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# Cycles to compute the full set of many-to-many stable matchings\*

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## Abstract

In a many-to-many matching model in which agents' preferences satisfy substitutability and the law of aggregate demand, we present an algorithm to compute the full set of stable matchings. This algorithm relies on the idea of “cycles in preferences” and generalizes the algorithm presented in [Roth and Sotomayor \(1990\)](#) for the one-to-one model.

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*Keywords:* Stable matchings, cyclic matching, substitutable preferences.

## 1 Introduction

In many-to-many matching models, there are two disjoint sets of agents: firms and workers. Each firm wishes to hire a set of workers and each worker wishes to work for a set of firms. Many real-world markets are many-to-many, for instance, the market for medical interns in the UK ([Roth and Sotomayor, 1990](#)), the assignment of teachers to high schools in some countries (35% of teachers in Argentina work in more than one school). A matching is an assignment of sets of workers to firms, and of sets of firms to

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workers, so that a firm is assigned to a worker if and only if this worker is also assigned to that firm. In these models, the most studied solution is the set of stable matchings. A matching is stable if all agents are matched to an acceptable subset of partners and there is no unmatched firm-worker pair, both of which would prefer to add the other to their current subset of partners.<sup>1</sup> In their seminal paper, [Gale and Shapley \(1962\)](#) introduce the Deferred Acceptance (*DA*, from now on) algorithm to show the existence of a stable matching in the one-to-one model. This algorithm computes the optimal stable matching for one side of the market. Later, the *DA* algorithm is adapted to the many-to-many case by [Roth \(1984\)](#).

In this paper, we present an algorithm to compute the full set of many-to-many stable matchings. In the one-to-one model, beginning from a stable matching and through a procedure of reduction of preferences, [Roth and Sotomayor \(1990\)](#) define a “cycle in preferences” that allows them to generate a new matching, called a “cyclic matching”, that turns out to be stable.<sup>2</sup> They present an algorithm that, starting from an optimal stable matching for one side of the market and by constructing all cycles and its corresponding cyclic matchings, computes the full set of one-to-one stable matchings (see [Irving and Leather, 1986](#); [Gusfield and Irving, 1989](#); [Roth and Sotomayor, 1990](#), for more details). The purpose of our paper is to extend Roth and Sotomayor’s construction to a many-to-many environment.

Our general framework assumes substitutability on all agents’ preferences. This condition, first introduced by [Kelso and Crawford \(1982\)](#), is the weakest requirement in preferences in order to guarantee the existence of many-to-many stable matchings. An agent has substitutable preferences when she wants to continue being matched to an agent of the other side of the market even if other agents become unavailable. Given an agent’s preference, [Blair \(1988\)](#) defines a partial order over subsets of agents of the other side of the market as follows: one subset is Blair-preferred to another subset if, when all agents of both subsets are available, only the agents of the first subset are chosen.<sup>3</sup> When preferences are substitutable, the set of stable matchings has a lattice structure with respect to the unanimous Blair order for any side of the market.<sup>4</sup>

In addition to substitutability, we require that agents’ preferences satisfy the “law of aggregate demand” (*LAD*, from now on).<sup>5</sup> This condition says that when an agent chooses from an expanded set, it selects at least as many agents as before. Under these two assumptions on preferences, the set of stable matchings satisfies the so-called Ru-

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<sup>1</sup>This notion of stability is known in the literature as “pairwise stability”.

<sup>2</sup>[Roth and Sotomayor \(1990\)](#) adapt the algorithm presented in [Irving and Leather \(1986\)](#).

<sup>3</sup>Blair’s order of an agent is more restrictive than the individual preference of that agent.

<sup>4</sup>For instance, a set of workers is Blair-preferred to another set of workers for the firms if the first set is Blair-preferred to the latter set for each firm.

<sup>5</sup>This property is first studied by [Alkan \(2002\)](#) under the name of “cardinal monotonicity”. See also [Hatfield and Milgrom \(2005\)](#).

ral Hospitals Theorem, which states that each agent is matched with the same number of partners in every stable matching. Substitutability of preferences and the law of aggregate demand ensure that suitable generalizations of the concepts of “cycle” and “cyclic matching” can be defined. To do this, given a substitutable preference profile and two stable matchings that are unanimously Blair-comparable (for one side of the market), we define a “reduced preference profile” with respect to these two stable matchings and show that this profile is also substitutable and satisfies *LAD*. Next, we adapt Roth and Sotomayor’s notion of a cycle for our reduced preference profile and use this many-to-many notion of a cycle to define a cyclic matching. This new matching turns out to be stable not only for this reduced preference profile but also for the original preference profile. With all these ingredients we can describe our algorithm as follows. Given a preference profile, by the *DA* algorithm compute the two optimal stable matchings, one for each side of the market. Pick one side of the market, say the firms’ side, and obtain the reduced preference profile with respect to the firms’ optimal and the workers’ optimal stable matchings. In each of the following steps, for each reduced preference profile obtained in the previous step, compute: (i) each cycle for this profile, (ii) its corresponding cyclic matching, and (iii) the reduced preference profile with respect to this cyclic matching and the worker optimal stable matching. The algorithm stops in the step where all the cyclic matchings computed are equal to the worker optimal stable matching. The firms’ optimal stable matching together with all the cyclic matchings obtained by the algorithm encompass the full set of stable matchings.

Several papers calculate the full set of stable matchings in two-sided matching models. [McVitie and Wilson \(1971\)](#) are the first to present an algorithm that computes the full set of one-to-one stable matchings. This algorithm starts at the optimal stable matching for one side of the market and then, at each step, breaks some matched pair and applies the *DA* algorithm to the new preference profile in which the broken matched pair is no longer acceptable. This algorithm is generalized by [Martínez et al. \(2004\)](#) to a many-to-many matching market in which agents’ preferences satisfy substitutability. Extending the approach in [Irving and Leather \(1986\)](#), [Cheng et al. \(2008\)](#) compute the full set of stable matchings for a many-to-one model in which all agents have a strict ordering over the agents of the other side of the market.<sup>6</sup> They make use of some results first presented by [Bansal et al. \(2007\)](#). [Dworczak \(2021\)](#), in the one-to-one model, generalizes the *DA* algorithm allowing both sides of the market to make offers in a specific ordering. He proposes a generalized *DA* algorithm with “compensation chains” and proves that: (i) for each order of the agents, the algorithm obtains a stable

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<sup>6</sup>This setting is equivalent to the one defined by [Roth \(1985\)](#) for the many-to-one model in which firms have responsive preferences over subsets of workers.

matching, and (ii) each stable matching can be obtained as the output of the algorithm for some order of the agents.

The paper is organized as follows. In Section 2 we present the preliminaries. The reduction procedure of preferences is presented in Section 3. Section 4 contains the definition of a cycle in preferences together with the algorithm that computes the many-to-many stable set. Concluding remarks are gathered in Section 5. All proofs are relegated to Appendix A.

## 2 Preliminaries

We consider many-to-many matching markets where there are two disjoint sets of agents: the set of firms  $F$  and the set of workers  $W$ . Each firm  $f \in F$  has a strict preference relation  $P_f$  over the set of all subsets of  $W$ . Each worker  $w \in W$  has a strict preference relation  $P_w$  over the set of all subsets of  $F$ . We denote by  $P$  the preference profile for all agents: firms and workers. A (many-to-many) matching market is denoted by  $(F, W, P)$ . Since the sets  $F$  and  $W$  are kept fixed throughout the paper, we often identify the market  $(F, W, P)$  with the preference profile  $P$ . Given an agent  $a \in F \cup W$ , a set  $S$  in the opposite side of the market is **acceptable for  $a$  under  $P$**  if  $SP_a \neq \emptyset$ . A pair  $(f, w) \in F \times W$  is **mutually acceptable under  $P$**  if  $\{f\}$  is acceptable for  $w$  under  $P$  and  $\{w\}$  is acceptable for  $f$  under  $P$ . In this paper, the preference relation  $P_a$  is represented by the ordered list of its acceptable sets (from most to least preferred).<sup>7</sup> Given a set of workers  $W' \subseteq W$  and a firm  $f \in F$ , let  $C_f(W')$  (the choice set for  $f$ ) denote firm  $f$ 's most preferred subset of  $W'$  according to the preference relation  $P_f$ . Symmetrically, given a set of firms  $F' \subseteq F$  and a worker  $w \in W$ , let  $C_w(F')$  (the choice set for  $w$ ) denote worker  $w$ 's most preferred subset of  $F'$  according to the preference relation  $P_w$ .

**Definition 1** A *matching*  $\mu$  is a function from the set  $F \cup W$  into  $2^{F \cup W}$  such that for each  $w \in W$  and for each  $f \in F$ :

- (i)  $\mu(w) \subseteq F$ ,
- (ii)  $\mu(f) \subseteq W$ ,
- (iii)  $w \in \mu(f)$  if and only if  $f \in \mu(w)$ .

Agent  $a \in F \cup W$  is **matched** if  $\mu(a) \neq \emptyset$ , otherwise she is **unmatched**. For the following definitions, fix a preference profile  $P$ . A matching  $\mu$  is **blocked by agent  $a$**  if  $\mu(a) \neq C_a(\mu(a))$ . A matching is **individually rational** if it is not blocked by any individual agent. A matching  $\mu$  is **blocked by a firm-worker pair  $(f, w)$**  if  $w \notin \mu(f), w \in$

<sup>7</sup>For instance,  $P_f : w_1 w_2, w_3, w_1, w_2$  indicate that  $\{w_1, w_2\} P_f \{w_3\} P_f \{w_1\} P_f \{w_2\} P_f \emptyset$  and  $P_w : f_1 f_3, f_3, f_1$  indicates that  $\{f_1, f_3\} P_w \{f_3\} P_w \{f_1\} P_w \emptyset$ .

$C_f(\mu(f) \cup \{w\})$ , and  $f \in C_w(\mu(w) \cup \{f\})$ . A matching  $\mu$  is **stable** if it is not blocked by any individual agent or any firm-worker pair. The set of stable matchings for a preference profile  $P$  is denoted by  $S(P)$ .

Agent  $a$ 's preference relation satisfies **substitutability** if, for each subset  $S$  of the opposite side of the market (for instance, if  $a \in F$  then  $S \subseteq W$ ) that contains agent  $b$ ,  $b \in C_a(S)$  implies that  $b \in C_a(S' \cup \{b\})$  for each  $S' \subseteq S$ . Moreover, if agent  $a$ 's preference relation is substitutable then it holds that

$$C_a(S \cup S') = C_a(C_a(S) \cup S') \quad (1)$$

for each pair of subsets  $S$  and  $S'$  of the opposite side of the market.<sup>8</sup>

Given a firm  $f$ , Blair (1988) defines a partial order for  $f$  over subsets of workers as follows: given firm  $f$ 's preference relation  $P_f$  and two subsets of workers  $S$  and  $S'$ , we write  $S \succeq_f S'$  whenever  $S = C_f(S \cup S')$ , and  $S \succ_f S'$  whenever  $S \succeq_f S'$  and  $S \neq S'$ . The partial orders  $\succeq_w$  and  $\succ_w$  for worker  $w$  are defined analogously. Given a preference profile  $P$  and two matchings  $\mu$  and  $\mu'$ , we write  $\mu \succeq_F \mu'$  whenever  $\mu(f) \succeq_f \mu'(f)$  for each  $f \in F$ , and we write  $\mu \succ_F \mu'$  if, in addition,  $\mu \neq \mu'$ .<sup>9</sup> Similarly, we define  $\succeq_W$  and  $\succ_W$ .

The set of stable matchings under substitutable preferences is very well structured. Blair (1988) proves that this set has two lattice structures, one with respect to  $\succeq_F$  and the other one with respect to  $\succeq_W$ . Furthermore, it contains two distinctive matchings: the firm-optimal stable matching  $\mu_F$  and the worker-optimal stable matching  $\mu_W$ . The matching  $\mu_F$  is unanimously considered by all firms to be the best among all stable matchings and  $\mu_W$  is unanimously considered by all workers to be the best among all stable matchings, according to the respective Blair's partial orders (see Roth, 1984; Blair, 1988, for more details).

Agent  $a$ 's preference relation satisfies the **law of aggregate demand (LAD)** if for all subsets  $S$  of the opposite side of the market and all  $S' \subseteq S$ ,  $|C_a(S')| \leq |C_a(S)|$ .<sup>10</sup> When preferences are substitutable and satisfy LAD, the lattices  $(S(P), \succeq_F)$  and  $(S(P), \succeq_W)$  are dual; that is,  $\mu \succeq_F \mu'$  if and only if  $\mu' \succeq_W \mu$  for  $\mu, \mu' \in S(P)$ . This is known as the "polarization of interests" result (see Alkan, 2002; Li, 2014, among others). Throughout this paper, we assume that the preferences of all agents are substitutable and satisfy LAD.

<sup>8</sup>See Proposition 2.3 in Blair (1988), for more details.

<sup>9</sup>We call  $\succeq_F$  the unanimous Blair order for the firms.

<sup>10</sup> $|S|$  denotes the number of agents in  $S$ .

### 3 The reduction procedure

In this section, we present a reduction procedure that will allow us to define a cycle in preferences, a concept that is essential for developing our algorithm. Given a substitutable preference profile and two Blair-comparable (for the firms) stable matchings, this reduction procedure generates a new preference profile, in which the most Blair-preferred stable matching is the firm-optimal matching and the least Blair-preferred stable matching is the worker-optimal matching, for the market identified with this new preference profile. The reduction procedure is described as follows. Let  $\mu$  and  $\tilde{\mu}$  be stable matchings for matching market  $(F, W, P)$  such that  $\mu \succeq_F \tilde{\mu}$ .

- Step 1:** (a) For each  $f \in F$ , each  $W' \subset W$  such that  $W' \succ_f \mu(f)$ , and each  $\tilde{w} \in W' \setminus \mu(f)$ , remove each  $\tilde{W} \subset W$  such that  $\tilde{w} \in \tilde{W}$  from  $f$ 's list of acceptable sets of workers.
- (b) For each  $w \in W$ , each  $F' \subset F$  such that  $F' \succ_w \tilde{\mu}(w)$ , and each  $\tilde{f} \in F' \setminus \tilde{\mu}(w)$ , remove each  $\tilde{F} \subset F$  such that  $\tilde{f} \in \tilde{F}$  from  $w$ 's list of acceptable sets of firms.
- Step 2:** (a) For each  $f \in F$ , each  $W' \subset W$  such that  $\tilde{\mu}(f) \succ_f W'$ , and each  $\tilde{w} \in W' \setminus \tilde{\mu}(f)$ , remove each  $\tilde{W} \subset W$  such that  $\tilde{w} \in \tilde{W}$  from  $f$ 's list of acceptable sets of workers.
- (b) For each  $w \in W$ , each  $F' \subset F$  such that  $\mu(w) \succ_w F'$ , and each  $\tilde{f} \in F' \setminus \mu(w)$ , remove each  $\tilde{F} \subset F$  such that  $\tilde{f} \in \tilde{F}$  from  $w$ 's list of acceptable sets of firms.
- Step 3:** After Steps 1 and 2 are performed, if  $f$  is not acceptable for  $w$  (that is, if  $\{f\}$  is not on  $w$ 's preference list as now modified), remove each  $W' \subset W$  such that  $w \in W'$  from  $f$ 's list of acceptable sets of workers. If  $w$  is not acceptable for  $f$  (that is, if  $\{w\}$  is not on  $f$ 's preference list as now modified), remove each  $F' \subset F$  such that  $f \in F'$  from  $w$ 's list of acceptable sets of firms.

The profile obtained by this procedure is called the **reduced preference profile with respect to  $\mu$  and  $\tilde{\mu}$** , and is denoted by  $P^{\mu, \tilde{\mu}}$ . When  $\tilde{\mu} = \mu_W$ , the profile is simply called the **reduced preference profile with respect to  $\mu$** , and is denoted by  $P^\mu$ .

Let us put in words how the reduction procedure works. In Step 1 (a), for each  $f \in F$ , if a worker is not in  $\mu(f)$  but belongs to a subset that is Blair-preferred to  $\mu(f)$ , the procedure eliminates each subset that contains this worker from firm  $f$ 's list of acceptable subsets. Step 1 (b) performs an analogous elimination in each worker's preference list. In Step 2 (a), for each  $f \in F$ , if a worker is not in  $\tilde{\mu}(f)$  and  $\tilde{\mu}(f)$  is Blair-preferred to a subset that includes this worker, the procedure eliminates each subset that contains this worker from firm  $f$ 's list of acceptable subsets. Step 2 (b) performs an analogous elimination in each worker's preference list. In Step 3, after Step 1 and



Step 2 are performed, the procedure eliminates all subsets of agents needed in order to make all pairs of agents mutually acceptable.

By  $C_f^{\mu, \tilde{\mu}}(W')$  we denote the firm  $f$ 's most preferred subset of  $W'$  according to the preference relation  $P_f^{\mu, \tilde{\mu}}$ . Similar notation is used for the choice sets according to the preference relations  $P_w^{\mu, \tilde{\mu}}$ ,  $P_f^\mu$ , and  $P_w^\mu$ . Some remarks on the reduced preference relations are in order.

**Remark 1** Let  $P$  be a market and assume  $\mu, \tilde{\mu} \in S(P)$ . Then the following statements hold.

- (i)  $\mu(f)$  is the most preferred subset of workers in  $f$ 's reduced preference relation (i.e.  $\mu(f) = C_f^{\mu, \tilde{\mu}}(W)$ ) and  $\tilde{\mu}(w)$  is the most preferred subset of firms in  $w$ 's reduced preference relation (i.e.  $\tilde{\mu}(w) = C_w^{\mu, \tilde{\mu}}(F)$ ).
- (ii)  $\mu$  is the firm-optimal stable matching under  $P^{\mu, \tilde{\mu}}$  and  $\tilde{\mu}$  is the worker-optimal stable matching under  $P^{\mu, \tilde{\mu}}$ . Furthermore,  $\tilde{\mu}$  is the firm-pessimal stable matching under  $P^{\mu, \tilde{\mu}}$  and  $\mu$  is the worker-pessimal stable matching under  $P^{\mu, \tilde{\mu}}$ .
- (iii)  $f$  is acceptable to  $w$  if and only if  $w$  is acceptable to  $f$  under  $P^{\mu, \tilde{\mu}}$ .

The following lemma states that the properties of substitutability and LAD are preserved by the reduction procedure.

**Lemma 1** Let  $\mu, \tilde{\mu} \in S(P)$  and  $a \in F \cup W$ . If  $P_a$  is substitutable and satisfies LAD, then the reduced preference relation  $P_a^{\mu, \tilde{\mu}}$  is substitutable and satisfies LAD.

The following example illustrates the reduction procedure for a matching market.

**Example 1** Let  $(F, W, P)$  be a matching market where  $F = \{f_1, f_2, f_3\}$ ,  $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ , and the preference profile is given by:

$$P_{f_1} : w_1w_2, w_1w_5, w_2w_5, w_1w_3, w_4w_5, w_2w_4, w_1w_4, w_3w_4, w_3w_5, w_2w_3, w_1, w_4, w_3, w_2, w_5$$

$$P_{f_2} : w_3w_6, w_3w_5, w_5w_6, w_2w_5, w_1w_3, w_2w_6, w_1w_5, w_1w_2, w_2w_3, w_1w_6, w_1, w_2, w_3, w_5, w_6$$

$$P_{f_3} : w_2w_4, w_1w_2, w_3w_4, w_2w_3, w_1w_3, w_1w_4, w_1, w_2, w_3, w_4$$

$$P_{w_1} : f_3, f_1, f_2$$

$$P_{w_2} : f_2f_3, f_1f_3, f_1f_2, f_1, f_2, f_3$$

$$P_{w_3} : f_1, f_2$$

$$P_{w_4} : f_1, f_3, f_2$$

$$P_{w_5} : f_2, f_3$$

$$P_{w_6} : f_1f_3, f_3, f_1$$

It is easy to check that these preference relations are substitutable and satisfy LAD. By the DA algorithm, we obtain the two optimal stable matchings:

$$\mu_F = \begin{pmatrix} f_1 & f_2 & f_3 & \emptyset \\ w_1w_2 & w_3w_5 & w_2w_4 & w_6 \end{pmatrix} \text{ and } \mu_W = \begin{pmatrix} f_1 & f_2 & f_3 & \emptyset \\ w_3w_4 & w_2w_5 & w_1w_2 & w_6 \end{pmatrix}.$$

Now, after the reduction procedure is performed, we obtain the reduced preference profile with respect to  $\mu_F, P^{\mu_F}$ :<sup>11</sup>

$$P_{f_1}^{\mu_F} : \mathbf{w_1w_2}, w_1w_3, w_2w_4, w_1w_4, \mathbf{w_3w_4}, w_2w_3, w_1, w_4, w_3, w_2$$

$$P_{f_2}^{\mu_F} : \mathbf{w_3w_5}, \mathbf{w_2w_5}, w_2w_3, w_2, w_3, w_5$$

$$P_{f_3}^{\mu_F} : \mathbf{w_2w_4}, \mathbf{w_1w_2}, w_1w_4, w_1, w_2, w_3, w_4$$

$$P_{w_1}^{\mu_F} : \mathbf{f_3}, \mathbf{f_1}$$

$$P_{w_2}^{\mu_F} : \mathbf{f_2f_3}, \mathbf{f_1f_3}, f_1f_2, f_1, f_2, f_3$$

$$P_{w_3}^{\mu_F} : \mathbf{f_1}, \mathbf{f_2}$$

$$P_{w_4}^{\mu_F} : \mathbf{f_1}, \mathbf{f_3}$$

$$P_{w_5}^{\mu_F} : \mathbf{f_2}$$

$$P_{w_6}^{\mu_F} : \emptyset$$

In order to show how each stage of the procedure works, we turn our attention to preferences  $P_{f_1}$  and  $P_{f_2}$ . At Step 1 of the reduction procedure we remove the following subsets of agents:

$$P_{f_1} : \mathbf{w_1w_2}, w_1w_5, w_2w_5, w_1w_3, w_4w_5, w_2w_4, w_1w_4, \mathbf{w_3w_4}, w_3w_5, w_2w_3, w_1, w_4, w_3, w_2, w_5$$

$$P_{f_2} : \mathbf{w_3w_5}, \mathbf{w_3w_5}, \mathbf{w_5w_6}, \mathbf{w_2w_5}, w_1w_3, \mathbf{w_2w_6}, w_1w_5, w_1w_2, w_2w_3, \mathbf{w_1w_6}, w_1, w_2, w_3, w_5, \mathbf{w_6}.$$

At Step 2 of the reduction procedure we remove the following subsets of agents:

$$P_{f_1} : \mathbf{w_1w_2}, w_1w_5, w_2w_5, w_1w_3, w_4w_5, w_2w_4, w_1w_4, \mathbf{w_3w_4}, w_3w_5, w_2w_3, w_1, w_4, w_3, w_2, w_5$$

$$P_{f_2} : \mathbf{w_3w_5}, \mathbf{w_2w_5}, \mathbf{w_1w_3}, \mathbf{w_1w_5}, \mathbf{w_1w_2}, w_2w_3, \mathbf{w_1}, w_2, w_3, w_5.$$

Since  $f_1$  is not acceptable for  $w_5$  at the original preferences,  $f_1$  is not acceptable for  $w_5$  after Steps 1 and 2 are performed. So at Step 3 we remove the following subsets of agents:

$$P_{f_1} : \mathbf{w_1w_2}, \mathbf{w_1w_5}, \mathbf{w_2w_5}, w_1w_3, \mathbf{w_4w_5}, w_2w_4, w_1w_4, \mathbf{w_3w_4}, \mathbf{w_3w_5}, w_2w_3, w_1, w_4, w_3, w_2, \mathbf{w_5}$$

$$P_{f_2} : \mathbf{w_3w_5}, \mathbf{w_2w_5}, w_2w_3, w_2, w_3, w_5.$$

In this way we obtain  $P_{f_1}^{\mu_F}$  and  $P_{f_2}^{\mu_F}$ . ◇

The following theorem states that the stability of a matching is preserved by the reduction procedure and that there are no new stable matchings for the reduced preference profile. This means that a stable matching in the original preference profile is in between (according to Blair's partial order) of the two stable matchings used to generate the reduced preference profile if and only if it is also stable in the reduced preference profile.<sup>12</sup>

<sup>11</sup>Notice that the subsets assigned in the optimal stable matchings are in bold.

<sup>12</sup>Recall that  $\succeq_F$  and  $\succeq_W$  are dual orders only in the set of stable matchings.

**Theorem 1** Let  $\mu, \tilde{\mu} \in S(P)$  with  $\mu \succeq_F \tilde{\mu}$ . Then,  $\mu' \in S(P)$  and  $\mu \succeq_F \mu' \succeq_F \tilde{\mu}$  if and only if  $\mu' \in S(P^{\mu, \tilde{\mu}})$ .

Notice that by optimality of  $\mu_F$  and  $\mu_W$ , any stable matching  $\mu \in S(P)$  satisfies  $\mu_F \succeq_F \mu$  and  $\mu_W \succeq_W \mu$ . Furthermore, by the polarization of interests,  $\mu \succeq_F \mu_W$ . Then,  $\mu_F \succeq_F \mu \succeq_F \mu_W$ . Thus, as a consequence of Theorem 1 we can state the following corollary.

**Corollary 1**  $S(P) = S(P^{\mu_F})$ .

## 4 Cycles and algorithm

In this section, we present the algorithm to compute the full set of many-to-many stable matchings. First, we introduce its key ingredients: the notion of a cycle in preferences and its corresponding cyclic matching.

### 4.1 Cycles and cyclic matchings

In the one-to-one model, Roth and Sotomayor (1990) present the notion of a cycle in preferences.<sup>13</sup> Their construction can be roughly explained as follows. Consider a one-to-one matching market  $(M, W, P)$  and a stable matching  $\mu \in S(P)$ . A reduced preference profile with respect to  $\mu$  and the worker-optimal stable matching  $\mu_W$ , say  $P^{\mu, \mu_W}$ , is obtained. The important facts about this reduced profile are that: (i)  $\mu(f)$  is  $f$ 's most preferred partner and  $\mu_W(f)$  is  $f$ 's least preferred partner according to  $P_f^{\mu, \mu_W}$ , for each  $f \in F$ ; and (ii)  $\mu_W(w)$  is  $w$ 's most preferred partner and  $\mu(w)$  is  $w$ 's least preferred partner according to  $P_w^{\mu, \mu_W}$ , for each  $w \in W$ . A cycle for  $P^{\mu, \mu_W}$  in the one-to-one model can be seen as an ordered sequence of worker-firm pairs  $\{(w_1, f_1), (w_2, f_2), \dots, (w_r, f_r)\}$  such that  $w_i = \mu(f_i)$  and  $w_{i+1}$  is  $f_i$ 's most-preferred worker of  $W \setminus \{w_i\}$  according to  $P_{f_i}^{\mu, \mu_W}$ . Our definition of a cycle generalizes this idea to the many-to-many environment. Formally,

**Definition 2** Let  $\mu, \tilde{\mu} \in S(P)$  with  $\mu \succ_F \tilde{\mu}$ . A *cycle  $\sigma$  for  $P^{\mu, \tilde{\mu}}$*  is an ordered sequence of worker-firm pairs  $\sigma = \{(w_1, f_1), (w_2, f_2), \dots, (w_r, f_r)\}$  such that, for  $i = 1, \dots, r$ , we have:

$$(i) \ w_i \in \mu(f_i) \setminus \tilde{\mu}(f_i),$$

$$(ii) \ C_{f_i}^{\mu, \tilde{\mu}}(W \setminus \{w_i\}) = (\mu(f_i) \setminus \{w_i\}) \cup \{w_{i+1}\}, \text{ with } w_{r+1} = w_1, \text{ and}$$

$$(iii) \ C_{w_i}^{\mu, \tilde{\mu}}(\mu(w_i) \cup \{f_{i-1}\}) = (\mu(w_i) \setminus \{f_i\}) \cup \{f_{i-1}\}, \text{ with } f_0 = f_r.$$

<sup>13</sup>Roth and Sotomayor (1990) adapt the notion of *rotation* presented in Irving and Leather (1986), and refer to it as a cycle in preferences.

Condition (i) states that worker  $w_i$  is matched with  $f_i$  under  $\mu$  but not under  $\tilde{\mu}$ . Condition (ii) states that the set obtained from  $\mu(f_i)$  by eliminating worker  $w_i$  and adding worker  $w_{i+1}$  is the most Blair-preferred subset of workers of  $W \setminus \{w_i\}$  that contains  $w_{i+1}$ , according to  $P_{f_i}^{\mu, \tilde{\mu}}$ . Condition (iii) mimics Condition (ii) for the other side of the market: it states that the set obtained from  $\mu(w_i)$  by eliminating firm  $f_i$  and adding firm  $f_{i-1}$  is the least Blair-preferred subset of firms among those that are Blair-preferred to  $\mu(w)$  and contains  $f_{i-1}$ , according to  $P_{w_i}^{\mu, \tilde{\mu}}$ . Notice that Condition (iii) is not needed in the one-to-one model.

In the rest of this section, we state four propositions that are essential to show that the algorithm computes the full set of stable matchings. All the proofs are relegated to the appendix. The following proposition gives a necessary and sufficient condition for the existence of a cycle in a reduced preference profile.

**Proposition 1** *Let  $\mu, \tilde{\mu} \in S(P)$  with  $\mu \succeq_F \tilde{\mu}$ . There is a cycle for  $P^{\mu, \tilde{\mu}}$  if and only if  $\mu \neq \tilde{\mu}$ .*

In the one-to-one model, a cycle  $\{(w_1, f_1), (w_2, f_2), \dots, (w_r, f_r)\}$  for  $P^{\mu, \mu^W}$  can be used to obtain a new matching from matching  $\mu$  by breaking the partnership between firm  $f_i$  and worker  $w_i$  and establishing a new partnership between firm  $f_i$  and worker  $w_{i+1}$  for each  $i = 1, \dots, r$  (modulo  $r$ ), keeping all remaining partnerships in  $\mu$  unaffected. This new matching is called a cyclic matching. Using our many-to-many version of a cycle, we generalize the concept of cyclic matching in a straightforward way:

**Definition 3** *Let  $\mu, \tilde{\mu} \in S(P)$  with  $\mu \succ_F \tilde{\mu}$ , and let  $\sigma = \{(w_1, f_1), (w_2, f_2), \dots, (w_r, f_r)\}$  be a cycle for  $P^{\mu, \tilde{\mu}}$ . The **cyclic matching  $\mu_\sigma$  under  $P^{\mu, \tilde{\mu}}$**  is defined as follows: for each  $f \in F$*

$$\mu_\sigma(f) = \begin{cases} [\mu(f) \setminus \{w_i : f = f_i\}] \cup \{w_{i+1} : f = f_i\} & \text{if } f \in \sigma \\ \mu(f) & \text{if } f \notin \sigma, \end{cases}$$

and for each  $w \in W$ ,  $\mu_\sigma(w) = \{f \in F : w \in \mu_\sigma(f)\}$ .

For Example 1, we illustrate how to compute a cycle and its corresponding cyclic matching.

**Example 1 (Continued)**  $\sigma_1 = \{(w_1, f_1), (w_4, f_3)\}$  is a cycle for  $P^{\mu_F}$  in Example 1. To see this, we show that each worker-firm pair in  $\sigma_1$  satisfies (i), (ii) and (iii) of Definition 2.

(i) By inspection,  $w_1 \in \mu_F(f_1) \setminus \mu_W(f_1)$  and  $w_3 \in \mu_F(f_3) \setminus \mu_W(f_3)$ .

(ii)  $C_{f_1}^{\mu_F}(W \setminus \{w_1\}) = C_{f_1}^{\mu_F}(\{w_2, w_3, w_4, w_5, w_6\}) = \{w_2, w_4\} = (\mu_F(f_1) \setminus \{w_1\}) \cup \{w_2\}$ ,

$C_{f_3}^{\mu_F}(W \setminus \{w_4\}) = C_{f_3}^{\mu_F}(\{w_1, w_2, w_3, w_5, w_6\}) = \{w_1, w_2\} = (\mu_F(f_3) \setminus \{w_4\}) \cup \{w_1\}$ .

$$(iii) \ C_{w_1}^{\mu_F}(\mu_F(w_1) \cup \{f_3\}) = C_{w_1}^{\mu_F}(\{f_1, f_3\}) = \{f_3\} = (\mu_F(w_1) \setminus \{f_1\}) \cup \{f_3\},$$

$$C_{w_4}^{\mu_F}(\mu_F(w_4) \cup \{f_1\}) = C_{w_4}^{\mu_F}(\{f_3, f_1\}) = \{f_1\} = (\mu_F(w_4) \setminus \{f_3\}) \cup \{f_1\}.$$

Now, we compute its associated cyclic matching  $\mu_{\sigma_1}$ . Since  $f_1$  and  $f_3$  are firms in  $\sigma_1$ , then  $\mu_{\sigma_1}(f_1) = (\mu_F(f_1) \setminus \{w_1\}) \cup \{w_2\} = \{w_2, w_4\}$  and  $\mu_{\sigma_1}(f_3) = (\mu_F(f_3) \setminus \{w_4\}) \cup \{w_1\} = \{w_1, w_2\}$ . Thus,  $\mu_{\sigma_1} = \begin{pmatrix} f_1 & f_2 & f_3 & \emptyset \\ w_2w_4 & w_3w_5 & w_1w_2 & w_6 \end{pmatrix}$ .  $\diamond$

In the next proposition, we state that each cyclic matching under a reduced preference profile is stable for that same reduced preference profile.

**Proposition 2** *Let  $\mu, \tilde{\mu} \in S(P)$  with  $\mu \succ_F \tilde{\mu}$  and let  $\mu'$  be a cyclic matching under  $P^{\mu, \tilde{\mu}}$ . Then,  $\mu' \in S(P^{\mu, \tilde{\mu}})$ .*

The following proposition says that, given two Blair-comparable stable matchings, there is a cyclic matching under the reduced preference profile with respect to the Blair-preferred one that is, either the least preferred of the two given stable matchings, or a matching in between the two (with respect to the unanimous Blair order).

**Proposition 3** *Let  $\mu, \mu' \in S(P)$  with  $\mu \succ_F \mu'$ . Then, there is a cyclic matching  $\mu_\sigma$  under  $P^\mu$  such that  $\mu \succeq_F \mu_\sigma \succeq_F \mu'$ .*

Finally, we state the last proposition before presenting the algorithm. It says that each stable matching for the original preference profile, different from the firm-optimal stable matching, is always a cyclic matching under a reduced preference profile with respect to some other stable matching.

**Proposition 4** *Let  $\mu' \in S(P) \setminus \{\mu_F\}$ . Then, there is  $\mu \in S(P)$  such that  $\mu'$  is a cyclic matching under  $P^\mu$ .*

## 4.2 The algorithm

We are now in a position to present our algorithm. Before that, we briefly explain it. Given a matching market  $(F, W, P)$ , by the DA algorithm we compute the two optimal stable matchings,  $\mu_F$  and  $\mu_W$ . If the two optimal stable matchings are equal, the algorithm stops and the market has only this stable matching. If they are different, for the firms' side, we obtain the reduced preference profile with respect to  $\mu_F$ ,  $P^{\mu_F}$ . In each of the following steps, proceed as follows. For each reduced preference profile obtained in the previous step, we compute the following things: (i) each cycle for this profile; (ii) for each cycle, its corresponding cyclic matching; and (iii) for each cyclic matching, the reduced preference profile with respect to this cyclic matching. The algorithm stops at the step in which all the cyclic matchings computed are equal to the worker optimal stable matching. Formally,

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**Algorithm:**

**Input** A many-to-many matching market  $(F, W, P)$

**Output** The set of stable matchings  $S(P)$

**Step 1** Find  $\mu_F$  and  $\mu_W$  (by the DA algorithm)

and set  $S(P) := \{\mu_F, \mu_W\}$

IF  $\mu_F = \mu_W$ ,

THEN STOP.

ELSE obtain  $P^{\mu_F}$  and continue to Step 2.

**Step  $t$**  For each reduced preference profile  $P^\mu$  obtained in Step  $t - 1$ , find all cycles for  $P^\mu$  and for each cycle obtain its cyclic matching under  $P^\mu$  and include it in  $S(P)$ .

IF each cyclic matching obtained in this step is equal to  $\mu_W$ ,

THEN STOP.

ELSE for each cyclic matching  $\mu' \neq \mu_W$ , obtain the reduced preference profile  $P^{\mu'}$  and continue to Step  $t + 1$ .

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Notice that this algorithm stops in a finite number of steps by the finiteness of the market. Now, we present the main result of the paper. It states that the firms' optimal stable matching together with all the cyclic matchings obtained by the algorithm encompass the full set of stable matchings.

**Theorem 2** *For a market  $(F, W, P)$ , the algorithm computes the full set of stable matchings  $S(P)$ .*

The following example illustrates the algorithm.

**Example 1 (Continued)** *We apply the algorithm to the market of Example 1. In what follows, we detail each of its steps:*

**Step 1** *By the DA algorithm, we compute the two optimal stable matchings:*

$$\mu_F = \begin{pmatrix} f_1 & f_2 & f_3 & \emptyset \\ w_1w_2 & w_3w_5 & w_2w_4 & w_6 \end{pmatrix}, \text{ and } \mu_W = \begin{pmatrix} f_1 & f_2 & f_3 & \emptyset \\ w_3w_4 & w_2w_5 & w_1w_2 & w_6 \end{pmatrix}.$$

*Since  $\mu_F \neq \mu_W$ , we apply the reduction procedure to  $P$  to obtain  $P^{\mu_F}$  which we already computed in Example 1.*

**Step 2** *We find all cycles for  $P^{\mu_F}$ . There are only two cycles:  $\sigma_1 = \{(w_1, f_1), (w_4, f_3)\}$  and  $\sigma_2 = \{(w_2, f_1), (w_3, f_2)\}$ . Their corresponding cyclic matchings are:*

$$\mu_{\sigma_1} = \begin{pmatrix} f_1 & f_2 & f_3 & \emptyset \\ w_2w_4 & w_3w_5 & w_1w_2 & w_6 \end{pmatrix}, \text{ and } \mu_{\sigma_2} = \begin{pmatrix} f_1 & f_2 & f_3 & \emptyset \\ w_1w_3 & w_2w_5 & w_2w_4 & w_6 \end{pmatrix}.$$

Since  $\mu_{\sigma_1} \neq \mu_W$ , we apply the reduction procedure to  $P^{\mu_F}$  to obtain the reduced preference profile with respect to  $\mu_{\sigma_1}$ ,  $P^{\sigma_1}$ ; and since  $\mu_{\sigma_2} \neq \mu_W$ , we apply the reduced preference profile with respect to  $\mu_{\sigma_2}$ ,  $P^{\sigma_2}$ . These two profiles are the following:

$$\begin{array}{ll}
P_{f_1}^{\sigma_1} : w_2w_4, w_3w_4, w_2w_3, w_4, w_3, w_2 & P_{f_1}^{\sigma_2} : w_1w_3, w_1w_4, w_3w_4, w_1, w_4, w_3 \\
P_{f_2}^{\sigma_1} : w_3w_5, w_2w_5, w_2w_3, w_2, w_3, w_5 & P_{f_2}^{\sigma_2} : w_2w_5, w_2, w_5 \\
P_{f_3}^{\sigma_1} : w_1w_2, w_1, w_2 & P_{f_3}^{\sigma_2} : w_2w_4, w_1w_2, w_1w_4, w_1, w_2, w_4 \\
P_{w_1}^{\sigma_1} : f_3 & P_{w_1}^{\sigma_2} : f_3, f_1 \\
P_{w_2}^{\sigma_1} : f_2f_3, f_1f_3, f_1f_2, f_1, f_2, f_3 & P_{w_2}^{\sigma_2} : f_2f_3, f_1f_3, f_1, f_2, f_3 \\
P_{w_3}^{\sigma_1} : f_1, f_2 & P_{w_3}^{\sigma_2} : f_1 \\
P_{w_4}^{\sigma_1} : f_1, f_3 & P_{w_4}^{\sigma_2} : f_1, f_3 \\
P_{w_5}^{\sigma_1} : f_2 & P_{w_5}^{\sigma_2} : f_2 \\
P_{w_6}^{\sigma_1} : \emptyset & P_{w_6}^{\sigma_2} : \emptyset
\end{array}$$

**Step 3** Lastly, we find all cycles for  $P^{\sigma_1}$  and  $P^{\sigma_2}$ . The only cycle for  $P^{\sigma_1}$  is  $\sigma_2 = \{(w_2, f_1), (w_3, f_2)\}$ . Similarly, the only cycle for  $P^{\sigma_2}$  is  $\sigma_1 = \{(w_1, f_1), (w_4, f_3)\}$ . Their corresponding cyclic matchings are both equal to  $\mu_W$ . Then, the algorithm stops and  $S(P) = \{\mu_F, \mu_{\sigma_1}, \mu_{\sigma_2}, \mu_W\}$ .  $\diamond$

## 5 Concluding remarks

For a many-to-many matching market in which agents' preferences satisfy substitutability and *LAD*, we presented an algorithm to compute the full set of stable matchings. Our approach extends the notion of cycles and cyclic matchings presented in [Roth and Sotomayor \(1990\)](#). Given any stable matching  $\mu$ , each adjacent stable matching  $\mu'$  is obtained as a cyclic matching under the reduced preference profile  $P^\mu$ .<sup>14</sup> Even though our results make no use of the lattice structure of the stable set, our algorithm travels through this lattice from the firm-optimal to the worker-optimal stable matching, finding all stable matchings in between.

A paper closely related to ours is [Martínez et al. \(2004\)](#), which also computes the full set of many-to-many stable matchings. An important difference between the algorithm of [Martínez et al. \(2004\)](#) and ours is that theirs is based on the one-to-one algorithm presented by [McVitie and Wilson \(1971\)](#). The *DA* algorithm must be applied to a reduced preference profile in *each* step of the algorithm of [Martínez et al. \(2004\)](#), while in our algorithm we use the *DA* algorithm only twice (to calculate the firm-optimal and worker-optimal stable matchings in the first step) and afterward we only seek for cycles in a reduced preference profile and compute their corresponding cyclic matchings.

<sup>14</sup>Stable matchings  $\mu$  and  $\mu'$  are *adjacent* if  $\mu \succ_F \mu'$  and there is no other stable matching  $\mu''$  such that  $\mu \succ_F \mu'' \succ_F \mu'$ .

Another difference is that [Martínez et al. \(2004\)](#) only assume substitutability on agents' preferences, while we assume in addition LAD. However, we provide an example that shows that the algorithm in [Martínez et al. \(2004\)](#) has an error (the algorithm does not compute the full set of stable matchings). Before presenting this example, we roughly explain the algorithm in [Martínez et al. \(2004\)](#). Let  $(F, W, P)$  be a matching market. By using the DA algorithm, compute  $\mu_F$  and  $\mu_W$  and set  $S^*(P) = \{\mu_F, \mu_W\}$ . In Step 1, for each pair  $(f, w)$  such that  $w \in \mu_F(f) \setminus \mu_W(f)$ , (i) compute the  $w$ -truncation of  $P_f$  and consider the new preference profile  $P^{(f,w)}$  obtained from  $P$  by replacing  $P_f$  by the  $w$ -truncation of  $P_f$ ,<sup>15</sup> (ii) compute, by the DA algorithm, the firm-optimal stable matching for the new market  $(F, W, P^{(f,w)})$ , denoted by  $\mu_F^{(f,w)}$ ; (iii) if  $C_{w'}(\mu_F(w') \cup \mu_F^{(f,w)}(w')) = \mu_F^{(f,w)}(w')$  for each  $w' \in W$ , then add  $\mu_F^{(f,w)}$  to  $S^*(P)$ . In Step  $t$ , for each matching added to  $S^*(P)$  in Step  $t - 1$ , repeat items (i), (ii), and (iii) of Step 1 for each pair  $(f, w)$  such that  $w$  is matched to  $f$  under this new matching but not under the original worker-optimal stable matching. The algorithm stops in the step in which no matching is added to  $S^*(P)$ . [Martínez et al. \(2004\)](#) wrongly state that  $S^*(P) = S(P)$ .

Now, we are in a position to present the example<sup>16</sup> showing that: (i) algorithm of [Martínez et al. \(2004\)](#) stops before computing all stable matchings, and (ii) our algorithm computes the full set of stable matchings.

**Example 2** Let  $(F, W, P)$  be a one-to-one matching market in which  $F = \{f_1, f_2, f_3, f_4\}$ ,  $W = \{w_1, w_2, w_3, w_4\}$ , and the preference profile is given by:

$$\begin{array}{ll} P_{f_1} : w_2, w_1, w_3, w_4 & P_{w_1} : f_2, f_1, f_4, f_3 \\ P_{f_2} : w_4, w_2, w_3, w_1 & P_{w_2} : f_4, f_3, f_2, f_1 \\ P_{f_3} : w_4, w_2, w_3, w_1 & P_{w_3} : f_3, f_1, f_4, f_2 \\ P_{f_4} : w_3, w_1, w_4, w_2 & P_{w_4} : f_1, f_3, f_4, f_2 \end{array}$$

Agents' preferences in a one-to-one matching market satisfy substitutability and LAD because they are linear orderings among single agents. For this market, there are three stable matchings:

$$\mu_F = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1 & w_2 & w_4 & w_3 \end{pmatrix}, \mu = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_3 & w_1 & w_4 & w_2 \end{pmatrix}, \text{ and } \mu_W = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_4 & w_1 & w_3 & w_2 \end{pmatrix}.$$

Following the algorithm of [Martínez et al. \(2004\)](#), the pairs  $(f, w)$  such that  $w \in \mu_F(f) \setminus \mu_W(f)$  are:  $(f_1, w_1)$ ,  $(f_2, w_2)$ ,  $(f_3, w_4)$  and  $(f_4, w_3)$ . For each of these pairs  $(f, w)$ , the firm-optimal stable matching for the  $w$ -truncation of  $P_f$  are:

$$\mu_F^{(f_1, w_1)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_3 & w_2 & w_4 & w_1 \end{pmatrix}, \mu_F^{(f_2, w_2)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_2 & w_1 & w_4 & w_3 \end{pmatrix},$$

<sup>15</sup>See Definition 4 in the Appendix.

<sup>16</sup>This example was provided to one of the authors of this paper by Xuan Zhang.



$$\mu^{(f_3, w_4)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1 & w_4 & w_2 & w_3 \end{pmatrix}, \text{ and } \mu^{(f_4, w_3)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1 & w_3 & w_4 & w_2 \end{pmatrix}.$$

Notice that

$$\begin{aligned} C_{w_1}(\mu_F(w_1) \cup \mu_F^{(f_1, w_1)}(w_1)) &= C_{w_1}(\{f_1, f_4\}) = \{f_1\} \neq \mu_F^{(f_1, w_1)}(w_1), \\ C_{w_2}(\mu_F(w_2) \cup \mu_F^{(f_2, w_2)}(w_2)) &= C_{w_2}(\{f_2, f_1\}) = \{f_2\} \neq \mu_F^{(f_2, w_2)}(w_2), \\ C_{w_4}(\mu_F(w_4) \cup \mu_F^{(f_3, w_4)}(w_4)) &= C_{w_4}(\{f_3, f_2\}) = \{f_3\} \neq \mu_F^{(f_3, w_4)}(w_4), \text{ and} \\ C_{w_3}(\mu_F(w_3) \cup \mu_F^{(f_4, w_3)}(w_3)) &= C_{w_3}(\{f_4, f_2\}) = \{f_4\} \neq \mu_F^{(f_4, w_3)}(w_3). \end{aligned}$$

Thus, the algorithm does not incorporate any matching to  $S^*(P)$  and, therefore, stops without computing stable matching  $\mu$ .

Now we show how our algorithm computes all of these three stable matchings. Once we compute  $\mu_F$  and  $\mu_W$  by the DA algorithm, the reduced preference profile  $P^{\mu_F}$  is given by:

$$\begin{array}{ll} P_{f_1}^{\mu_F} : w_1, w_3, w_4 & P_{w_1}^{\mu_F} : f_2, f_1 \\ P_{f_2}^{\mu_F} : w_2, w_1 & P_{w_2}^{\mu_F} : f_4, f_3, f_2 \\ P_{f_3}^{\mu_F} : w_4, w_2, w_3 & P_{w_3}^{\mu_F} : f_3, f_1, f_4 \\ P_{f_4}^{\mu_F} : w_3, w_2 & P_{w_4}^{\mu_F} : f_1, f_3 \end{array}$$

It is easy to check that there is only one cycle for  $P^{\mu_F}$ ,  $\sigma_1 = \{(w_1, f_1), (w_3, f_4), (w_2, f_2)\}$ . Its corresponding cyclic matching is  $\mu_{\sigma_1} = \mu$ . Now, the reduced preference profile  $P^{\mu_{\sigma_1}}$  is given by:

$$\begin{array}{ll} P_{f_1}^{\mu_{\sigma_1}} : w_3, w_4 & P_{w_1}^{\mu_{\sigma_1}} : f_2 \\ P_{f_2}^{\mu_{\sigma_1}} : w_1 & P_{w_2}^{\mu_{\sigma_1}} : f_4 \\ P_{f_3}^{\mu_{\sigma_1}} : w_4, w_3 & P_{w_3}^{\mu_{\sigma_1}} : f_3, f_1 \\ P_{f_4}^{\mu_{\sigma_1}} : w_2 & P_{w_4}^{\mu_{\sigma_1}} : f_1, f_3 \end{array}$$

Finally, it is easy to check that there is only one cycle for  $P^{\mu_{\sigma_1}}$ ,  $\sigma_2 = \{(w_3, f_1), (w_4, f_3)\}$ . Its corresponding cyclic matching is  $\mu_{\sigma_2} = \mu_W$ . In this way, our algorithm computes the full set of stable matchings for the market  $(F, W, P)$ .  $\diamond$

## A Appendix

In order to prove Lemma 1, we first define a  $w$ -truncation of preference  $P_f$  and adapt a lemma of [Martínez et al. \(2004\)](#) to our setting.

**Definition 4 (Martínez et al., 2004)** We say that the preference  $P_f^w$  is the  $w$ -truncation of  $P_f$  if:

- (i) All sets containing  $w$  are unacceptable to  $f$  according to  $P_f^w$ . That is, if  $w \in S$  then  $\emptyset P_f^w S$ .

(ii) The preferences  $P_f$  and  $P_f^w$  coincide on all sets that do not contain  $w$ . That is, if  $w \notin S_1 \cup S_2$  then  $S_1 P_f S_2$  if and only if  $S_1 P_f^w S_2$ .

Similarly, we define  $P_w^f$  as an  $f$ -truncation of  $P_w$ .

**Remark 2** Given a  $w$ -truncation of  $P_f$  and any subset of workers  $S$ ,  $C_f(S \setminus \{w\}) = C_f^w(S)$ . Similarly, given a  $f$ -truncation of  $P_w$  and any subset of firms  $S$ ,  $C_w(S \setminus \{f\}) = C_w^f(S)$ .

**Lemma 2** Let  $f \in F$  and  $w \in W$  with their respective preference relations  $P_f$  and  $P_w$ . If  $P_f$  is substitutable and satisfies LAD, then  $P_f^w$  is substitutable and satisfies LAD. Similarly, if  $P_w$  is substitutable and satisfies LAD, then  $P_w^f$  is substitutable and satisfies LAD.

*Proof.* Let  $f \in F$ ,  $w \in W$ , and  $P_f$  be substitutable and satisfies LAD. Let  $P_f^w$  be the  $w$ -truncation of  $P_f$ . We only prove that if  $P_f$  is substitutable and satisfies LAD, then  $P_f^w$  is substitutable and satisfies LAD. The other implication is analogous. To see that  $P_f^w$  is substitutable, let  $\tilde{w}, w' \in S$  be arbitrary and assume that  $\tilde{w} \in C_f^w(S)$ .<sup>17</sup> If  $w \notin S$ , then  $\tilde{w} \in C_f^w(S \setminus \{w'\})$  because  $C_f^w(S) = C_f(S)$ ,  $C_f^w(S \setminus \{w'\}) = C_f(S \setminus \{w'\})$ , and because of the substitutability of  $P_f$ . If  $w \in S$ , then we have that  $C_f^w(S) = C_f(S \setminus \{w\})$ ; therefore, by assumption  $\tilde{w} \in C_f(S \setminus \{w'\})$ . By the substitutability of  $P_f$ , we have that  $\tilde{w} \in C_f([S \setminus \{w\}] \setminus \{w'\})$ . But, the equality  $C_f([S \setminus \{w\}] \setminus \{w'\}) = C_f^w(S \setminus \{w'\})$  implies that  $\tilde{w} \in C_f^w(S \setminus \{w'\})$ . Therefore,  $P_f^w$  is substitutable.

To see that  $P_f^w$  satisfies LAD, let  $S'$  and  $S$  be two subsets of workers such that  $S' \subset S$ . Note that,  $S' \setminus \{w\} \subset S \setminus \{w\}$ . Then, by Remark 2 and the fact that  $P_f$  satisfies LAD we have,

$$|C_f^w(S')| = |C_f(S' \setminus \{w\})| \leq |C_f(S \setminus \{w\})| = |C_f^w(S)|.$$

Therefore,  $P_f^w$  satisfies LAD.  $\square$

*Proof of Lemma 1.* W.l.o.g. assume that agent  $a$  is a firm, say  $f \in F$ . Let  $P_f$  be a substitutable preference that satisfies LAD. Let  $\tilde{W}_f$  be the set of workers selected in Step 1 (a), Step 2 (a) or Step 3 of the reduction procedure for firm  $f$ . Take any  $w \in \tilde{W}_f$  and consider the  $w$ -truncation  $P_f^w$ . By Lemma 2, preference  $P_f^w$  is substitutable and satisfies LAD. Now, any take  $w' \in \tilde{W}_f \setminus \{w\}$  and consider the  $w'$ -truncation of  $P_f^w$ . Again by Lemma 2, the  $w'$ -truncation of  $P_f^w$  is substitutable and satisfies LAD. Continuing in the same way for each worker of  $\tilde{W}_f$  not yet considered, we construct the corresponding truncation of the previously obtained truncated preference. By Lemma 2, each one of these truncated preferences is substitutable and satisfies LAD. By the finiteness of the set  $\tilde{W}_f$ , this process will end. Moreover, by definition of  $\tilde{W}_f$ , the last truncated preference obtained in this process is  $P_f^{\mu, \tilde{\mu}}$ . Therefore, preference  $P_f^{\mu, \tilde{\mu}}$  is substitutable and satisfies LAD.  $\square$

<sup>17</sup>Denote by  $C_f^w(S)$  to  $f$ 's most preferred subset of  $S$  according to the  $w$ -truncation of  $P_f$ .

In order to prove Theorem 1, we first show in Lemma 3 that, under certain conditions, individual rationality of a matching under the original preference profile is equivalent to individual rationality under a reduced preference profile.

**Lemma 3** *Let  $\mu, \tilde{\mu} \in S(P)$  with  $\mu \succ_F \tilde{\mu}$  and let  $\mu'$  be a matching. The matching  $\mu'$  is individually rational under  $P$  with  $\mu \succeq_F \mu' \succeq_F \tilde{\mu}$  and  $\tilde{\mu} \succeq_W \mu' \succeq_W \mu$  if and only if  $\mu'$  is individually rational under  $P^{\mu, \tilde{\mu}}$ .<sup>18</sup>*

*Proof.* Let  $\mu, \tilde{\mu} \in S(P)$  with  $\mu \succ_F \tilde{\mu}$  and let  $\mu'$  be a matching.

( $\implies$ ) Assume that the matching  $\mu'$  is individually rational under  $P$  with  $\mu \succeq_F \mu' \succeq_F \tilde{\mu}$  and  $\tilde{\mu} \succeq_W \mu' \succeq_W \mu$ . We claim that  $\mu'(f)$  and  $\mu'(w)$  are not eliminated in the reduction to obtain  $P^{\mu, \tilde{\mu}}$  for each  $f \in F$  and  $w \in W$ . Since  $\mu \succeq_F \mu' \succeq_F \tilde{\mu}$ , we have  $\mu(f) = C_f(\mu(f) \cup \mu'(f))$  and  $\mu'(f) = C_f(\tilde{\mu}(f) \cup \mu'(f))$  for each  $f \in F$ . Moreover, since  $\tilde{\mu} \succeq_W \mu' \succeq_W \mu$ , we have  $\tilde{\mu}(w) = C_w(\mu'(w) \cup \tilde{\mu}(w))$  and  $\mu'(w) = C_w(\mu'(w) \cup \mu(w))$  for each  $w \in W$ . Therefore,  $\mu'(f)$  and  $\mu'(w)$  are not eliminated at Step 1, and Step 2 of the reduction procedure. Let  $(f, w)$  be a pair assigned in  $\mu'$ . Since  $\mu'$  is individually rational, the pair  $(f, w)$  is mutually acceptable under  $P$ . Moreover, since  $\mu'(f)$  and  $\mu'(w)$  were not eliminated at Step 1, or Step 2, then  $(f, w)$  is mutually acceptable under  $P^{\mu, \tilde{\mu}}$ . Thus, no pair of agents assigned in  $\mu'$  is eliminated in Step 3 of the reduction procedure. Then,

$$C_f^{\mu, \tilde{\mu}}(\mu'(f)) = C_f(\mu'(f)) = \mu'(f)$$

and

$$C_w^{\mu, \tilde{\mu}}(\mu'(w)) = C_w(\mu'(w)) = \mu'(w).$$

Therefore,  $\mu'$  is an individuality rational matching under  $P^{\mu, \tilde{\mu}}$ .

( $\impliedby$ ) Assume that the matching  $\mu'$  is individually rational under  $P^{\mu, \tilde{\mu}}$ . The definition of reduced preference  $P^{\mu, \tilde{\mu}}$  implies that  $\mu \succeq_F \mu' \succeq_F \tilde{\mu}$  and  $\tilde{\mu} \succeq_W \mu' \succeq_W \mu$ . Let  $f \in F$ . Notice that, by the reduction procedure, if  $w \in C_f^{\mu, \tilde{\mu}}(\mu'(f))$  then,  $w \in C_f(\mu'(f))$ . Since  $\mu'(f) = C_f^{\mu, \tilde{\mu}}(\mu'(f)) \subseteq C_f(\mu'(f)) \subseteq \mu'(f)$ , we have that  $C_f(\mu'(f)) = \mu'(f)$ . Similarly,  $C_w(\mu'(w)) = \mu'(w)$  for each  $w \in W$ . Therefore,  $\mu'$  is an individually rational matching under  $P$ .  $\square$

*Proof of Theorem 1.* Let  $\mu, \tilde{\mu} \in S(P)$  with  $\mu \succ_F \tilde{\mu}$ .

( $\impliedby$ ) Let  $\mu' \in S(P^{\mu, \tilde{\mu}})$ . By Lemma 3, we have that  $\mu'$  is individually rational under  $P$ . Assume that  $\mu' \notin S(P)$ . Thus, there is a blocking pair of  $\mu'$  under  $P$ , i.e., there is  $(f, w) \in F \times W$  such that  $w \notin \mu'(f)$ ,  $w \in C_f(\mu'(f) \cup \{w\})$  and  $f \in C_w(\mu'(w) \cup \{f\})$ .

<sup>18</sup>Recall that  $\succeq_F$  and  $\succeq_W$  are dual orders only in the set of stable matchings, so both  $\mu \succeq_F \mu' \succeq_F \tilde{\mu}$  and  $\tilde{\mu} \succeq_W \mu' \succeq_W \mu$  need to be required.

**Claim: neither  $C_f(\mu'(f) \cup \{w\})$  nor  $C_w(\mu'(w) \cup \{f\})$  is eliminated in the reduction procedure.** First, assume w.l.o.g. that  $C_f(\mu'(f) \cup \{w\})$  is eliminated in Step 1 the reduction procedure. There are two cases to consider:

**Case 1:  $C_f(\mu'(f) \cup \{w\}) \not\preceq_f \mu(f)$ .** Thus, there are  $W'$  and  $\tilde{w}$  such that  $\tilde{w} \in W' \setminus \mu(f)$ ,  $\tilde{w} \in C_f(\mu'(f) \cup \{w\})$  and  $W' = C_f(W' \cup \mu(f))$ . Then, if  $\tilde{w} \in \mu'(f)$ ,  $\mu'(f)$  is eliminated in Step 1 of the reduction procedure. Therefore,  $\mu'(f) \neq C_f^{\mu, \tilde{\mu}}(\mu'(f))$ , and  $\mu'$  is not individually rational under  $P^{\mu, \tilde{\mu}}$ , contradicting Lemma 3. If  $\tilde{w} = w$ ,  $w \in C_f(W' \cup \mu(f))$  and, by substitutability,

$$w \in C_f(\mu(f) \cup \{w\}). \quad (2)$$

Moreover, by definition of Blair's partial order and (1)

$$C_w(\mu'(w) \cup \{f\}) \succeq_w C_w(\mu'(w)). \quad (3)$$

Since  $\mu'$  is individually rational under  $P^{\mu, \mu'}$ ,  $\mu'$  is individually rational under  $P$  and  $\tilde{\mu} \succeq_W \mu' \succeq_W \mu$  by Lemma 3. Then,  $\mu'(w) = C_w(\mu'(w)) \succeq_w \mu(w)$  and, by (3) and transitivity of  $\succeq_w$ , we have  $C_w(\mu'(w) \cup \{f\}) \succeq_w \mu(w)$ . Thus,  $C_w(\mu'(w) \cup \{f\}) = C_w(\mu(w) \cup C_w(\mu'(w) \cup \{f\}))$ . Applying (1), we have  $C_w(\mu(w) \cup C_w(\mu'(w) \cup \{f\})) = C_w(\mu(w) \cup \mu'(w) \cup \{f\})$ . Recall that  $(f, w)$  is a blocking pair for  $\mu'$  under  $P$ . Hence,  $f \in C_w(\mu'(w) \cup \{f\}) = C_w(\mu(w) \cup \mu'(w) \cup \{f\})$ . Since  $w = \tilde{w} \in W' \setminus \mu(f)$ ,  $f \notin \mu(w)$ . Thus, by substitutability,  $f \in C_w(\mu(w) \cup \mu'(w) \cup \{f\})$  implies that

$$f \in C_w(\mu(w) \cup \{f\}). \quad (4)$$

Furthermore, since  $w = \tilde{w}$ ,  $w \notin \mu(f)$ . This, together with (2) and (4) imply that  $(f, w)$  is a blocking pair for  $\mu$  under  $P$ . This is a contradiction.

**Case 2:  $C_f(\mu'(f) \cup \{w\}) \succeq_f \mu(f)$ .** Since we assume that  $(f, w)$  is a blocking pair for  $\mu'$  under  $P$ ,  $w \in C_f(\mu'(f) \cup \{w\})$ . By this case's hypothesis,  $w \in C_f(\mu(f) \cup C_f(\mu'(f) \cup \{w\}))$ . By (1),  $w \in C_f(\mu(f) \cup \mu'(f) \cup \{w\}) = C_f(C_f(\mu(f) \cup \mu'(f)) \cup \{w\})$ . Since  $\mu \succeq_F \mu'$ ,

$$w \in C_f(\mu(f) \cup \{w\}). \quad (5)$$

Now, we claim that  $w \notin \mu(f)$ . First, note that  $f \in C_w(\mu'(w) \cup \{f\})$  and  $f \notin \mu'(w)$  implies that  $C_w(\mu'(w) \cup \{f\}) \succ_w \mu'(w)$ . Second,  $\mu' \in S(P^{\mu, \tilde{\mu}})$  implies, by Lemma 3,  $\mu' \succeq_W \mu$ . Thus,  $\mu'(w) = C_w(\mu'(w) \cup \mu(w))$  for each  $w \in W$ . Lastly, since  $f \notin \mu'(w)$  and assuming that  $f \in \mu(w)$ , we conclude that

$$\mu'(w) = C_w(\mu'(w) \cup \mu(w)) \succeq_w C_w(\mu'(w) \cup \{f\}) \succ_w \mu'(w),$$

and this is a contradiction. Then,

$$w \notin \mu(f). \quad (6)$$

Moreover, by the same argument used to obtain (4),  $f \in C_w(\mu'(w) \cup \{f\})$  implies that

$$f \in C_w(\mu(w) \cup \{f\}). \quad (7)$$

Hence, by (5), (6), and (7),  $(f, w)$  is a blocking pair for  $\mu$  under  $P$ , and this is a contradiction. Therefore, by Case 1 and Case 2,  $C_f(\mu'(f) \cup \{w\})$  is not eliminated in Step 1.

Second, assume w.l.o.g. that  $C_f(\mu'(f) \cup \{w\})$  is eliminated in Step 2 the reduction procedure. Note that this cannot happen, because  $C_f(\mu'(f) \cup \{w\}) \succeq_f \mu'(f) \succeq_f \tilde{\mu}(f)$  for each  $f \in F$ . By a symmetrical argument, we have that  $C_w(\mu'(w) \cup \{f\})$  can not be eliminated in Step 1 or Step 2 either.

Now, we show that neither  $C_f(\mu'(f) \cup \{w\})$  nor  $C_w(\mu'(w) \cup \{f\})$  is eliminated in Step 3. Assume w.l.o.g. that  $C_f(\mu'(f) \cup \{w\})$  is eliminated in Step 3. Thus, there is  $\tilde{w} \in C_f(\mu'(f) \cup \{w\})$  such that  $\tilde{w}$  is not acceptable for  $f$  after Steps 1 and 2 are performed. Note that this implies that  $C_f^{\mu, \mu'}(\{\tilde{w}\}) \neq \{\tilde{w}\}$ . By definition of  $C_f$  we have  $C_f(\mu'(f) \cup \{w\}) \subseteq \mu'(f) \cup \{w\}$ . Thus,  $\tilde{w} \in \mu'(f)$  or  $\tilde{w} = w$ . If  $\tilde{w} \in \mu'(f)$ , since  $\mu'$  is individually rational under  $P^{\mu, \tilde{\mu}}$ ,  $\tilde{w} \in \mu'(f) = C_f^{\mu, \tilde{\mu}}(\mu'(f))$  and, by substitutability,  $\tilde{w} \in C_f^{\mu, \tilde{\mu}}(\{\tilde{w}\}) \neq \{\tilde{w}\}$ , which is absurd. Therefore,  $\tilde{w} = w$ . Since  $(f, w)$  is a blocking pair of  $\mu'$  under  $P$ ,  $w \in C_f(\mu'(f) \cup \{w\})$ . Since  $w$  is not acceptable for  $f$  after Step 1 and Step 2, this implies that any set that contains agent  $w$  is removed from  $f$ 's preference list at Step 1 or Step 2. Thus,  $C_f(\mu'(f) \cup \{w\})$  is removed from  $f$ 's preference list in Step 1 or Step 2, and this is a contradiction. Therefore,  $C_f(\mu'(f) \cup \{w\})$  is not eliminated in Step 3. A similar argument proves that  $C_w(\mu'(w) \cup \{f\})$  is not eliminated either in Step 3. This completes the proof of the Claim.

In order to finish the proof, since by the Claim neither  $C_f(\mu'(f) \cup \{w\})$  nor  $C_f(\mu'(w) \cup \{f\})$  is eliminated by the reduction procedure, we have that  $C_f(\mu'(f) \cup \{w\}) = C_f^{\mu, \tilde{\mu}}(\mu'(f) \cup \{w\})$  and  $C_f(\mu'(w) \cup \{f\}) = C_f^{\mu, \tilde{\mu}}(\mu'(w) \cup \{f\})$ . Then,  $(f, w)$  is a blocking pair for  $\mu'$  under  $P^{\mu, \tilde{\mu}}$ , and this is a contradiction. Therefore,  $\mu' \in S(P)$ .

( $\implies$ ) Let  $\mu' \in S(P)$  with  $\mu \succeq_F \mu' \succeq_F \tilde{\mu}$ . This implies that  $\tilde{\mu} \succeq_W \mu' \succeq_W \mu$ . By Lemma 3, we have that  $\mu'$  is individually rational in  $P^{\mu, \tilde{\mu}}$ . Assume that  $\mu' \notin S(P^{\mu, \tilde{\mu}})$ . Thus, there is a pair  $(f, w) \in F \times W$  such that  $w \notin \mu'(f)$ ,  $w \in C_f^{\mu, \tilde{\mu}}(\mu'(f) \cup \{w\})$  and  $f \in C_f^{\mu, \tilde{\mu}}(\mu'(w) \cup \{f\})$ . By the reduction procedure  $w \in C_f(\mu'(f) \cup \{w\})$  and  $f \in C_f(\mu'(w) \cup \{f\})$ . Therefore the pair  $(f, w)$  blocks  $\mu'$  under  $P$ , and this is a contradiction of  $\mu' \in S(P)$ . Thus,  $\mu' \in S(P^{\mu, \tilde{\mu}})$ .  $\square$

*Proof of Proposition 1.* ( $\implies$ ) This implication is straightforward from Definition 2, since there is a cycle only if there is a firm  $f$  such that  $\mu(f) \neq \tilde{\mu}(f)$  under  $P^{\mu, \tilde{\mu}}$ .

( $\impliedby$ ) Assume that  $\mu \neq \tilde{\mu}$ . We construct a bipartite oriented the digraph  $D^{\mu, \tilde{\mu}}$  with sets

of nodes

$$V_1 = \{(w, f) \in W \times F : w \in \mu(f) \setminus \tilde{\mu}(f)\}$$

and

$$V_2 = (F \times W) \setminus \{(f, w) : (w, f) \in V_1\}.$$

Since  $\mu \neq \tilde{\mu}$ , both  $V_1$  and  $V_2$  are non-empty. The oriented arcs are defined as follows. There is an arc from  $(w, f) \in V_1$  to  $(f', w') \in V_2$  if

$$f = f' \text{ and } C_f^{\mu, \tilde{\mu}}(W \setminus \{w\}) = (\mu(f) \setminus \{w\}) \cup \{w'\}.$$

There is an arc from  $(f', w') \in V_2$  to  $(w, f) \in V_1$  if

$$w' = w \text{ and } C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\}) = (\mu(w) \setminus \{f\}) \cup \{f'\}.$$

It is easy to see that there is an oriented cycle in the digraph  $D^{\mu, \tilde{\mu}}$  if and only if there is a cycle for preference  $P^{\mu, \tilde{\mu}}$ . In fact, if  $\{(w_1, f_1), (f_1, w_2), (w_2, f_2), (f_2, w_3), \dots, (w_r, f_r), (f_r, w_1)\}$  is a cycle for  $D^{\mu, \tilde{\mu}}$ , then  $\{w_1, f_1, w_2, f_2, \dots, w_r, f_r\}$  is a cycle for  $P^{\mu, \tilde{\mu}}$ . Assume that there is no cycle for  $P^{\mu, \tilde{\mu}}$ . Then, there is no cycle in digraph  $D^{\mu, \tilde{\mu}}$ . Let  $p$  be a maximal path in  $D^{\mu, \tilde{\mu}}$ . There are two cases to consider:

**Case 1: the terminal node  $(w, f)$  of  $p$  belongs to  $V_1$ .** Then  $w \in \mu(f) \setminus \tilde{\mu}(f)$  and there is no  $w' \in W$  such that  $w' \notin \mu(f) \setminus \tilde{\mu}(f)$  and  $w' \in C_f^{\mu, \tilde{\mu}}(W \setminus \{w\})$ . Therefore,  $C_f^{\mu, \tilde{\mu}}(W \setminus \{w\}) \subsetneq C_f^{\mu, \tilde{\mu}}(W) = \mu(f)$ . By LAD,

$$|C_f^{\mu, \tilde{\mu}}(W \setminus \{w\})| < |\mu(f)|. \quad (8)$$

Moreover, since  $w \in \mu(f) \setminus \tilde{\mu}(f)$ , we have  $\tilde{\mu}(f) \subseteq W \setminus \{w\}$ . Thus,  $\tilde{\mu}(f) = C_f^{\mu, \tilde{\mu}}(\tilde{\mu}(f)) \subseteq C_f^{\mu, \tilde{\mu}}(W \setminus \{w\})$  and, by LAD,

$$|\tilde{\mu}(f)| \leq |C_f^{\mu, \tilde{\mu}}(W \setminus \{w\})|. \quad (9)$$

By the Rural Hospitals Theorem,<sup>19</sup>  $|\mu(f)| = |\tilde{\mu}(f)|$ . This, together with (8) and (9) implies that  $|\tilde{\mu}(f)| \leq |C_f^{\mu, \tilde{\mu}}(W \setminus \{w\})| < |\mu(f)| = |\tilde{\mu}(f)|$ , which is absurd.

**Case 2: the terminal node  $(f', w)$  of  $p$  belongs to  $V_2$ .** Then,  $f' \notin \mu(w) \setminus \tilde{\mu}(w)$ . First, we claim that  $|C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\})| = |\mu(w)|$ . Since  $C_w^{\mu, \tilde{\mu}}(F) = \tilde{\mu}(w)$  by Remark 1 (i) and  $\mu(w) \cup \{f'\} \subseteq F$ , by LAD it follows that

$$|\tilde{\mu}(w)| \geq |C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\})|. \quad (10)$$

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<sup>19</sup>The *Rural Hospitals Theorem* states that each agent is matched with the same number of partners in every stable matching. That is,  $|\mu(a)| = |\mu'(a)|$  for each  $\mu, \mu' \in S(P)$  and for each  $a \in F \cup W$  (see [Alkan, 2002](#), for more detail).

Furthermore, by LAD and individual rationality of  $\mu$ ,

$$|C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\})| \geq |C_w^{\mu, \tilde{\mu}}(\mu(w))| = |\mu(w)|. \quad (11)$$

Assume  $|C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\})| > |\mu(w)|$ . By (11) and (10), it follows that  $|\tilde{\mu}(w)| > |\mu(w)|$ . This contradicts the Rural Hospitals Theorem. Therefore,  $|C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\})| = |\mu(w)|$ , and the proof of the claim is completed. Now, we have two subcases to consider:

**Subcase 2.1:**  $f' \in C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\})$ . As  $|C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\})| = |\mu(w)|$ , there is  $f \in \mu(w)$  such that

$$C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\}) = (\mu(w) \setminus \{f\}) \cup \{f'\}. \quad (12)$$

Furthermore,  $f \notin \tilde{\mu}(w)$ . To see this, notice that if  $f \in \tilde{\mu}(w) = C_w^{\mu, \tilde{\mu}}(F)$  then, by substitutability,  $f \in C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\})$ , contradicting (12). Therefore,  $f \in \mu(w) \setminus \tilde{\mu}(w)$  and (12) imply that there is an arc from  $(f', w) \in V_2$  to  $(w, f) \in V_1$ . This contradicts that  $(f', w)$  is a terminal node of  $p$ .

**Subcase 2.2:**  $f' \notin C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\})$ . First, assume that  $f' \notin C_w(\mu(w) \cup \{f'\})$ . Since  $(f', w)$  is the terminal node of path  $p$ , there are  $(w', f') \in V_1$  and an arc from  $(w', f')$  to  $(f', w)$ . Also,  $f' \notin \mu(w)$ , implying that  $C_w(\mu(w) \cup \{f'\}) = \mu(w)$ . Thus, by Step 2 (b) of the reduction procedure,  $\{f'\}$  is eliminated from  $w$ 's preference list. Thus, by Step 3 of the reduction procedure, all subsets of workers containing  $w$  are eliminated from preference list of  $f'$  as well. This contradicts that  $(f', w) \in V_2$ . Second, assume that  $f' \in C_w(\mu(w) \cup \{f'\})$ . Since  $f' \notin \mu(w)$  and  $C_w(\mu(w) \cup \{f'\}) \neq \mu(w)$ , then  $\{f'\}$  is not eliminated on Step 2 (b) of the reduction procedure. Moreover, Step 3 of the reduction procedure neither eliminates  $f'$  nor  $w$  from each other's preference lists, because  $(f', w) \in V_2$ . Then,  $C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\}) = C_w(\mu(w) \cup \{f'\})$ , implying that  $f' \in C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\})$ , contradicting this subcase's hypothesis.

Therefore, by Cases 1 and 2, path  $p$  has no terminal node so it is a cycle in digraph  $D^{\mu, \tilde{\mu}}$ . As a consequence, a cycle for  $P^{\mu, \tilde{\mu}}$  must also exist.  $\square$

*Proof of Proposition 2.* Let  $\mu'$  be a cyclic matching under  $P^{\mu, \tilde{\mu}}$ . Let  $\sigma$  be the cycle associated with  $\mu'$ . First, we prove that  $\mu'$  is an individually rational matching under  $P^{\mu, \tilde{\mu}}$ . If  $a \in F \cup W$  with  $a \notin \sigma$ , we have that  $\mu'(a) = \mu(a)$ . Then, by the individual rationality of  $\mu$  under  $P^{\mu, \tilde{\mu}}$  we have that  $C_a^{\mu, \tilde{\mu}}(\mu'(a)) = \mu'(a)$ . If  $f \in \sigma$ , there is  $w' \in \sigma$  such that  $\mu'(f) = C_f^{\mu, \tilde{\mu}}(W \setminus \{w'\})$ . Thus,  $C_f^{\mu, \tilde{\mu}}(\mu'(f)) = C_f^{\mu, \tilde{\mu}}(C_f^{\mu, \tilde{\mu}}(W \setminus \{w'\})) = C_f^{\mu, \tilde{\mu}}(W \setminus \{w'\}) = \mu'(f)$ . If  $w \in \sigma$ , there is  $f \in \mu(w)$  and  $f' \in \sigma$  such that  $\mu'(w) = (\mu(w) \setminus \{f\}) \cup \{f'\}$ . Then,  $C_w^{\mu, \tilde{\mu}}(\mu'(w)) = C_w^{\mu, \tilde{\mu}}((\mu(w) \setminus \{f\}) \cup \{f'\})$ . By definition of a cycle,  $C_w^{\mu, \tilde{\mu}}((\mu(w) \setminus \{f\}) \cup \{f'\}) = C_w^{\mu, \tilde{\mu}}(C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\})) = C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\}) = \mu'(w)$ . Therefore,  $\mu'$  is individually rational under  $P^{\mu, \tilde{\mu}}$ .

Second, assume that there is a blocking pair  $(f, w)$  for  $\mu'$  under  $P^{\mu, \tilde{\mu}}$ . We claim that both  $f$  and  $w$  belong to  $\sigma$ . Furthermore,  $w$  immediately precedes  $f$  in cycle  $\sigma$ . In order

to see this, first assume that  $f \notin \sigma$ . Thus, by the definition of cyclic matching,  $\mu'(f) = \mu(f)$  and since, by Remark 1 (i),  $\mu'(f)$  is the most preferred subset of workers in  $P_f^{\mu, \tilde{\mu}}$ , there is no  $w' \notin \mu'(f)$  such that  $w' \in C_f^{\mu, \tilde{\mu}}(\mu'(f) \cup \{w'\})$ . When  $w' = w$ , this contradicts that  $(f, w)$  is a blocking pair for  $\mu'$ . Therefore,  $f \in \sigma$ .

Also, as  $(f, w)$  is a blocking pair for  $\mu'$ ,  $w \in C_f^{\mu, \tilde{\mu}}(\mu'(f) \cup \{w\})$ . By the definition of cycle, there is  $w'$  such that  $C_f^{\mu, \tilde{\mu}}(W \setminus \{w'\}) = \mu'(f)$  and thus  $w \in C_f^{\mu, \tilde{\mu}}(C_f^{\mu, \tilde{\mu}}(W \setminus \{w'\}) \cup \{w\})$  which in turn, by (1), becomes

$$w \in C_f^{\mu, \tilde{\mu}}((W \setminus \{w'\}) \cup \{w\}). \quad (13)$$

To see that  $w$  immediately precedes  $f$  in cycle  $\sigma$ , i.e.  $w = w'$ , assume that  $w \neq w'$ . Then,  $w \in W \setminus \{w'\}$  and, therefore, (13) implies  $w \in C_f^{\mu, \tilde{\mu}}(W \setminus \{w'\}) = \mu'(f)$ . Thus,  $w \in \mu'(f)$ , which contradicts  $(f, w)$  being a blocking pair for  $\mu'$ . Hence,  $w = w'$ . This completes our claim.

To finish our proof, notice that by definition of cyclic matching and the fact that  $w = w' \in \sigma$ , there is  $f'$  such that

$$C_w^{\mu, \tilde{\mu}}(\mu(w) \cup \{f'\}) = (\mu(w) \setminus \{f\}) \cup \{f'\} = \mu'(w). \quad (14)$$

Since  $\mu(w) \cup \{f'\} = \mu'(w) \cup \{f\}$ , using (14) and  $f \notin \mu'(w)$  (that follows from  $(f, w)$  being a blocking pair for  $\mu'$ ) we have that  $f \notin C_w^{\mu, \tilde{\mu}}(\mu'(w) \cup \{f\})$ . But then again we contradict that  $(f, w)$  is a blocking pair for  $\mu'$ . Hence,  $\mu' \in S(P^{\mu, \tilde{\mu}})$ .  $\square$

*Proof of Proposition 3.* Let  $\mu, \mu' \in S(P)$  with  $\mu \succ_F \mu'$ . Consider the reduced preference profile  $P^{\mu, \mu'}$ . By Proposition 1, there is a cycle  $\sigma$  for  $P^{\mu, \mu'}$ . Let  $\mu_\sigma$  be its corresponding cyclic matching under  $P^{\mu, \mu'}$ . By Proposition 2,  $\mu_\sigma \in S(P^{\mu, \mu'})$  and, consequently,  $\mu_\sigma \succeq_F \mu'$  by Lemma 1. Furthermore,  $\mu \succeq_F \mu_\sigma$  follows straightforward from the fact that  $\mu_\sigma \in S(P^\mu)$  and that  $\mu$  is the firm-optimal stable matching for  $P^\mu$ .  $\square$

**Lemma 4** *Let  $\mu, \tilde{\mu} \in S(P)$  with  $\mu \succeq_F \tilde{\mu}$ . If  $\tilde{\mu}$  is a cyclic matching under  $P^{\mu, \tilde{\mu}}$ , then  $\tilde{\mu}$  is a cyclic matching under  $P^\mu$ .*

*Proof.* Let  $\mu, \tilde{\mu} \in S(P)$  with  $\mu \succeq_F \tilde{\mu}$  and let  $\tilde{\mu}$  be a cyclic matching under  $P^{\mu, \tilde{\mu}}$ . By Theorem 1,  $\tilde{\mu} \in S(P^\mu)$ . Let  $\sigma = \{(w_1, f_1), (w_2, f_2), \dots, (w_r, f_r)\}$  be the cycle associated with  $\tilde{\mu}$ . We only need to prove that  $\sigma$  is a cycle for  $P^\mu$ . First, notice that for each  $(w_i, f_i) \in \sigma$ ,  $w_i \in \mu(f_i) \setminus \tilde{\mu}(f_i)$  implies that  $w_i \in \mu(f_i) \setminus \mu_W(f_i)$ . Otherwise,  $w_i \in \mu_W(f_i)$  and  $w_i \in \mu(f_i) = C_{f_i}^\mu(\mu(f_i) \cup \tilde{\mu}(f_i) \cup \mu_W(f_i))$  imply, by substitutability, that  $w_i \in C_{f_i}^\mu(\tilde{\mu}(f_i) \cup \mu_W(f_i)) = \tilde{\mu}(f_i)$ , a contradiction. Second, by definition of cycle for  $P^{\mu, \tilde{\mu}}$  and Definition 3,  $\tilde{\mu}(f_i) = C_{f_i}^{\mu, \tilde{\mu}}(W \setminus \{w_i\}) = (\mu(f_i) \setminus \{w_i\}) \cup \{w_{i+1}\}$ . By Proposition 2,  $\tilde{\mu}$  is individually rational under  $P^{\mu, \tilde{\mu}}$ . By Lemma 3,  $\tilde{\mu}$  is individually rational



under  $P^\mu$ . Thus,  $C_{f_i}^\mu(\tilde{\mu}(f_i)) = \tilde{\mu}(f_i)$ . Hence,

$$C_{f_i}^\mu(\tilde{\mu}(f_i)) = (\mu(f_i) \setminus \{w_i\}) \cup \{w_{i+1}\}. \quad (15)$$

Lastly, again by definition of cycle for  $P^{\mu, \tilde{\mu}}$ , we have

$$C_{w_i}^{\mu, \tilde{\mu}}(\mu(w_i) \cup \{f_{i-1}\}) = (\mu(w_i) \setminus \{f_i\}) \cup \{f_{i-1}\} = \tilde{\mu}(w_i).$$

Now, we prove that  $C_{w_i}^{\mu, \tilde{\mu}}(\mu(w_i) \cup \{f_{i-1}\}) = C_{w_i}^\mu(\mu(w_i) \cup \{f_{i-1}\})$ . By the reduction procedure, we have that  $C_{w_i}^{\mu, \tilde{\mu}}(\mu(w_i) \cup \{f_{i-1}\}) \subseteq C_{w_i}^\mu(\mu(w_i) \cup \{f_{i-1}\})$ . Now, assume that  $C_{w_i}^\mu(\mu(w_i) \cup \{f_{i-1}\}) \neq C_{w_i}^{\mu, \tilde{\mu}}(\mu(w_i) \cup \{f_{i-1}\})$ . This implies that  $C_{w_i}^\mu(\mu(w_i) \cup \{f_{i-1}\})$  is eliminated in the reduction procedure to obtain  $P^{\mu, \tilde{\mu}}$ . Since  $\mu \in S(P^{\mu, \tilde{\mu}})$ , then the only possibility is that the firm selected by the reduction procedure to eliminate from  $C_{w_i}^\mu(\mu(w_i) \cup \{f_{i-1}\})$  be  $f_{i-1}$ . This contradicts that  $\tilde{\mu}$  is individually rational under  $P^{\mu, \tilde{\mu}}$ , because  $f_{i-1} \in \tilde{\mu}(w_i)$ .  $\square$

*Proof of Proposition 4.* Let  $\mu' \in S(P) \setminus \{\mu_F\}$  and consider the reduced preference profile  $P^{\mu_F, \mu'}$ . If  $\mu'$  is a cyclic matching under  $P^{\mu_F, \mu'}$ , then by Lemma 4  $\mu'$  is a cyclic matching under  $P^{\mu_F}$  and the proof is complete. If not, by Proposition 3 there is a cyclic matching under  $P^{\mu_F, \mu'}$ , say  $\mu_1$ , such that  $\mu_1 \succ_F \mu'$ . By Lemma 1 and Proposition 2,  $\mu_1 \in S(P)$ , so we can consider the reduced preference profile  $P^{\mu_1, \mu'}$ . If  $\mu'$  is a cyclic matching under  $P^{\mu_1, \mu'}$ , then by Lemma 4  $\mu'$  is a cyclic matching under  $P^{\mu_1}$ , and the proof is complete. If not, continue this process until, by the finiteness of  $S(P)$ , there is  $\mu_k \in S(P)$  such that  $\mu'$  is a cyclic matching under  $P^{\mu_k, \mu'}$ , then by Lemma 4  $\mu'$  is a cyclic matching under  $P^{\mu_k}$ .  $\square$

*Proof of Theorem 2.* Let  $(F, W, P)$  be a matching market. First, notice that by Proposition 1, for each reduced profile obtained in Step  $t - 1$ , there is at least a cycle. Proposition 2 and Theorem 1 show that each cyclic matching obtained by the algorithm belongs to  $S(P)$ . To see that each stable matching is computed by the algorithm, assume that it is not the case for  $\mu \in S(P) \setminus \{\mu_F\}$ . By Proposition 4, there is another  $\mu' \in S(P)$  such that  $\mu$  is a cyclic matching under  $P^{\mu'}$  (remember that, as  $\mu$  is a cyclic matching under  $P^{\mu'}$ ,  $\mu' \succ_F \mu$ ). Thus,  $\mu'$  is not computed by the algorithm either (otherwise, if  $\mu'$  is computed by the algorithm in Step  $t$ ,  $\mu$  necessarily is computed in Step  $t + 1$ ). Thus, again by Proposition 4, there is another  $\mu'' \in S(P)$  such that  $\mu'$  is a cyclic matching under  $P^{\mu''}$  with  $\mu'' \succ_F \mu'$  and  $\mu''$  is not computed by the algorithm either. Continuing this line of reasoning, by the finiteness of the set  $S(P)$ , we eventually reach  $\mu_F$  and conclude that the algorithm cannot compute it either, which is absurd.  $\square$

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