcal{T}$ and a doctor d, a preference $P'_d \in \mathcal{T}_d$ is an **obvious manipulation** of φ at P_d if:

(i) There is $P_{-d} \in \mathcal{T}_{-d}$ such that $\varphi_d(P'_d, P_{-d})\ P_d\ \varphi_d(P_d, P_{-d})$; and

(ii) $W_d(P_d, O^{\varphi}(P'_d))P_i W_d(P_d, O^{\varphi}(P_d))$.

P'_d is a **manipulation** of φ at P_i when it satisfies condition (i). A manipulation becomes obvious if (ii) holds, i.e., if the worst possible outcome under the manipulation is strictly better than the worst possible outcome under truth-telling.

Remark 1 *Two remarks regarding the definition of obvious manipulations are relevant:*[7]*.*

(a) Condition (i) is not strictly necessary because we could impose only requirement (ii) on any arbitrary preference, and the resulting definition would remain unchanged, as (ii) implies (i).

(b) The original definition in Troyan and Morrill (2020) also considers that P'_d is an obvious manipulation if $C_d(P_d, O^{\varphi}(P'_d))P_d C_d(P_d, O^{\varphi}(P_d))$, i.e., if the best possible outcome under the manipulation is strictly better than the best possible outcome under truth-telling. However, such a condition can be omitted in our context because it never holds for a deviation P'_d in stable-dominating (or individually rational) mechanisms.

Definition 2 *A mechanism $\varphi : \mathcal{T} \to \mathcal{X}$ is **not obviously manipulable (NOM)** if it does not admit obvious manipulations on $\mathcal{T}$. Otherwise, φ is **obviously manipulable (OM)** on $\mathcal{T}$.*

Before concluding this subsection, we formally introduce two prominent stable mechanisms: the doctor-proposing DA and the hospital-proposing DA mechanisms, both of which have special and classical relevance in the literature. The doctor-proposing DA mechanism can be computed using a deferred acceptance algorithm, where doctors make offers. Given a profile $P \in \mathcal{P}$, each doctor proposes a choice set based on the set of contracts that have not been rejected in previous steps, while each hospital accepts from its choice set, considering all accumulated offers. The algorithm terminates when all offers are accepted. The output of the algorithm is the set of contracts accepted by the hospitals in the final iteration. Here, we present the formal definition.

The doctor-proposing Deferred Acceptance Algorithm (DDA algorithm)

[7] For details, seeArribillaga and Risma (2025)

Input:
A market (X, P).
Begin:
Set $X^1 = X$ and $t := 1$.
Repeat:

Step 1: Determine the set of contracts that doctors offer in the iteration t, this is,
$O^t := C_D(X^t)$.

Step 2: From the set of accumulated offers, $O^t_A := \cup_{k=1}^t O^k$, determine $C_H(O^t_A)$. This is the set of contracts (provisionally) accepted by hospitals in the iteration t.
If $C_H(O^t_A) = O^t$, the algorithm stops with output $C_H(O^t_A)$.
If $C_H(O^t_A) \neq O^t$, define $X^{t+1} := X^t \setminus (O^t \setminus C_H(O^t_A))$, this is, the set of contracts that have not been rejected yet; set $t := t + 1$; and repeat Steps 1 and 2.
End

Let $DDAM : \mathcal{T} \to \mathcal{X}$ be the **doctor-proposing DA mechanism**, i.e., the mechanism that returns the stable allocation obtained by the DDA algorithm for each preference profile $P \in \mathcal{T}$. Symmetrically, we have the **hospital-proposing Deferred Acceptance (HDA) algorithm**, where hospitals make offers, and the roles of doctors and hospitals are interchanged. Let $HDAM : \mathcal{T} \to \mathcal{X}$ be the **hospital-proposing DA mechanism**.

Given two mechanisms $\varphi, \varphi' : \mathcal{T} \to \mathcal{X}$, we say that φ' **Pareto-dominates** φ if $\varphi' \neq \varphi$ and $\varphi'_d(P) R_d \varphi_d(P)$ for each $d \in D$ and each $P \in \mathcal{T}$. It is known that any stable-dominating mechanism Pareto-dominates the $HDAM$; we state this in a remark for future reference.

Remark 2 *Any stable-dominating mechanism $\varphi' : \mathcal{T} \to \mathcal{X}$ such that $\varphi' \neq HDAM$ Pareto-dominates $HDAM$.*

3 Three Different Results for Three Different Models

In this section, we present the three main results, each corresponding to a different many-to-many matching model: the simplest model without contracts, an intermediate well-known model with unitary contracts, and a general model with multiple contracts. We argue that these results highlight significant differences between the models regarding the strategic behavior of agents. Specifically, we transition from a model without contracts, where all stable-dominating mechanisms are NOM, to a model with multiple contracts, where all stable mechanisms and all efficient stable-dominating mechanisms are OM.

To provide a clearer structure, we describe each model by focusing on a distinct preference domain within $\mathcal{P}$.

3.1 Three Different Domains and Their Associated Models

Given X, consider a subset $X^o \subseteq X$ such that, for each doctor-hospital pair (d, h), there exists at most one contract $x \in X^o_d \cap X^o_h$. Fixing X^o, we say that a preference $P_d \in \mathcal{P}$ is an X^o**-choice** preference if, for all $Y \subseteq X$,

$$C_d(P_d, Y) \subseteq X^o$$

In such a domain, under individual rationality, each doctor can have at most one acceptable contract with each hospital —specifically, the contract in $X^o_d \cap X^o_h$ in case it exists. Let $\mathcal{P}^o_d$ denote the set of all X^o-choice preferences for doctor d in $\mathcal{P}_d$, and let $\mathcal{P}^o$ be the corresponding set of preference profiles. Clearly, $\mathcal{P}^o$ depends on X^o; however, since we will assume that X^o is fixed in what follows, we omit explicit reference to it.

The **(classical) many-to-many matching model without contracts** —or, equivalently, a model where each doctor-hospital pair has at most one contract— can be represented within our general framework by restricting doctors' preference profiles to $\mathcal{P}^o$. An alternative approach to describing this model would be to assume directly that the set of feasible contracts X contains at most one contract per doctor-hospital pair. However, for clarity and consistency in our presentation, we prefer to introduce this restriction through the preference domain.

Given a doctor d, a preference $P_d \in \mathcal{P}$ is said to be **unitary** if, for all $Y \subseteq X$,

$$x, x' \in C_d(P_d, Y) \text{ with } x \neq x', \text{ then } x_H \neq x'_H.$$

The term "unitary" for this type of preference was first introduced by Kominers (2012). When doctors have unitary preferences, each doctor-hospital pair can sign at most one contract in any individually rational allocation. Let $\mathcal{P}^u_d$ denote the set of all unitary preferences for doctor d in $\mathcal{P}_d$, and let $\mathcal{P}^u$ be the corresponding set of preference profiles. The **many-to-many matching model with unitary contracts** —that is, a model where each doctor-hospital pair can sign at most one contract— can be represented within our general framework by restricting doctors' preference profiles to $\mathcal{P}^u$.

Observe that $\mathcal{P}^o \subseteq \mathcal{P}^u$, meaning that the many-to-many model without contracts can be viewed as a special case of the many-to-many matching model with unitary contracts.[8]

Finally, we consider the most general model, in which multiple contracts can be signed between each doctor-hospital pair. This occurs when doctors' preference profiles span the entire set of substitutable preferences, denoted by $\mathcal{P}$. To ensure clarity in the presentation of our results, we refer to this general framework as the **many-to-many matching model with multiple contracts**.

3.2 Main Results

In every statement, we assume that all hospitals have substitutable preferences, and we omit explicitly mentioning this general hypothesis.

Our first result pertains to the model without contracts, where we show that any stable-dominating mechanism is NOM. The result is based on three key findings. First, we establish that any mechanism that Pareto-dominates a stable and NOM mechanism is also NOM. Next, we focus on the $HDAM$ mechanism and demonstrate that it is NOM. Finally, since all stable-dominating mechanisms Pareto-dominate $HDAM$, we conclude that they must also be NOM. Arribillaga and Risma (2025) presents a similar result for the many-to-one model. Although our proof shares a similar structure, numerous technical modifications and additional observations were required to adapt it to the many-to-many setting. The complete proofs of this result and the following are extensive and can be found in the Appendix.

Theorem 1 *Assume that the domain of doctors' preference profile is $\mathcal{P}^o$, i.e., consider a many-to-many matching model with no contracts. Then:*

(i) Any mechanism that Pareto-dominates a NOM stable mechanism is also NOM.

(ii) The $HDAM$ is NOM.

(iii) Any stable-dominating mechanism is NOM.

Proof. See Appendix. □

[8] It is also straightforward to see that the general many-to-one matching model with contracts can be viewed as a special case of the many-to-many matching model with unitary contracts.

Next, we examine the unitary model. Our second theorem consists of three key statements. First, the $DDAM$ mechanism remains NOM in these markets. Second, the $HDAM$ mechanism becomes obviously manipulable. Third, any efficient stable-dominating mechanism is obviously manipulable. Notably, the last two results hold even in the simplest one-to-one model. For an efficient stable-dominating mechanism, a doctor d can misreport their preferences by falsely declaring an "unacceptable" contract with a hospital as "acceptable." This misrepresentation may disrupt an allocation where another doctor d' holds a contract with the same hospital. Consequently, the previously "unacceptable" contract may become the only stable option, albeit inefficient, allowing d to secure a more preferred contract in an efficient stable-dominating mechanism. We provide an example demonstrating how such manipulations are both possible and obvious. In the case of $HDAM$, a doctor can misreport their preferences by falsely declaring an "acceptable" contract as "unacceptable." This leads the contract to be rejected when proposed by the hospital. As a result, the hospital may later offer more favorable terms to the same doctor in a subsequent round.This contrasts with $DDAM$, where doctors initiate the process by proposing their most preferred contracts. Manipulating $DDAM$—though possible for some substitutable preferences—requires triggering a more complex "rejection chain" involving other doctors.

To prove that $DDAM$ remains, we use two lemmas (see Appendix):

1. At the end of each DDA iteration, every hospital is assigned to its choice set given all offers it received up to that time.

2. Any arbitrary allocation strictly preferred to the $DDAM$ outcome must contain at least a contract that is rejected in some iteration of the DDA algorithm.

Then, given a doctor d, we consider the worst outcome it can obtain under $DDAM$ by truthfully reporting their preferences. If this outcome is not their top choice (otherwise, manipulation would not be possible), we construct a sub-profile of preferences for the remaining doctors such that any false declaration by d leads to an outcome no better than their worst truthful outcome. This establishes that $DDAM$ is NOM. This result extends one in Arribillaga and Risma (2025) from a many-to-one model with contracts to a many-to-many model with unitary contracts. However, unlike the approach in that paper, our proof follows a different structure and introduces new technical arguments and observations tailored to the many-to-many setting.

Theorem 2 *Assume that the domain of doctors' preference profile is $\mathcal{P}^u$, i.e., consider the many-to-many matching model with unitary contracts. Then,*

(i) The $DDAM$ is NOM.

(ii) The $HDAM$ is OM, even in the one-to-one model.

(iii) Any efficient stable-dominating mechanism is OM, even in the one-to-one model.

Proof. See Appendix. □

Remark 3 *In light of Theorem 2, none of the three statements in Theorem 1 remain valid when moving from a model without contracts to one with (unitary) contracts.*

Finally, we present our results concerning the general many-to-many matching model with multiple contracts. Surprisingly, the $DDAM$ also becomes obviously manipulable in this context, such as any stable mechanism. This happens because, in this scenario, a doctor can manipulate the mechanism by misreporting their preferences—falsely declaring one of their two (or more) contracts in their top allocation as "unacceptable." As a result, the doctor does not propose that contract, leading the hospital to offer a more favorable contract to the same doctor instead.

Theorem 3 *Assume that the domain of doctors' preference profile is $\mathcal{P}$, i.e., consider the many-to-many matching model with multiple contracts. Then,*

(i) Any stable mechanism is OM —in particular, both $DDAM$ and $HDAM$ are OM— even when doctors' and hospitals' preferences are responsive.

(ii) Any efficient stable-dominating mechanism is OM, even in the one-to-one case.

Proof.

Let $D = \{d_1, d_2, d_3\}$, $H = \{h_1, h_2\}$ and $\mathbf{X} = \{v, w, x, y, z\}$ be the sets of doctors, hospitals, and contracts, respectively. Where $v_H = w_H = x_H = h_1$, $y_H = z_H = h_2$, $v_D = w_D = d_1$, $x_D = d_2$, and $y_D = z_D = d_3$. The hospitals' preferences are given by:

$$P_{h_1} = \{v\}, \{x\}, \{w\}, \emptyset$$
$$P_{h_2} = \{y\}, \{z\}, \emptyset$$

For clarity, we denote the preferences of doctor d_i as P_i. Consider the preference profile $(\overline{P}_1, \overline{P}_2, \overline{P}_3)$, where:

$$\begin{aligned} \overline{P}_1 &= \{w\}, \{v\}, \emptyset \\ \overline{P}_2 &= \{x\}, \emptyset \\ \overline{P}_3 &= \{y, z\}, \{z\}, \{y\}, \emptyset \end{aligned}$$

Under this preference profile, the allocation $\{v, y\}$ is the unique stable allocation, while $\{w, z\}$ is the only efficient stable-dominating allocation.

The proof now proceeds by considering two cases corresponding to (i) and (ii) of our theorem:

Case 1 (Part i): Assume that φ is a stable mechanism. Then,

$$\varphi(\overline{P}_1, \overline{P}_2, \overline{P}_3) = \{v, y\}.$$

Now, suppose that doctor d_3 misreports their preferences as $P'_3 \in \mathcal{P}_3$, where $P'_3 = \{z\}, \emptyset$. Under $(\overline{P}_1, \overline{P}_2, P'_3)$, the only stable allocation is $\{v, z\}$. Thus,

$$\varphi_{d_3}(\overline{P}_1, \overline{P}_2, P'_3) = \{z\}.$$

Since $\varphi_{d_3}(\overline{P}_1, \overline{P}_2, P'_3) = z\overline{P}_3 y = \varphi_{d_3}(\overline{P}_1, \overline{P}_2, \overline{P}_3)$, it follows that P'_3 is a manipulation of φ at $\overline{P}_3$. Furthermore, for any preference profile $(P_1, P_2) \in \mathcal{P}_1 \times \mathcal{P}_2$ for doctors d_1 and d_2, any stable allocation at (P_1, P_2, P'_3) contains z. Thus, P'_3 is an obvious manipulation of φ at $\overline{P}_3$.

Case 2 (Part ii):[9] Assume that φ is an efficient stable-dominating mechanism. Then,

$$\varphi(\overline{P}_1, \overline{P}_2, \overline{P}_3) = \{w, z\}.$$

Now, consider the alternative preference declaration of d_1, $P_1 := \{w\}, \emptyset$. Under $(P_1, \overline{P}_2, \overline{P}_3)$, the only stable and stable-dominating allocations are $\{x, y\}$ and $\{x, z\}$, respectively. Thus,

$$\varphi(P_1, \overline{P}_2, \overline{P}_3) = \{x, z\}.$$

which implies

$$\varphi_{d_1}(P_1, \overline{P}_2, \overline{P}_3) = \emptyset.$$

[9] Note that in this part of the proof, doctor d_3 can be omitted, making the argument applicable even in the one-to-one setting.

However, if d_1 instead declares $\overline{P}_1$, then

$$\varphi_{d_1}(\overline{P}_1, \overline{P}_2, \overline{P}_3) = \{w\}.$$

This implies that $\overline{P}_1$ is a manipulation of φ at P_1. Furthermore, for any $(P_2, P_3) \in \mathcal{P}_2 \times \mathcal{P}_3$, the only stable allocations at $(\overline{P}_1, P_2, P_3)$ must take the form $\{v\} \cup Y$ or $\{w\} \cup Y$ for some $Y \subseteq \{y, z\}$. Since φ is an efficient stable-dominating mechanism, it follows that

$$\varphi_{d_1}(\overline{P}_1, P_2, P_3) = \{w\}.$$

Thus, $\overline{P}_1$ is an obvious manipulation of φ at P_1. □

Remark 4 *In the proof of the previous theorem, we construct a matching market where any mechanism that selects either a stable allocation or an efficient stable-dominating allocation is obviously manipulable. Note that this class of mechanisms is broader than the union of stable mechanisms and efficient stable-dominating mechanisms.*

4 Final Remarks

Table 1 summarizes our main findings, providing a comprehensive characterization of NOM in many-to-many matching models. It highlights a clear and significant distinction between models with no contracts, those with unitary contracts, and those with multiple contracts, in terms of agents' strategic behavior. Notably, these results hold for both substitutable and responsive preferences.

	NOM (for doctors)		
	no contracts	unitary contracts	multiple contracts
DDAM	*yes*	*yes*	*no*
HDAM	*yes*	*no*	*no*
Stable	*yes*	*yes/no*	*no*
Efficient and stable-dominating	*yes*	*no*	*no*

Table 1: *Summary of results.*

Finally, we observe that if hospitals' preferences fail to satisfy substitutability, any stable-dominating mechanism (including stable ones and without requiring efficiency) becomes obviously manipulable. The following example illustrates this point. Let $D = \{d_1, d_2\}$, $H = \{h\}$ and $X = \{x, y, z\}$ be the sets of doctors, hospitals, and contracts, respectively, where: $x_H = y_H = z_H = h$, $x_D = y_D = d_1$, and $z_D = d_2$. The preference of hospital h is given by $P_h = \{y, z\}, \{x\}, \{y\}, \{z\}, \emptyset$. For clarity, we denote the preferences of doctor d_i as P_i. Consider the preference profile (P_1, P_2), where:

$$P_1 = \{x, y\}, \{x\}, \{y\}, \emptyset$$
$$P_2 = \{z\}, \emptyset$$

Under this profile, the unique stable-dominating allocation is $\{y, z\}$. Suppose that φ is a stable-dominating mechanism. Then, $\varphi(P_1, P_2) = \{y, z\}$. Now, if doctor d_1 misreports their preferences as $P'_1 \in \mathcal{P}_1$, where $P'_1 = \{x\}, \emptyset$, then the unique stable-dominating allocation under (P'_1, P_2) is $\{x\}$. Thus, $\varphi_{d_1}(P'_1, P_2) = \{x\}$. Since $\{x\} P_1 \{y\} = \varphi_{d_1}(P_1, P_2)$, this implies that P'_1 constitutes a manipulation of φ at P_1. Furthermore, for any $P_2 \in \mathcal{P}_2$, $\{x\}$ remains the only stable-dominating allocation under (P'_1, P_2). Hence, P'_1 is an obvious manipulation of φ at P_1.

Appendix. Proofs

Some well-known properties of choice sets follow almost directly from their definition. We state them in the following remark, as they will be frequently used in the proofs of our results.

Remark 5 *For all $Z, Y \subseteq X$ and $i \in F$: (I) $C_i(Y) \subseteq Y$; (II) $C_i(Y) \subseteq Z \subseteq Y$ implies $C_i(Z) = C_i(Y)$; (III) $C_i(C_i(Y)) = C_i(Y)$; and (IV) If i's preferences satisfy substitutability, then $C_i(Z \cup Y) = C_i(C_i(Z) \cup Y)$.*

Theorem 1 *Assume that the domain of doctors' preference profile is $\mathcal{P}^o$, i.e., consider a many-to-many matching model without contracts. Then,*

(i) Any mechanism that Pareto-dominates a NOM stable mechanism is also NOM.

(ii) The HDAM is NOM.

(iii) Any stable-dominating mechanism is NOM.

Proof of Theorem 1

Part (i).

Let $\varphi : \mathcal{P}^o \to \mathcal{X}$ a stable and NOM mechanism. Let $\varphi' : \mathcal{P}^o \to \mathcal{X}$ be any mechanism that Pareto-dominates φ. We will prove that φ' is NOM.

Given $d \in D$ and $P_d \in \mathcal{P}^o_d$. As φ' Pareto-dominates φ

$$W(P_d, O^{\varphi'}(P_d)) R_d W(P_d, O^{\varphi}(P_d)) \tag{1}$$

Furthermore, as φ is NOM, given an arbitrary $P'_d \in \mathcal{P}^o_d$, there exists a sub-profile $P_{-d} \in \mathcal{P}^o_{-d}$ of substitutable preferences such that

$$W(P_d, O^{\varphi}(P_d)) R_d \varphi_d(P'_d, P_{-d}) \tag{2}$$

We denote $\widetilde{X} := \varphi(P'_d, P_{-d})$, and consider the sub-profile $\widetilde{P}_{-d} \in \mathcal{P}^o_{-d}$ defined as follows: Given $i \in D \setminus \{d\}$, set $\widetilde{P}_i := \widetilde{X}_i, \widetilde{X}_{i1}, ..., \widetilde{X}_{ik}, \varnothing$ where $\widetilde{X}_{i1}, ..., \widetilde{X}_{ik}$ denote the non-empty proper subsets of $\widetilde{X}_i$, ordered as in P_i. Observe that $\widetilde{P}_i$ satisfies substitutability because otherwise P_i would fail to be substitutable.

($\widetilde{P}_i := \varnothing$ in case $\widetilde{X}_i = \varnothing$)

Next, we will state a claim about the profile $(P'_d, \widetilde{P}_{-d})$, which will be useful to prove our theorem.

<u>Claim 1:</u> $\varphi(P'_d, P_{-d}) = \varphi(P'_d, \widetilde{P}_{-d}) = \varphi'(P'_d, \widetilde{P}_{-d})$

Let $\widetilde{P} = (P'_d, \widetilde{P}_{-d})$. We will show that $\widetilde{X}$ is the only stable allocation in $\widetilde{P}$. Then, the first equality will follow as a direct consequence.

From the definition of $\widetilde{P}$ and the fact that $\widetilde{X}$ is stable at (P'_d, P_{-d}), it is clear that $\widetilde{X}$ is stable at $\widetilde{P}$. Assume that there exists an allocation Y such that $Y \neq \widetilde{X}$ and Y is also stable at $\widetilde{P}$. By the definition of $\widetilde{P}_{-d}$ and the individual rationality of Y, we have

$$Y_i \subseteq \widetilde{X}_i \text{ for all } i \in D \setminus \{d\} \tag{3}$$

Moreover, $Y_d \setminus \widetilde{X}_d \neq \varnothing$, since otherwise $Y \subsetneq \widetilde{X}$, and by considering any contract $w \in \widetilde{X} \setminus Y$, we would have a blocking contract for Y: in fact, $w \in C_{w_D}(\widetilde{X}) \cap C_{w_H}(\widetilde{X})$, because $\widetilde{X}$ is individually rational in $\widetilde{P}$. Then, due to substitutability, $w \in C_{w_D}(Y \cup \{w\}) \cap C_{w_H}(Y \cup \{w\})$.

Next, we will prove that

$$Y_d P'_d \widetilde{X}_d \tag{4}$$

In the case that $\widetilde{X}_d = \varnothing$, 4 follows trivially because Y is individually rational in $\widetilde{P}$.

Otherwise, if $\widetilde{X}_d \neq \varnothing$, since $Y_d \neq \widetilde{X}_d$, we will assume that $\widetilde{X}_d P'_d Y_d$ in order to reach a contradiction. From our previous assumption, it follows that $C_d(\widetilde{X}_d \cup Y_d) R'_d \widetilde{X}_d P'_d Y_d$. This implies $C_d(\widetilde{X}_d \cup Y_d) \setminus Y_d \neq \varnothing$, because Y is individually rational in $\widetilde{P}$. Then, consider a contract $x \in C_d(\widetilde{X}_d \cup Y_d) \setminus Y_d$. By substitutability, $x \in C_d(\{x\} \cup Y_d)$. Moreover, all contracts in Y_{x_H} involve doctors in $D \setminus \{d\}$ because $x \notin Y$ and x is the only contract between d and x_H that can be acceptable for d in $\widetilde{P}$ (as well as in any preference profile in $\mathcal{P}^o$). Consequently, $Y_{x_H} \subseteq \widetilde{X}_{x_H}$ according to 3. In addition, $x \in C_{x_H}(\widetilde{X}_{x_H})$ since $\widetilde{X}$ is individually rational in $\widetilde{P}$ and then, by substitutability, $x \in C_{x_H}(Y_{x_H} \cup \{x\})$. Therefore, x would be a blocking contract for Y, which leads to a contradiction.

So, 4 holds. Consequently, $C_d(\widetilde{X}_d \cup Y_d) R'_d Y_d P'_d \widetilde{X}_d$. Observe that the last implies $C_d(\widetilde{X}_d \cup Y_d) \setminus \widetilde{X}_d \neq \varnothing$ because $\widetilde{X}$ is individually rational in $\widetilde{P}$. Now, consider a contract $y \in C_d(\widetilde{X}_d \cup Y_d) \setminus \widetilde{X}_d$. Because of substitutability, $y \in C_d(\widetilde{X}_d \cup \{y\})$, and consequently,

$$y \notin C_{y_H}(\widetilde{X}_{y_H} \cup \{y\}) \tag{5}$$

since otherwise, y would block $\widetilde{X}$ at $\widetilde{P}$. As Y is individually rational, we have $y \in C_{y_H}(Y_{y_H})$. Then, by 5 and the fact that $Y_{y_H} \subseteq \widetilde{X}_{y_H} \cup \{y\}$ due to 3, a contract $z \in \widetilde{X}_{y_H} \setminus Y_{y_H}$ must exist, such that $z \in C_{y_H}(\widetilde{X}_{y_H} \cup \{y\})$. Because of substitutability, the last implies

$$z \in C_{y_H}(Y_{y_H} \cup \{z\}) \tag{6}$$

(Observe that $z_H = y_H$). Furthermore, due to $z \in \widetilde{X}_{y_H} \setminus Y_{y_H}$, we have $z \neq y$, and so $z_D \neq d$, since y is the only contract between d and y_H that can be acceptable for d in $\widetilde{P}$. Consequently, $Y_{z_D} \subseteq \widetilde{X}_{z_D}$ due to 3. Then, given that $z \in C_{z_D}(\widetilde{X}_{z_D})$ because $\widetilde{X}$ is individually rational in $\widetilde{P}$, it follows that

$$z \in C_{z_D}(Y_{z_D} \cup \{z\}) \tag{7}$$

due to substitutability. So, z is a blocking contract for Y at $\widetilde{P}$, which leads to a contradiction. The contradiction arises from the assumption that there exists an allocation Y such that $Y \neq \widetilde{X}$ and Y is stable in $\widetilde{P}$. Therefore, $\widetilde{X}$ is the only stable allocation in $\widetilde{P}$.

In order to prove the second equality in Claim 1, we will show that if X' is an individually rational allocation in $\widetilde{P}$ and $X'_i \widetilde{R}_i \widetilde{X}_i$ for all $i \in D$, then $X' = \widetilde{X}$

In fact, from definition of $\widetilde{P}_{-d}$, it follows that $\widetilde{X}_i \widetilde{R}_i X'_i$ for each $i \in D \setminus \{d\}$. Then,

$$X'_i = \widetilde{X}_i \text{ for all } i \in D \setminus \{d\}. \tag{8}$$

Now, suppose that $X'_d P'_d \widetilde{X}_d R'_d \varnothing$. Then, $C_d(X'_d \cup \ \widetilde{X}_d) R'_d X'_d P'_d \widetilde{X}_d$, and consequently, $C_d(X'_d \cup \ \widetilde{X}_d) \setminus \widetilde{X}_d \neq \varnothing$ because $\widetilde{X}$ is individually rational in $\widetilde{P}$. Then, consider a contract $w \in C_d(X'_d \cup \ \widetilde{X}_d) \setminus \widetilde{X}_d$. Due to substitutability, we have $w \in C_d(\widetilde{X} \cup \{w\})$. Because $w \notin \widetilde{X}$ and there is no other contract between w_H and d that is acceptable for d in $\widetilde{P}$, we have $\widetilde{X}_{w_H} \subseteq \cup_{i \in D \setminus \{d\}} \widetilde{X}_i = \cup_{i \in D \setminus \{d\}} X'_i$, where the last equality is due to 8. So, $\widetilde{X}_{w_H} \cup \{w\} \subseteq X'$. Moreover, $w \in C_{w_H}(X')$ since X' is individually rational in $\widetilde{P}$. Then, because of substitutability, $w \in C_{w_H}(\widetilde{X} \cup \{w\})$. Therefore, w is a blocking contract for $\widetilde{X}$ at $\widetilde{P}$, which leads to a contradiction. This completes the proof of Claim 1.

Finally, from 1, 2 and Claim 1, it follows that $W(P_d, O^{\varphi'}(P_d)) R_d W(P_d O^{\varphi'}(P'_d))$ for each $P'_d \in \mathcal{P}^o_d$. Therefore, φ' is NOM.

This completes the proof of *Part (i)*.

Part (ii).

Given $d \in D$ and $P_d \in \mathcal{P}^o_d$, let $\widehat{X}$ be an allocation obtained through the $HDAM$, which matches d with the worst outcome it can obtain by reporting its true preferences, i.e., $\widehat{X}_d = W(P_d, O^{HDAM}(P_d))$. Because X is finite, there exists a sub-profile $P_{-d} \in \mathcal{P}^o_{-d}$ such that $HDAM_d(P_d, P_{-d}) = \widehat{X}$. Consider the sub-profile $\widehat{P}_{-d} \in \mathcal{P}^o_{-d}$ defined as follows:

Given $i \in D \setminus \{d\}$, set $\widehat{P}_i := \widehat{X}_i, \widehat{X}_{i1}, ..., \widehat{X}_{ik}, \varnothing$ where $\widehat{X}_{i1}, ..., \widehat{X}_{ik}$ denote the non-empty proper subsets of $\widehat{X}_i$, ordered as in P_i. Observe that $\widehat{P}_i$ satisfies substitutability, because otherwise, P_i would fail to be substitutable.

($\widehat{P}_i := \varnothing$ in case $\widehat{X}_i = \varnothing$)

<u>Claim 2</u> :

$$\widehat{X}_d R_d HDAM_d(P'_d, \widehat{P}_{-d}) \text{ for all } P'_d \in \mathcal{P}^o_d \setminus \{P_d\}. \tag{9}$$

Assume, for the sake of contradiction, that there exists $P'_d \in \mathcal{P}^o_d \setminus \{P_d\}$ such that

$$HDAM_d(P'_d, \widehat{P}_{-d}) P_d \widehat{X}_d R_d \varnothing. \tag{10}$$

Then, $HDAM_d(P'_d, \widehat{P}_{-d}) = Z$ for some non-empty set $Z \subseteq \boldsymbol{X}_d$, and by the definition of choice set, $C_d(Z \cup \widehat{X}_d) P_d \widehat{X}_d$. Consequently, there exists at least one contract $z \in C_d(Z \cup \widehat{X}_d) \setminus \widehat{X}_d$, and due to substitutability, we have

$$z \in C_d(\{z\} \cup \widehat{X}) \tag{11}$$

Now, consider the hospital z_H and observe that $\widehat{X}_{z_H} \subseteq \cup_{i \in D \setminus \{d\}} \widehat{X}_i$, because $z \notin \widehat{X}$ and z is the only contract involving both d and z_H that is acceptable for d in P'_d (or any other preference in $\mathcal{P}^o_d$) . Hence, by the definition of $\widehat{P}_{-d}$, contracts in $\widehat{X}_{z_H}$ are never rejected by doctors along the HDA algorithm at $(P'_d, \widehat{P}_{-d})$. Let T' be the total number of iterations that HDA algorithm requires to converge at $(P'_d, \widehat{P}_{-d})$, and let $X^{T'-1}$ be the set of contracts that have not been rejected until the end of stage $T'-1$, as defined in the HDA algorithm. Then, $\widehat{X}_{z_H} \subseteq X^{T'-1}$, and since $z \in Z = HDAM_d(P'_d, \widehat{P}_{-d})$, it follows that z was offered by z_H in the last iteration. Therefore, $z \in C_H(X^{T'-1})$, and consequently, due to substitutability, we have

$$z \in C_{z_H}(\{z\} \cup \widehat{X}_{z_H}) = C_{z_H}(\{z\} \cup \widehat{X}) \tag{12}$$

But (11) and (12) imply that z is a blocking contract for $\widehat{X} = HDAM(P_d, \widehat{P}_{-d})$. This contradiction concludes the proof of Claim 2.

Therefore, for all $P'_d \in \mathcal{P}^o_d \setminus \{P_d\}$, (9) implies

$$W(P_d, O^{HDA}(P_d)) = \widehat{X}_d R_d HDAM(P'_d, \widehat{P}_{-d}) R_d W(P_d, O^{HDAM}(P'_d)). \tag{13}$$

Thus, $HDAM$ is NOM.

This concludes the proof of Part (ii).

Part (iii)

The proof of (iii) follows from Remark 2 , (i), and (ii). □

Theorem 2 *Assume that the domain of doctors' preference profile is $\mathcal{P}^u$, i.e., consider a many-to-many matching model with unitary contracts. Then,*

(i) The $DDAM$ is not obviously manipulable.

(ii) The $HDAM$ is obviously manipulable.

(iii) Any efficient and stable-dominating mechanism is obviously manipulable.

The following lemmas are necessary in order to prove Theorem 2.

Lemma 1 *Given $P \in \mathcal{P}$, let O^t and O^t_A be defined as in DDA algorithm. Then, for every $h \in H$*

$$C_h(O^t_A) = C_h(O^t) \text{ for all } t = 1, ...T \tag{14}$$

where T is the number of iterations of the DDA algorithm at P.

The proof presented in Arribillaga and Risma (2025), for a many-to-one model, also extends directly to the many-to-many setting considered in this paper, and is therefore omitted.

Lemma 2 *Given $P \in \mathcal{P}$ and $d \in D$, assume that $Z \subseteq X_d$ is an arbitrary allocation such that Z P_d $DDAM(P)_d$. Then, there exists a contract $z \in Z \setminus DDAM(P)_d$ such that z is offered by d and rejected by z_H in some iteration of the DDA.*

Proof. Suppose that the conclusion of Lemma 2 does not hold. Then, $Z \subseteq X^T$ where T is the number of iterations required for $DDAM(P)$ to converge, and X^T is the set of contracts that have not been rejected at the beginning of iteration T. By the definition of $DDAM$, the set of offers made by d in the last iteration is $(O^T)_d = C_d(X^T)$, and by Lemma 1, it coincides with the final outcome $DDAM(P)_d$. Thus, $C_d(X^T) = DDAM(P)_d$. This contradicts the definition of the choice set, since $Z \subseteq X^T$ and Z P_d $DDAM(P)_d$. □

Proof of Theorem 2

Part (i).

Given $d \in D$, and $P_d \in \mathcal{P}^u_d$, let $\widehat{X}$ be an allocation produced by $DDAM$ that assigns d its worst possible outcome obtainable by truthfully reporting its preferences, i.e., $\widehat{X}_d = W(P_d, O^{DDAM}(P_d))$. Since X is finite, there exists a profile $\widehat{P}_{-d} \in \mathcal{P}^u_{-d}$ such that $DDAM(P_d, \widehat{P}_{-d}) = \widehat{X}$.

If $\widehat{X}_d$ were d's first-ranked option, then $DDAM$ would not admit manipulations by d at P_d.

Otherwise, suppose that d´s true preference is as follows: $P_d : Y^1, ..., Y^{n-1}, \widehat{X}_d, ...$ so that, $\widehat{X}_d$ ranks in the n-th position (with n>1). We will prove that no allocation that d prefers over $\widehat{X}_d$ can be its worst possible outcome obtainable by declaring (any) false preferences. Let T be the number of iterations that DDA algorithm needs to converge to $DDAM(P_d, \widehat{P}_{-d})$, and let $K \in \{2, ..., T\}$ be the integer such that d offers $\widehat{X}_d$ for the first time in the K-th iteration of the $DDAM(P_d, \widehat{P}_{-d})$. Let O^t be the set of accumulated offers made by doctors before and during iteration t of $DDAM(P_d, \widehat{P}_{-d})$; and let O^t_A be the set of accumulated offers performed by doctors before and during iteration t of the $DDAM(P_d, \widehat{P}_{-d})$ for $t = 1, ..., T$.

Next, we identify a particular profile of preferences $\widetilde{P}_{-d} \in \mathcal{P}^u_{-d}$ that leads d to obtain an an outcome that is equal to or worse than $\widehat{X}_d$ by declaring any false preferences P'_d. Given $i \in D \setminus \{d\}$, define:

$$\widetilde{P}_i := C_H((O^{K-1}_A)_{-d})_i, A^1_i, A^2_i, \cdots,$$

where $A^1_i, A^2_i, \cdots$ denote the proper subsets of $C_H((O^{K-1}_A)_{-d})_i$, ordered as they appear in $\widehat{P}_{-d}$. ($\widetilde{P}_i := \varnothing$ in case $C_H((O^{K-1}_A)_{-d})_i = \varnothing$)

Consider an arbitrary false preference $P'_d \in \mathcal{P}^u_d \setminus \{P_d\}$. Suppose that $DDAM(P'_d, \widetilde{P}_{-d})_d = Y^j$ for some $j \in \{1, ..., n-1\}$, i.e., $DDAM(P'_d, \widetilde{P}_{-d})_d) P_d \widehat{X}_d$. Let $\widetilde{T}$ be the number of iterations required for the DDA algorithm to converge to $DDAM(P'_d, \widetilde{P}_{-d})$; and let $\widetilde{O}^t_A$ denote the set of contracts offered by doctors before and during iteration $t = 1, ..., \widetilde{T}$ of the $DDAM$ when applied under the profile $(P'_d, \widetilde{P}_{-d})$. Note that $(\widetilde{O}^{\widetilde{T}}_A)_{-d} = C_H((O^{K-1}_A)_{-d})$. Then, by definition of DDA algorithm $DDAM(P'_d, \widetilde{P}_{-d}) = C_H(\widetilde{O}^{\widetilde{T}}_A) = C_H((\widetilde{O}^{\widetilde{T}}_A))_d \cup (\widetilde{O}^{\widetilde{T}}_A)_{-d}) = C_H(\widetilde{O}^{\widetilde{T}}_A)_d \cup C_H((O^{K-1}_A)_{-d}))$. Therefore, since $Y^j \subseteq (\widetilde{O}^{\widetilde{T}}_A)_d$ and by substitutability, we obtain

$$y \in C_{y_H}((\{y\} \cup C_{y_H}((O^{K-1}_A)_{-d})) \tag{15}$$

for all $y \in Y^j$.

Now, according to Lemma 2, there exists a contract $y \in Y^j$ which is offered by d and rejected by the hospital y_H during some iteration $\widehat{k} \in \{1, ..., K-1\}$ of the DDA algorithm, when applied under the profile $(P_d, \widehat{P}_{-d})$.

Since $P_d \in \mathcal{P}^u_d$, observe that such a contract is the only one offered by d to y_H during the iteration $\widehat{k}$. Therefore, $O^{\widehat{k}}_{y_H} = \{y\} \cup (O^{\widehat{k}}_{-d})_{y_H}$, and $y \notin C_{y_H}(O^{\widehat{k}})$. Next, by substitutability and the fact that $O^{\widehat{k}}_{-d} \subseteq (O^{K-1}_A)_{-d}$

$$y \notin C_{y_H}(\{y\} \cup (O^{K-1}_A)_{-d})$$

Now, by a choice set property, the last can be rewritten as

$$y \notin C_{y_H}(\{y\} \cup C_{y_H}((O^{K-1}_A)_{-d})),$$

which contradicts 15. Thus, $DDAM(P'_d, \widetilde{P}_{-d})_d \neq Y^j$ for all $j \in \{1, ..., n-1\}$, i.e., $\widehat{X}_d \; R_d \; DDAM(P'_d, \widetilde{P}_{-d})_d$.

Therefore,

$$W(P_d, O^{DDAM}(P_d)) = \widehat{X}_d R_d W(P_d, O^{DDAM}(P'_d)).$$

Part (ii).

Since this is a negative result, the same example from the proof of Theorem 4 in Arribillaga and Risma (2025) , which demonstrates that $HDAM$ is obviously manipulable in a many-to-one model, can be applied to prove the result in our more general model.

Part (iii).

Our proof of part (iii) in Theorem 3 also applies in this case.

□

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